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# How to Make Systems of Nonlinear Autonomous ODEs with Attractor-Behavior, by First Making the General Solutions: Part One

**Magne Stensland**

Moldjord, Norway
Email: mag-ste@online.no

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## Abstract

In this paper, we will present a new method for making first-order systems of nonlinear autonomous ODEs that exhibit limit cycles with a specific geometric shape in two and three dimensions, or systems of ODEs where surfaces in three dimensions have attractor behavior. The method is to make the general solutions first by using the exponential function, sine and cosine. We are building up the general solutions bit for bit according to the constant terms that contain the formula of the desired limit cycle, and differentiating them. We will obtain a system of ODEs with the desired behavior. We design the general solutions for a distinct purpose. Using the methods described in this paper, it is possible to make some systems of nonlinear ODEs that are exhibiting limit cycles with a distinct geometric shape in two or three dimensions, and some surfaces having attractor behavior. The pictures show the result.

## Keywords

System of Nonlinear ODEs, Limit Cycle, General Solution, Attractor

## 1. Introduction

In the Introduction to [1], we can read: nonlinear dynamical systems exhibiting limit cycles are found in a large variety of fields including biology, chemistry, mechanics and electronics.

Over the past two decades, the theory of limit cycles, especially for quadratic differential systems, has progressed dramatically in China as well as in other countries [2].

If somebody wants to make a system of ODEs that is exhibiting a limit cycle (LC) with a certain geometric shape, or a distinct surface that has attractor behavior, how can we do it?

1) For example, making a system of ODEs exhibiting a LC with the three-dimensional shape determined by the projection of the closed curve $x^4 - x^2y^2 + y^4 = f^4$ onto the paraboloid $z = x^2 + y^2$. $f$ is a parameter.

2) Or making a system of ODEs and their general solutions that are exhibiting four LC, one in each quadrant.

3) Or making a system of ODEs where the surface of an ellipsoid has attractor-behavior, that is attracting the solution curves from both inside and outside the ellipsoid, and then following the surface of the ellipsoid as the variable $t$ goes to infinity.

In this paper, we will describe some techniques I have developed for making nonlinear first-order systems with some special behavior. For the most part, we are going to work with systems of ODEs exhibiting limit cycles in two and three dimensions. For the first, we will decide how to write the general solutions in order to make them easier to work with. Then, we will make general solutions to systems of ODEs that are exhibiting limit cycles, and give a two-dimensional LC a three-dimensional shape by projecting it onto a surface for which we know the formula to. No difficult or impossible integrals, just some "funny" differentiations.

The general solutions that we are building up in this paper are very logic and are giving a simple and logic explanation of why limit cycles appear, namely that all exponential functions have disappeared when the solution curves arrive a specific closed curve, whether this curve is in two or three dimensions. At a limit cycle, the general solutions have only the functions sine and cosine left. That explains why we see only rotation at a limit cycle.

The method that we will use in this paper is new to the literature, at least to my knowledge. My background for the investigations is the textbook Differential Equations [3]. The number I have given each system of equations, is the number they have in my collection. These techniques have been funny to develop, so let us make some funny limit cycles!

## 2. How to Write the Solutions

For the first we will look at the way I have chosen to write the solutions, in order to "simplify" the derivations.

The differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x - x^2 \tag{1}$$

has the general solution

$$x(t) = \frac{1}{1 - \frac{1}{x_0}(x_0 - 1)\mathrm{e}^{-t}} \tag{2}$$

This is how we find the solution in textbooks.

Multiply the numerator and the denominator with $x_0\mathrm{e}^t$:

$$x(t)=\frac{x_0\mathrm{e}^t}{x_0\mathrm{e}^t-x_0+1} \tag{3}$$

The drawback with the last expression is that we have got the same exponential function both in the numerator and the denominator. I prefer to use (3) because:

1) The exponent in the exponential function has the same sign than the coefficient in the differential equation. It is easier to see the connection between the solution and the equation.

2) It is easier to differentiate the solution when it is written in the form (3).

I have consequently chosen to write the solutions on the form (3). But when we want to analyze the solutions, we must use (2).

Derivation of fraction:

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{f(x)}{g(x)}=\frac{f'(x)g(x)-g'(x)f(x)}{g(x)^2}=\frac{f'(x)}{g(x)}-\frac{f(x)g'(x)}{g(x)^2} \tag{4}$$

We will use the last expression when derivation of solutions.

We can demonstrate by differentiating (3):

$$\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{x_0\mathrm{e}^t}{x_0\mathrm{e}^t-x_0+1}-\frac{x_0\mathrm{e}^t x_0\mathrm{e}^t}{\left(x_0\mathrm{e}^t-x_0+1\right)^2}=x-x^2 \tag{5}$$

Here, we see how simple the derivation becomes, and how easy it is to see the connection between the derivative and the solution $x(t)$, when we write the solution on the form (3).

The general expression of the solution (3) is:

$$x(t)=\frac{fx_0\mathrm{e}^{at}}{x_0\mathrm{e}^{at}-x_0+f} \tag{6}$$

$$\frac{\mathrm{d}x}{\mathrm{d}t}=ax-\frac{a}{f}x^2 \tag{7}$$

In order to practice the method that we are going to use later, it is crucial necessary that we write the solutions so that $x(0)=x_0, y(0)=y_0$ and $z(0)=z_0$ If we put $t=0$ in the solution (6), we will get $x(0)=x_0$.

Another solution we will use is when the denominator is square root:

$$x(t)=\frac{fx_0\mathrm{e}^{at}}{\sqrt{x_0^2\mathrm{e}^{2at}-x_0^2+f^2}} \tag{8}$$

$$x(0)=x_0$$

$$\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{afx_0\mathrm{e}^{at}}{\sqrt{x_0^2\mathrm{e}^{2at}-x_0^2+f^2}}-\frac{fx_0\mathrm{e}^{at}\frac{1}{2}2ax_0^2\mathrm{e}^{2at}}{\sqrt{x_0^2\mathrm{e}^{2at}-x_0^2+f^2}\left(x_0^2\mathrm{e}^{2at}-x_0^2+f^2\right)} \tag{9}$$
$$=ax-\frac{a}{f^2}x^3$$

And once again we can see how easy it is to see the connection between the derivative $\frac{\mathrm{d}x}{\mathrm{d}t}$ and the solution $x(t)$ when we write the solution in this way. Notice that all parts in the square root are of same degree, namely second degree. If the denominator is fourth root, all parts inside the root must be of fourth degree, and so on. This is important when we shall build up the solutions bit for bit, unless it will be wrong.

## 3. How to Make Limit Cycles (LC)

### 3.1. The Theory behind LC

A spiral contains two types of motion: One rotating motion according to the function sine and cosine, and one outwards or inwards motion according to the exponential function. Remember that we are using only these three functions in the solutions described in this paper. In order to make a stable limit cycle, the equilibrium point must be spiral source. If we choose initial values $(x_0, y_0)$ inside the LC, the solution curve will make an outwards rotating spiral ending up at the LC. And if we choose initial values outside the LC, the solution curve will make an inwards rotating spiral ending up at the LC. When the solution curves arrive the LC, either from outside or inside the LC, the outwards or inwards motion will stop. We have only rotating motion left. What has happened?

Regarding the general solutions this means that the exponential functions have disappeared, and the solutions contain only the functions sine and cosine, that provide rotation. If we choose initial values $(x_0, y_0)$ wherever on a stable LC, we will observe only rotation along the same closed curve over and over when the variable $t$ goes to infinity.

This tells us that we must make the general solutions so that if we choose initial values wherever on the LC, all exponential functions are gone. And if we choose initial values wherever else, the general solutions will contain exponential functions. The general solutions must contain a constant term made so that if we choose initial values on the LC, this constant term becomes zero.

### 3.2. LC in Two Dimensions

A well-known example is a circular LC that is the graph of $x^2 + y^2 = R^2$.

But let us try to make a system of two differential equations exhibiting a stable LC with the geometric shape of the ellipse $\frac{x^2}{g^2} + \frac{y^2}{h^2} = 1$.

When I am making system of differential equations exhibiting LC, I always start with the constant term of the general solutions, and building the solution according to the constant term.

When we want an elliptic LC, the constant term will be:

$$-\left(\frac{x_0^2}{g^2} + \frac{y_0^2}{h^2}\right) + 1 \tag{10}$$

This part of the general solution will become zero when we choose initial values $(x_0, y_0)$ on the ellipse $\frac{x^2}{g^2}+\frac{y^2}{h^2}=1$. But it will not become zero if we choose initial values outside or inside the ellipse.

The general solutions to a system of linear differential equations making spiral are:

$$x(t)=\mathrm{e}^{at}\left(x_0\cos(bt)+y_0\sin(bt)\right) \tag{11}$$

$$y(t)=\mathrm{e}^{at}\left(y_0\cos(bt)-x_0\sin(bt)\right) \tag{12}$$

These solutions give the system:

$$\frac{\mathrm{d}x}{\mathrm{d}t}=ax+by$$

$$\frac{\mathrm{d}y}{\mathrm{d}t}=ay-bx$$

Here is (0, 0) spiral source when $a>0$.

But we will try to make the general solutions so that the spiral is ending at an ellipse $\left(\frac{x^2}{g^2}+\frac{y^2}{h^2}\right)=1$ either we choose initial values inside or outside the ellipse. And the solution curves will follow the ellipse, and repeat the same closed curve over and over when the variable $t$ goes to infinity.

We start with placing the constant term $-\left(\frac{x_0^2}{g^2}+\frac{y_0^2}{h^2}\right)+1$ under the fraction line:

$$\overline{-\left(\frac{x_0^2}{g^2}+\frac{y_0^2}{h^2}\right)+1} \tag{13}$$

In order to use this as a solution we must have $\frac{x_0^2}{g^2}$ and $\frac{y_0^2}{h^2}$ in front of the constant term:

$$\overline{\sqrt{\frac{1}{g^2}\mathrm{e}^{2at}\left(x_0\cos(bt)+y_0\sin(bt)\right)^2+\frac{1}{h^2}\mathrm{e}^{2at}\left(y_0\cos(bt)-x_0\sin(bt)\right)^2-\left(\frac{x_0^2}{g^2}+\frac{y_0^2}{h^2}\right)+1}} \tag{14}$$

We must take the square root of the denominator, since it contains parts of second degree. When $t=0$, we will get 1 under the fraction line. Then we have the general solutions:

$$x(t)=\frac{\mathrm{e}^{at}\left(x_0\cos(bt)+y_0\sin(bt)\right)}{\sqrt{\frac{1}{g^2}\mathrm{e}^{2at}\left(x_0\cos(bt)+y_0\sin(bt)\right)^2+\frac{1}{h^2}\mathrm{e}^{2at}\left(y_0\cos(bt)-x_0\sin(bt)\right)^2-\left(\frac{x_0^2}{g^2}+\frac{y_0^2}{h^2}\right)+1}} \tag{15}$$

$$y(t)=\frac{\mathrm{e}^{at}\left(y_0\cos(bt)-x_0\sin(bt)\right)}{\sqrt{\frac{1}{g^2}\mathrm{e}^{2at}\left(x_0\cos(bt)+y_0\sin(bt)\right)^2+\frac{1}{h^2}\mathrm{e}^{2at}\left(y_0\cos(bt)-x_0\sin(bt)\right)^2-\left(\frac{x_0^2}{g^2}+\frac{y_0^2}{h^2}\right)+1}} \tag{16}$$

We test that $x(0) = x_0$ and $y(0) = y_0$ This is important, unless these methods will not work. We notice that when $\frac{x_0^2}{g^2} + \frac{y_0^2}{h^2} = 1$, the exponential functions will disappear, and the general solutions have only parts left containing the functions sine and cosine. This is what characterizes many of the general solutions to systems of differential equations exhibiting limit cycles. The exponential functions disappear when the solution curves arrive the ellipse. The general solutions are giving a logic explanation to a special behavior. Study the solutions well. We will do the same many times with longer fraction line, but the principle is the same.

We differentiate the solutions (15) and (16) and get the system (101):

$$\frac{\mathrm{d}x}{\mathrm{d}t} = ax + by - ax\left(\frac{x^2}{g^2} + \frac{y^2}{h^2}\right) - bx^2 y\left(\frac{1}{g^2} - \frac{1}{h^2}\right) \tag{17}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = ay - bx - ay\left(\frac{x^2}{g^2} + \frac{y^2}{h^2}\right) - bxy^2\left(\frac{1}{g^2} - \frac{1}{h^2}\right) \tag{18}$$

And once again we see how simple the differentiations becomes when we write the solutions in this way, and how easy it is to express the derivative by means of the solutions $x(t)$ and $y(t)$. Notice that when $g = h$, the last part in each equation become zero, and we get the same equations as for a circular LC. Notice also that the parameter *a* belongs to the exponential functions, and the parameter *b* belongs to the function sine and cosine.

When $\frac{x^2}{g^2} + \frac{y^2}{h^2} = 1$, all parts containing *a* disappear. That means that the exponential functions in the general solutions are gone. The parts that are left contain the parameter *b*. Only the sine and cosine parts in the solutions are left. That's why we observe only rotation along the same closed curve over and over when the parameter *t* goes to infinity, and why the spiral stop when it arrives the ellipse no matter from which side. When $a > 0$, the ellipse is a stable LC.

A more general ellipse is $\frac{x^2}{g^2} + \beta\frac{xy}{gh} + \frac{y^2}{h^2} = 1$, $-1 \le \beta \le 1$.

The constant term in these solutions is:

$$-\left(\frac{x_0^2}{g^2} + \beta\frac{x_0 y_0}{gh} + \frac{y_0^2}{h^2}\right) + 1 \tag{19}$$

The fraction line in the solutions $x(t)$ and $y(t)$ is too long for the page. We leave it as an exercise to make the solutions.

Differentiating the solutions give system (129), that makes a LC that can remind a bit of Van der Pool's LC.

$$\frac{\mathrm{d}x}{\mathrm{d}t} = ax + by - x\left[a\left(\frac{x^2}{g^2} + \beta\frac{xy}{gh} + \frac{y^2}{h^2}\right) + bxy\left(\frac{1}{g^2} - \frac{1}{h^2}\right) + \beta\frac{b}{2gh}\left(y^2 - x^2\right)\right] \tag{20}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = ay - bx - y\left[a\left(\frac{x^2}{g^2} + \beta\frac{xy}{gh} + \frac{y^2}{h^2}\right) + bxy\left(\frac{1}{g^2} - \frac{1}{h^2}\right) + \beta\frac{b}{2gh}\left(y^2 - x^2\right)\right] \tag{21}$$

See **Figure 1**, where $a = b = 1, g = 2, h = 3, \beta = -1$.

Let us try to make a system of differential equations and their general solutions where the closed curve $x^4 + y^4 = f^4$ becomes a LC. Place the formula for this closed curve under the fraction line in form of a constant term, as done above:

$$-\left(x_0^4 + y_0^4\right) + f^4 \tag{22}$$

Then we must have fourth degree parts of $x_0$ and $y_0$ in front of the constant term, and taking the fourth root of the denominator:

$$x(t) = \frac{f\,\mathrm{e}^{at}\left(x_0\cos(bt) + y_0\sin(bt)\right)}{\sqrt[4]{\mathrm{e}^{4at}\left(x_0\cos(bt) + y_0\sin(bt)\right)^4 + \mathrm{e}^{4at}\left(y_0\cos(bt) - x_0\sin(bt)\right)^4 - \left(x_0^4 + y_0^4\right) + f^4}} \tag{23}$$

$$y(t) = \frac{f\,\mathrm{e}^{at}\left(y_0\cos(bt) - x_0\sin(bt)\right)}{\sqrt[4]{\mathrm{e}^{4at}\left(x_0\cos(bt) + y_0\sin(bt)\right)^4 + \mathrm{e}^{4at}\left(y_0\cos(bt) - x_0\sin(bt)\right)^4 - \left(x_0^4 + y_0^4\right) + f^4}} \tag{24}$$

We differentiate these solutions and get the system (102):

$$\frac{\mathrm{d}x}{\mathrm{d}t} = ax + by - \frac{x}{f^4}\left[a\left(x^4 + y^4\right) + bx^3y - bxy^3\right] \tag{25}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = ay - bx - \frac{y}{f^4}\left[a\left(x^4 + y^4\right) + bx^3y - bxy^3\right] \tag{26}$$

In **Figure 2**, we see the result. Here are $a = 1, b = 1, f = 2$.

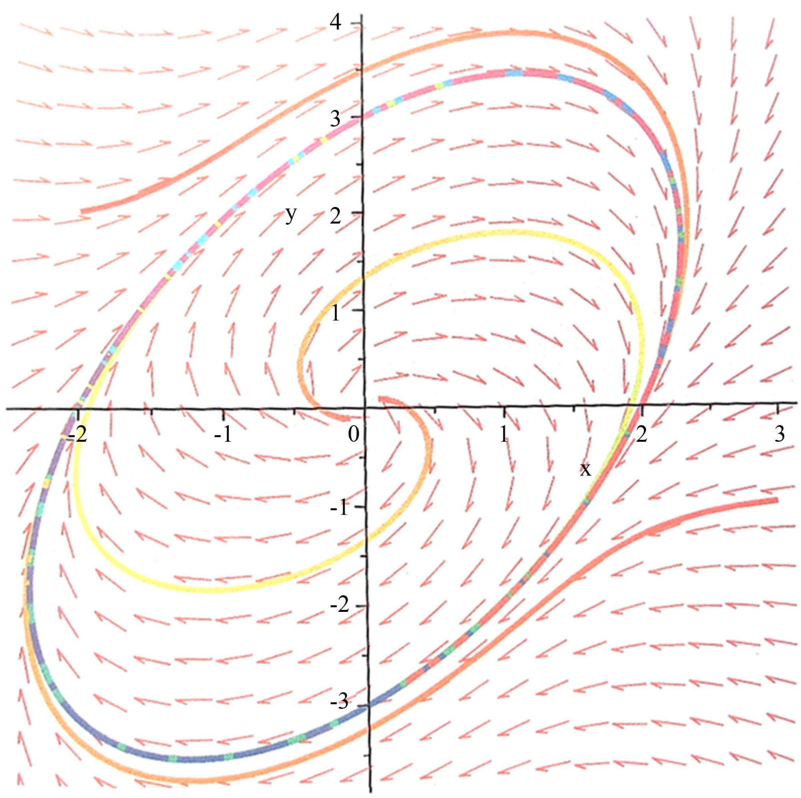


**Figure 1.** An elliptic LC.

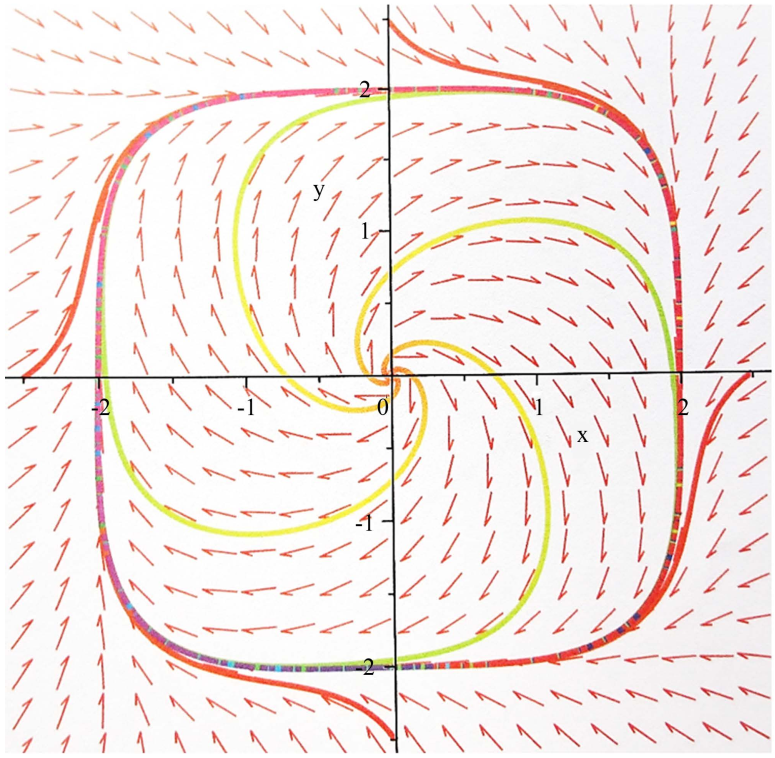


**Figure 2.** The closed curve $x^4 + y^4 = 16$ as a LC.

### 3.3. LC in Three Dimensions

We will try to place a two-dimensional LC on a surface that we know the formula to, so that the LC will completely follow this surface. The solution curves will then follow the surface, and repeat the same closed curve over and over when the variable $t$ goes to infinity. We are going to give a two-dimensional LC a three-dimensional shape.

We must make the solution $z(t)$ so that the exponential function belonging to $z(t)$ has disappeared when the solution curves have arrived this surface. We will make a constant term in a such way that when we choose initial values on this surface, this constant term will become zero, and for all other initial values is this constant term not zero.

In the first example we will try to make a LC with a geometric shape of the projection of an ellipse on a paraboloid or hyperbolic paraboloid $z = cx^2 + dy^2$. The solution $z(t)$ contain two constant terms: One for the ellipse $-\left(\frac{x_0^2}{g^2} + \frac{y_0^2}{h^2}\right) + 1$, as we earlier have placed under the fraction line in both solutions $x(t)$ and $y(t)$. And one constant term containing the formula for the paraboloid or hyperbolic paraboloid $+\left(z_0 - cx_0^2 - dy_0^2\right)$, and multiply this last constant term with an exponential function. When both constant terms are zero, are absolutely all exponential functions gone, and the general solutions contain only the functions sine and cosine. We have then obtained a three-dimensional LC.

$$z(t)=cx(t)^2+dy(t)^2+\left(z_0-cx_0^2-dy_0^2\right)\mathrm{e}^{kt} \tag{27}$$

The solutions $x(t)$ and $y(t)$ have we made earlier, (15) and (16).

We put in the expressions for the solutions $x(t)$ and $y(t)$:

$$z(t)=\frac{c\mathrm{e}^{2at}\left(x_0\cos(bt)+y_0\sin(bt)\right)^2+d\mathrm{e}^{2at}\left(y_0\cos(bt)-x_0\sin(bt)\right)^2}{\frac{\mathrm{e}^{2at}}{g^2}\left(x_0\cos(bt)+y_0\sin(bt)\right)^2+\frac{\mathrm{e}^{2at}}{h^2}\left(y_0\cos(bt)-x_0\sin(bt)\right)^2-\left(\frac{x_0^2}{g^2}+\frac{y_0^2}{h^2}\right)+1} \tag{28}$$
$$+\left(z_0-cx_0^2-dy_0^2\right)\mathrm{e}^{kt}$$

We take it step by step:

$$\begin{aligned}\frac{\mathrm{d}z}{\mathrm{d}t}&=2a\left(cx^2+dy^2\right)+2bxy(c-d)-\left(cx^2+dy^2\right)\left[2a\left(\frac{x^2}{g^2}+\frac{y^2}{h^2}\right)\right.\\&\left.+2bxy\left(\frac{1}{g^2}-\frac{1}{h^2}\right)\right]+k\left(z_0-cx_0^2-dy_0^2\right)\mathrm{e}^{kt}\end{aligned} \tag{29}$$

$$kz=k\left(cx^2+dy^2\right)+k\left(z_0-cx_0^2-dy_0^2\right)\mathrm{e}^{kt} \tag{30}$$

Subtract $kz$ from both sides of the equation $\frac{\mathrm{d}z}{\mathrm{d}t}$ and the result will be:

$$\begin{aligned}\frac{\mathrm{d}z}{\mathrm{d}t}&=kz+(2a-k)\left(cx^2+dy^2\right)+2bxy(c-d)\\&-\left(cx^2+dy^2\right)\left[2a\left(\frac{x^2}{g^2}+\frac{y^2}{h^2}\right)+2bxy\left(\frac{1}{g^2}-\frac{1}{h^2}\right)\right]\end{aligned} \tag{31}$$

Here again we see how simple the derivations become, and how easy it is to express the derivative $z(t)$ by means of the solutions $x(t)$ and $y(t)$, when we write the solutions in this way.

Together with the solutions (15) and (16) we have the system (369). **Figure 3** shows the result.

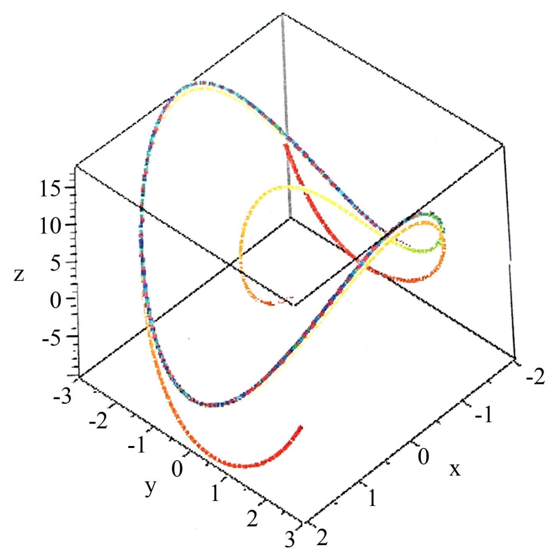


**Figure 3.** An elliptic LC on a hyperbolic paraboloid.

In Figure 3 are $a = b = 1, k = c = -1, d = 2$.

This system of three differential equations is exhibiting a LC with the desired geometric shape. Choose $k < 0$. Then the *xy*-plane will act as a sink, and prevent the solution curves to go to infinity parallel with the z-axis.

We see that when $\frac{x^2}{g^2} + \frac{y^2}{h^2} = 1$, all parts containing the parameter *a* are gone. And when $z = cx^2 + dy^2$, all parts containing the parameter *k* are also gone. When the solution curves arrive the projection of the ellipse $\frac{x^2}{g^2} + \frac{y^2}{h^2} = 1$ on the paraboloid or hyperbolic paraboloid $z = cx^2 + dy^2$, have all exponential functions disappeared. We have only parts containing the parameter *b* left. This parameter belongs to the function sine and cosine, that give rotation along this three-dimensional curve:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = by - bx^2 y\left(\frac{1}{g^2} - \frac{1}{h^2}\right) \tag{32}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -bx - bxy^2\left(\frac{1}{g^2} - \frac{1}{h^2}\right) \tag{33}$$

$$\frac{\mathrm{d}z}{\mathrm{d}t} = 2bxy\left(c - d\right) - 2bxyz\left(\frac{1}{g^2} - \frac{1}{h^2}\right) \tag{34}$$

Now we are able to solve problem a) in the Introduction: Make a LC with the geometric shape of the closed curve $x^4 - x^2y^2 + y^4 = f^4$ on the paraboloid $z = x^2 + y^2$. The constant term under the fraction line will be:

$$-\left(x_0^4 - x_0^2 y_0^2 + y_0^4\right) + f^4 \tag{35}$$

The fraction line will be too long for this page. We compress the solutions by putting $A = x_0 \cos(bt) + y_0 \sin(bt)$, $B = y_0 \cos(bt) - x_0 \sin(bt)$

$$x(t) = \frac{f\,\mathrm{e}^{at}(A)}{\sqrt[4]{\mathrm{e}^{4at}\left((A)^4 - (A)^2(B)^2 + (B)^4\right) - \left(x_0^4 - x_0^2 y_0^2 + y_0^4\right) + f^4}} \tag{36}$$

$$y(t) = \frac{f\,\mathrm{e}^{at}(B)}{\sqrt[4]{\mathrm{e}^{4at}\left((A)^4 - (A)^2(B)^2 + (B)^4\right) - \left(x_0^4 - x_0^2 y_0^2 + y_0^4\right) + f^4}} \tag{37}$$

$$z(t) = cx(t)^2 + dy(t)^2 + \left(z_0 - cx_0^2 - dy_0^2\right)\mathrm{e}^{kt} \tag{38}$$

Then we have the solution $z(t)$:

$$z(t) = \frac{f^2 \mathrm{e}^{2at}\left(c(A)^2 + d(B)^2\right)}{\sqrt{\mathrm{e}^{4at}\left((A)^4 - (A)^2(B)^2 + (B)^4\right) - \left(x_0^4 - x_0^2 y_0^2 + y_0^4\right) + f^4}} + \left(z_0 - cx_0^2 - dy_0^2\right)\mathrm{e}^{kt} \tag{39}$$

These solutions give system (507):

$$\frac{\mathrm{d}x}{\mathrm{d}t} = ax + by - \frac{x}{f^4}\left[a\left(x^4 - x^2y^2 + y^4\right) + \frac{3}{2}bx^3y - \frac{3}{2}bxy^3\right] \tag{40}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = ay - bx - \frac{y}{f^4}\left[a\left(x^4 - x^2y^2 + y^4\right) + \frac{3}{2}bx^3y - \frac{3}{2}bxy^3\right] \tag{41}$$

$$\begin{aligned}\frac{\mathrm{d}z}{\mathrm{d}t} &= kz + (2a-k)\left(cx^2 + dy^2\right) + 2bxy(c-d) \\ &\quad - \frac{1}{f^4}\left(cx^2 + dy^2\right)\left[2a\left(x^4 - x^2y^2 + y^4\right) + 3bx^3y - 3bxy^3\right]\end{aligned} \tag{42}$$

See Figure 4 for the result. Here are $a = 1, b = 1, f = 1, c = 1, d = 1, k = -1$

In the same way we can use the more general closed curve $\frac{x^4}{g^4} + \beta\frac{x^2}{g^2}\frac{y^2}{h^2} + \frac{y^4}{h^4} = 1$, $-1 \le \beta \le 1$, and make a three-dimensional LC by projecting it on whatever surface that we know the formula to.

In the next examples, we will try to project some limit cycles on surfaces, where the formula for the surface contains trigonometric functions. For the first we will try to place an elliptic LC on the surface $z = \sin x \sin y$.

We are using the solutions $x(t)$ and $y(t)$, (15) and (16) to the system (101). The constant term belonging to the solution $z(t)$ must be $+\left(z_0 - \sin(x_0)\sin(y_0)\right)$ Multiply this term with an exponential function, and we have the solution $z(t)$:

$$z(t) = \sin(x(t))\sin(y(t)) + \left(z_0 - \sin(x_0)\sin(y_0)\right)\mathrm{e}^{kt} \tag{43}$$

We will take it step by step:

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \sin y \cos x \frac{\mathrm{d}x}{\mathrm{d}t} + \sin x \cos y \frac{\mathrm{d}y}{\mathrm{d}t} + k\left(z_0 - \sin(x_0)\sin(y_0)\right)\mathrm{e}^{kt} \tag{44}$$

Subtract $kz$ from both sides of the equation $\frac{\mathrm{d}z}{\mathrm{d}t}$:

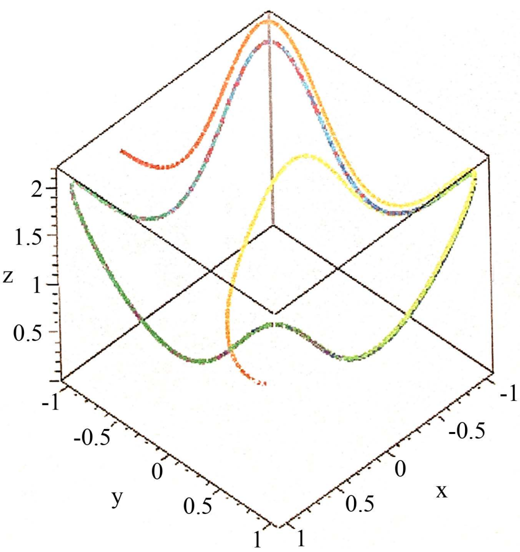


Figure 4. The LC $x^4 - x^2y^2 + y^4$ on $z = x^2 + y^2$.

$$\frac{\mathrm{d}z}{\mathrm{d}t}-kz=\sin y\cos x\frac{\mathrm{d}x}{\mathrm{d}t}+\sin x\cos y\frac{\mathrm{d}y}{\mathrm{d}t}-k\sin x\sin y \tag{45}$$

Put in the equation $\frac{\mathrm{d}x}{\mathrm{d}t}$ and $\frac{\mathrm{d}y}{\mathrm{d}t}$ and we become the equation $\frac{\mathrm{d}z}{\mathrm{d}t}$:

$$\begin{aligned}\frac{\mathrm{d}z}{\mathrm{d}t}&=kz-k\sin x\sin y+\sin y\cos x\left[ax+by-ax\left(\frac{x^2}{g^2}+\frac{y^2}{h^2}\right)-bx^2y\left(\frac{1}{g^2}-\frac{1}{h^2}\right)\right]\\&\quad+\sin x\cos y\left[ay-bx-ay\left(\frac{x^2}{g^2}+\frac{y^2}{h^2}\right)-bxy^2\left(\frac{1}{g^2}-\frac{1}{h^2}\right)\right]\end{aligned} \tag{46}$$

Together with the Equations (17) and (18) we have the system (1187).

See the result in **Figure 5**, where $a=b=1, k=-1, g=3, h=4$.

In the next example we will try to project a circular LC on the surface

$$z=\arctan\left[\sin(3x-y)+\cos(x+2y)\right] \tag{47}$$

By using the methods described earlier, you can place a two-dimensional LC on whatever surface, no matter how complicated the formula to this surface is. The general solution will become long, but not a big problem to differentiate. Not as difficult as it seems to be.

The constant term belonging to the solution $z(t)$ is

$$+\left(z_0-\arctan\left[\sin(3x_0-y_0)+\cos(x_0+2y_0)\right]\right) \tag{48}$$

$$\begin{aligned}\frac{\mathrm{d}z}{\mathrm{d}t}&=kz-k\arctan\left[\sin(3x-y)+\cos(x+2y)\right]\\&\quad+\frac{\cos(3x-y)\left(3\frac{\mathrm{d}x}{\mathrm{d}t}-\frac{\mathrm{d}y}{\mathrm{d}t}\right)-\sin(x+2y)\left(\frac{\mathrm{d}x}{\mathrm{d}t}+2\frac{\mathrm{d}y}{\mathrm{d}t}\right)}{1+\left(\sin(3x-y)+\cos(x+2y)\right)^2}\end{aligned} \tag{49}$$

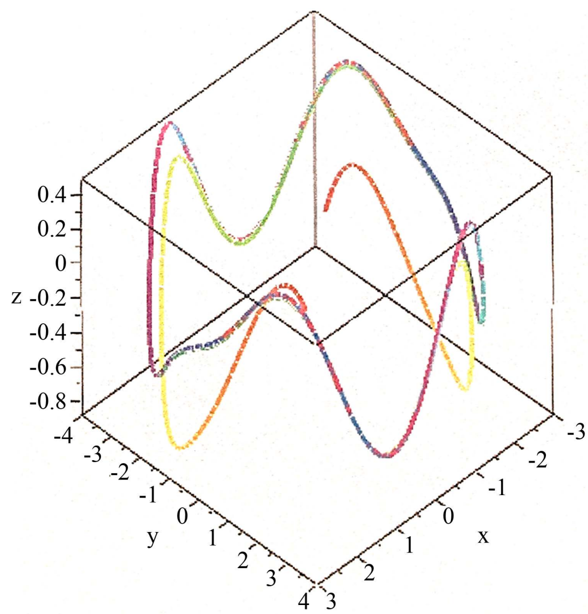


**Figure 5.** An elliptic LC on the surface $z=\sin x\sin y$.

For a circular LC is

$$\frac{\mathrm{d}x}{\mathrm{d}t} = ax + by - \frac{a}{R^2}\left(x^2 + y^2\right) \tag{50}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = ay - bx - \frac{a}{R^2}\left(x^2 + y^2\right) \tag{51}$$

By putting these expressions for $\frac{\mathrm{d}x}{\mathrm{d}t}$ and $\frac{\mathrm{d}y}{\mathrm{d}t}$ into (39) we have a solution to problem (37).

See **Figure 6** for the result.

### 3.4. Systems That Make 4 LC, One in Each Quadrant in Two and Three Dimensions

In all the systems of differential equations we have made so far exhibiting LC, has (0, 0) been the only equilibrium point. Any point can be equilibrium point for a system giving LC.

So now we will try to make the general solutions so that we will have a system of two differential equations giving 4 LC, one in each quadrant.

A formula for four closed curves is

$$\left(x^2 - p^2\right)^2 + \left(y^2 - q^2\right)^2 = f^4 \tag{52}$$

Centers for these 4 curves are $(\pm p, \pm q)$

The constant term under the fraction line is

$$-\left(\left(x_0^2 - p^2\right)^2 + \left(y_0^2 - q^2\right)^2\right) + f^4 \tag{53}$$

The solutions $x(t)$ and $y(t)$ must be as we see below:

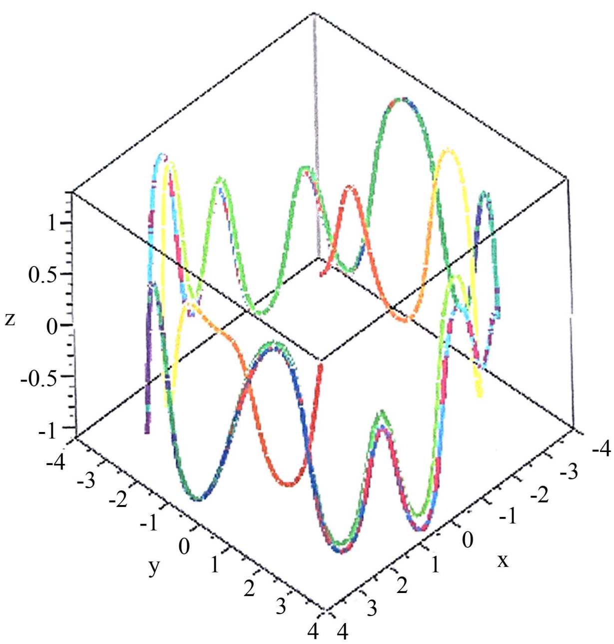


**Figure 6.** A circular LC on the surface $z = \arctan\left[\sin(3x - y) + \cos(x + 2y)\right]$.

$$x(t)=\sqrt{\frac{f^2e^{2at}\left(\left(x_0^2-p^2\right)\cos(bt)+\left(y_0^2-q^2\right)\sin(bt)+p^2\right)}{\sqrt{e^{4at}\left(\left(x_0^2-p^2\right)\cos(bt)+\left(y_0^2-q^2\right)\sin(bt)\right)^2+e^{4at}\left(\left(y_0^2-q^2\right)\cos(bt)-\left(x_0^2-p^2\right)\sin(bt)\right)^2-\left(\left(x_0^2-p^2\right)^2+\left(y_0^2-q^2\right)^2\right)+f^4}}} \tag{54}$$

$$y(t)=\sqrt{\frac{f^2e^{2at}\left(\left(y_0^2-q^2\right)\cos(bt)-\left(x_0^2-p^2\right)\sin(bt)+q^2\right)}{\sqrt{e^{4at}\left(\left(x_0^2-p^2\right)\cos(bt)+\left(y_0^2-q^2\right)\sin(bt)\right)^2+e^{4at}\left(\left(y_0^2-q^2\right)\cos(bt)-\left(x_0^2-p^2\right)\sin(bt)\right)^2-\left(\left(x_0^2-p^2\right)^2+\left(y_0^2-q^2\right)^2\right)+f^4}}} \tag{55}$$

Notice the addition $p^2$ above the fraction line in the solution $x(t)$, and the addition $q^2$ in the solution $y(t)$. And also that we have taken the square root of the denominator, and taken the square root of the hole solution. Put $t=0$ into these solutions, and convince yourself that $x(0)=x_0$ and that $y(0)=y_0$. Then you will understand why the solutions must be so.

We can also notice that when $\left(x_0^2-p^2\right)^2+\left(y_0^2-q^2\right)^2=f^4$, all exponential functions are gone. We will do the differentiation step by step. The expression inside the big square root is the same as $x^2$ in the first solution, and $y^2$ in the second solution.

$$\begin{aligned}\frac{dx}{dt}=\frac{1}{2\sqrt{x^2}}\Big\{&2ax^2+b\left(y^2-q^2\right)-\frac{x^2}{2f^4}\Big[4a\left(x^2-p^2\right)^2+4a\left(y^2-q^2\right)^2\\&+2\left(x^2-p^2\right)b\left(y^2-q^2\right)-2\left(y^2-q^2\right)b\left(x^2-p^2\right)\Big]\Big\}\end{aligned} \tag{56}$$

$$\frac{dx}{dt}=\frac{1}{2x}\left(2ax^2+b\left(y^2-q^2\right)-\frac{x^2}{f^4}\left[2a\left(x^2-p^2\right)^2+2a\left(y^2-q^2\right)^2\right]\right) \tag{57}$$

This gives the system (768):

$$\frac{dx}{dt}=ax+\frac{b}{2x}\left(y^2-q^2\right)-\frac{a}{f^4}x\left[\left(x^2-p^2\right)^2+\left(y^2-q^2\right)^2\right] \tag{58}$$

$$\frac{dy}{dt}=ay-\frac{b}{2y}\left(x^2-p^2\right)-\frac{a}{f^4}y\left[\left(x^2-p^2\right)^2+\left(y^2-q^2\right)^2\right] \tag{59}$$

Notice that when $\left(x^2-p^2\right)^2+\left(y^2-q^2\right)^2=f^4$, are all parts containing the parameter $a$ gone. And we will get the system:

$$\frac{dx}{dt}=\frac{b}{2x}\left(y^2-q^2\right) \tag{60}$$

$$\frac{dy}{dt}=-\frac{b}{2y}\left(x^2-p^2\right) \tag{61}$$

This system has 4 centers. No spirals. We see also that $\frac{dx}{dt}$ don't exist when $x=0$, and that also $\frac{dy}{dt}$ don't exist for $y=0$.

The geometric shape of the 4 limit cycles shows a good similarity with the picture of the 4 closed curves $\left(x^2-p^2\right)^2+\left(y^2-q^2\right)^2=f^4$ See **Figure 7** and compare with **Figure 8**.

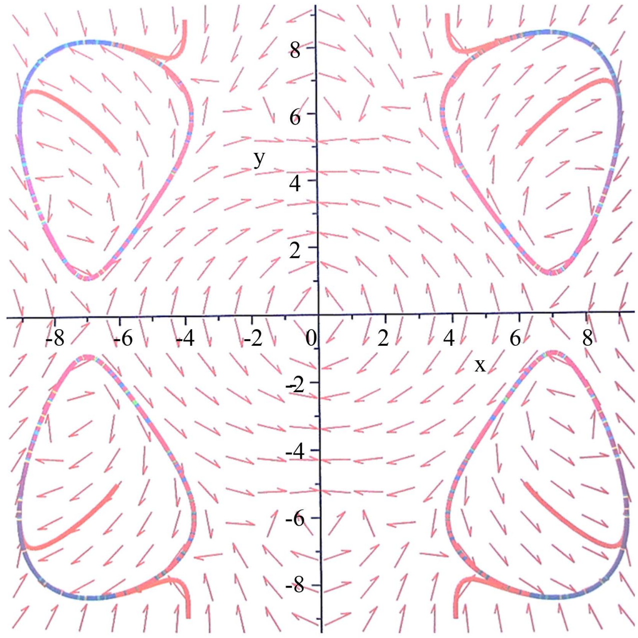


**Figure 7.** Four LC with equilibrium points $(\pm 7, \pm 6)$.

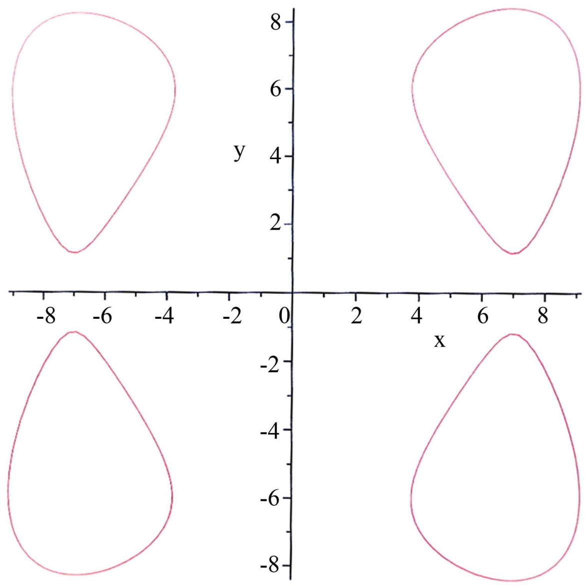


**Figure 8.** The graph of $\left(x^2-49\right)^2+\left(y^2-36\right)^2=1200$.

In **Figure 7** are $a=b=1, p=7, q=6, f^4=1200$.

Let us try to give these 4 limit cycles a three-dimensional shape, by projecting them on a surface. The method is as shown in the last section. Make a solution $z(t)$ containing the formula for this surface as a constant term, and multiply this constant term with an exponential function.

As an example, can we use the formula for a paraboloid or hyperbolic paraboloid. We use the solutions and differential equations to system (768). The solution $z(t)$ will be:

$$z(t)=cx(t)^2+dy(t)^2+\left(z_0-cx_0^2-dy_0^2\right)\mathrm{e}^{kt} \tag{62}$$

Put in the solutions $x(t)$ and $y(t)$, (44) and (45), and differentiate.

$$\begin{aligned}\frac{\mathrm{d}z}{\mathrm{d}t}&=kz+(2a-k)\left(cx^2+dy^2\right)+cb\left(y^2-q^2\right)-db\left(x^2-p^2\right)\\&-\frac{2a}{f^4}\left(cx^2+dy^2\right)\left[\left(x^2-p^2\right)^2+\left(y^2-q^2\right)^2\right]\end{aligned} \tag{63}$$

Together with the differential equations in system (768), we become system (773).

See Figure 9, where $a=1, b=1, c=1, d=1, p=7, q=6, f^4=1200$.

## 4. Some Other Attractors

An attractor can be a point, a curve or a surface [4]. An attractor does attract the solution curves and keep them fast to the attractor, in the same way as a stable limit cycle. The solution curves will follow the curve or the surface when the variable t goes to infinity. Along these curves or surfaces all exponential functions are gone, and the general solutions contain only the functions sine and cosine.

### 4.1. Curves as Attractors

We will make the general solutions as shown earlier, and place the formula for the open curve as a constant term under the fraction line in the solutions, so that if we choose initial values on the curve, this constant term will become zero. And if we choose initial values anywhere outside the curve, this constant term will not become zero.

For the straight line $y=cx+f$ the constant term will be

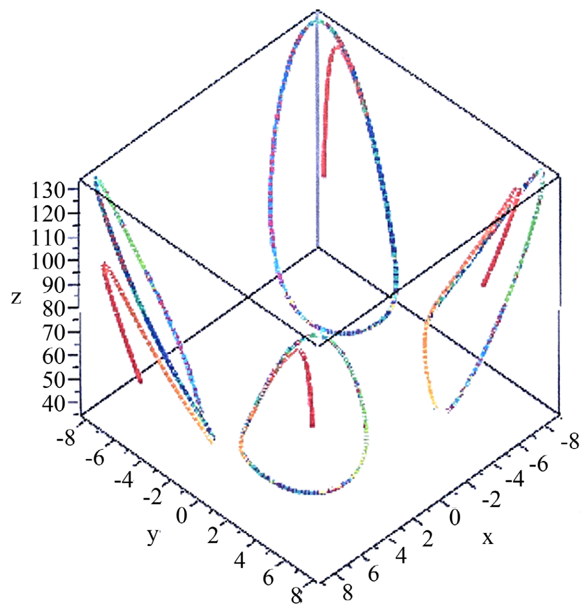


**Figure 9.** Four LC on a paraboloid.

$$-\left(y_0 - cx_0\right) + f \,. \tag{64}$$

When $y_0 - cx_0 = f$, is this term zero. Then we have the solutions:

$$x(t) = \frac{f\,\mathrm{e}^{at}\left(x_0\cos(bt) + y_0\sin(bt)\right)}{\mathrm{e}^{at}\left(y_0\cos(bt) - x_0\sin(bt)\right) - c\mathrm{e}^{at}\left(x_0\cos(bt) + y_0\sin(bt)\right) - \left(y_0 - cx_0\right) + f} \tag{65}$$

$$y(t) = \frac{f\,\mathrm{e}^{at}\left(y_0\cos(bt) - x_0\sin(bt)\right)}{\mathrm{e}^{at}\left(y_0\cos(bt) - x_0\sin(bt)\right) - c\mathrm{e}^{at}\left(x_0\cos(bt) + y_0\sin(bt)\right) - \left(y_0 - cx_0\right) + f} \tag{66}$$

Notice that all exponential functions have disappeared when $y_0 - cx_0 = f$. When $a > 0$, this line will attract the solution curves when *t* goes to infinity. Differentiation of these solutions gives system (99):

$$\frac{\mathrm{d}x}{\mathrm{d}t} = ax + by - \frac{x}{f}\left[a\left(y - cx\right) - bx - cby\right] \tag{67}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = ay - bx - \frac{y}{f}\left[a\left(y - cx\right) - bx - cby\right] \tag{68}$$

We see that when $y - cx = f$, all terms containing *a* are gone. That means that the exponential functions have disappeared when the solution curves arrive this straight line.

What about $x^2y^2 = f^4$ as attractor? The constant term will here be

$$-x_0^2y_0^2 + f^4 \tag{69}$$

$$x(t) = \frac{f\,\mathrm{e}^{at}\left(x_0\cos(bt) + y_0\sin(bt)\right)}{\sqrt[4]{\mathrm{e}^{4at}\left(x_0\cos(bt) + y_0\sin(bt)\right)^2\left(y_0\cos(bt) - x_0\sin(bt)\right)^2 - x_0^2y_0^2 + f^4}} \tag{70}$$

$$y(t) = \frac{f\,\mathrm{e}^{at}\left(y_0\cos(bt) - x_0\sin(bt)\right)}{\sqrt[4]{\mathrm{e}^{4at}\left(x_0\cos(bt) + y_0\sin(bt)\right)^2\left(y_0\cos(bt) - x_0\sin(bt)\right)^2 - x_0^2y_0^2 + f^4}} \tag{71}$$

These solutions give system (95):

$$\frac{\mathrm{d}x}{\mathrm{d}t} = ax + by - \frac{x}{f^4}\left[ax^2y^2 + \frac{b}{2}xy\left(y^2 - x^2\right)\right] \tag{72}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = ay - bx - \frac{y}{f^4}\left[ax^2y^2 + \frac{b}{2}xy\left(y^2 - x^2\right)\right] \tag{73}$$

The solution curves will follow $x^2y^2 = f^4$ to infinity along the axis.

It is easy to make many other systems of differential equations that have open curves as attractors.

### 4.2. Surfaces as Attractors

Common for the systems of differential equations that we are going to make in this section, is that a three-dimensional surface will act as an attractor. If we choose initial values inside or outside a closed surface, the surface will attract the solution curves, for example an ellipsoid, and then follow the surface of the ellipsoid over and over in the same orbit as the parameter $t$ goes to infinity.

The orbit is determined by the initial values. It is a necessary condition that the equilibrium point is spiral source. All exponential functions have disappeared along the whole surface. This is what characterizes a stable three-dimensional attractor. This kind of attractor can also be an open surface, where the solution curves follow the surface to infinity, no matter how complicated this surface is, provided it is continuous.

We will start by making a system of three differential equations and their general solutions, where the ellipsoid $\frac{x^2}{g^2}+\frac{y^2}{h^2}+\frac{z^2}{l^2}=1$ behave as an attractor. When we made the solutions to a system of differential equations exhibiting an elliptic limit cycle, we placed the formula for the ellipse under the fraction line in the solutions $x(t)$ and $y(t)$, as a constant term. Now we will place the formula for the ellipsoid under the fraction line to all the solutions $x(t), y(t)$ and $z(t)$. The constant term must then be:

$$-\left(\frac{x_0^2}{g^2}+\frac{y_0^2}{h^2}+\frac{z_0^2}{l^2}\right)+1 \tag{74}$$

$$x(t)=\frac{\mathrm{e}^{at}\left(x_0\cos(bt)+y_0\sin(bt)\right)}{\sqrt{\frac{1}{g^2}\mathrm{e}^{2at}\left(x_0\cos(bt)+y_0\sin(bt)\right)^2+\frac{1}{h^2}\mathrm{e}^{2at}\left(y_0\cos(bt)-x_0\sin(bt)\right)^2+\frac{z_0^2}{l^2}\mathrm{e}^{2at}-\left(\frac{x_0^2}{g^2}+\frac{y_0^2}{h^2}+\frac{z_0^2}{l^2}\right)+1}} \tag{75}$$

$$y(t)=\frac{\mathrm{e}^{at}\left(y_0\cos(bt)-x_0\sin(bt)\right)}{\sqrt{\frac{1}{g^2}\mathrm{e}^{2at}\left(x_0\cos(bt)+y_0\sin(bt)\right)^2+\frac{1}{h^2}\mathrm{e}^{2at}\left(y_0\cos(bt)-x_0\sin(bt)\right)^2+\frac{z_0^2}{l^2}\mathrm{e}^{2at}-\left(\frac{x_0^2}{g^2}+\frac{y_0^2}{h^2}+\frac{z_0^2}{l^2}\right)+1}} \tag{76}$$

$$z(t)=\frac{z_0\mathrm{e}^{at}}{\sqrt{\frac{1}{g^2}\mathrm{e}^{2at}\left(x_0\cos(bt)+y_0\sin(bt)\right)^2+\frac{1}{h^2}\mathrm{e}^{2at}\left(y_0\cos(bt)-x_0\sin(bt)\right)^2+\frac{z_0^2}{l^2}\mathrm{e}^{2at}-\left(\frac{x_0^2}{g^2}+\frac{y_0^2}{h^2}+\frac{z_0^2}{l^2}\right)+1}} \tag{77}$$

Notice that when $\frac{x_0^2}{g^2}+\frac{y_0^2}{h^2}+\frac{z_0^2}{l^2}=1$, all exponential functions belonging to $x(t), y(t)$ and $z(t)$ will disappear. The only functions left are sine and cosine, which give rotation.

Derivation of these solutions gives system (533):

$$\frac{\mathrm{d}x}{\mathrm{d}t}=ax+by-x\left[a\left(\frac{x^2}{g^2}+\frac{y^2}{h^2}+\frac{z^2}{l^2}\right)+bxy\left(\frac{1}{g^2}-\frac{1}{h^2}\right)\right] \tag{78}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t}=ay-bx-y\left[a\left(\frac{x^2}{g^2}+\frac{y^2}{h^2}+\frac{z^2}{l^2}\right)+bxy\left(\frac{1}{g^2}-\frac{1}{h^2}\right)\right] \tag{79}$$

$$\frac{\mathrm{d}z}{\mathrm{d}t}=az-z\left[a\left(\frac{x^2}{g^2}+\frac{y^2}{h^2}+\frac{z^2}{l^2}\right)+bxy\left(\frac{1}{g^2}-\frac{1}{h^2}\right)\right] \tag{80}$$

Here we can see that when $\frac{x^2}{g^2}+\frac{y^2}{h^2}+\frac{z^2}{l^2}=1$, all parts containing $a$ are gone.

The parameter *a* is the exponent in all three exponential functions. This tells us that these exponential functions have disappeared when the solution curves have arrived the surface of the ellipsoid.

See **Figure 10**, where $a=\frac{1}{4}, b=1, g=1, h=2, l=3$. We can see 9 LC on the surface of the ellipsoid, one for each initial value.

In the next example we will try to make the hyperboloid $\frac{x^2}{g^2}+\frac{y^2}{h^2}-\frac{z^2}{l^2}=1$ as an attractor.

The constant term will be

$$-\left(\frac{x_0^2}{g^2}+\frac{y_0^2}{h^2}-\frac{z_0^2}{l^2}\right)+1 \tag{81}$$

When $\frac{z_0^2}{l^2}=\frac{x_0^2}{g^2}+\frac{y_0^2}{h^2}+1$, is this constant term zero, and all exponential functions are gone.

The solutions will be almost the same as (64), (65) and (66), so we have the system (527):

$$\frac{\mathrm{d}x}{\mathrm{d}t}=ax+by-x\left[a\left(\frac{x^2}{g^2}+\frac{y^2}{h^2}-\frac{z^2}{l^2}\right)+bxy\left(\frac{1}{g^2}-\frac{1}{h^2}\right)\right] \tag{82}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t}=ay-bx-y\left[a\left(\frac{x^2}{g^2}+\frac{y^2}{h^2}-\frac{z^2}{l^2}\right)+bxy\left(\frac{1}{g^2}-\frac{1}{h^2}\right)\right] \tag{83}$$

$$\frac{\mathrm{d}z}{\mathrm{d}t}=az-z\left[a\left(\frac{x^2}{g^2}+\frac{y^2}{h^2}-\frac{z^2}{l^2}\right)+bxy\left(\frac{1}{g^2}-\frac{1}{h^2}\right)\right] \tag{84}$$

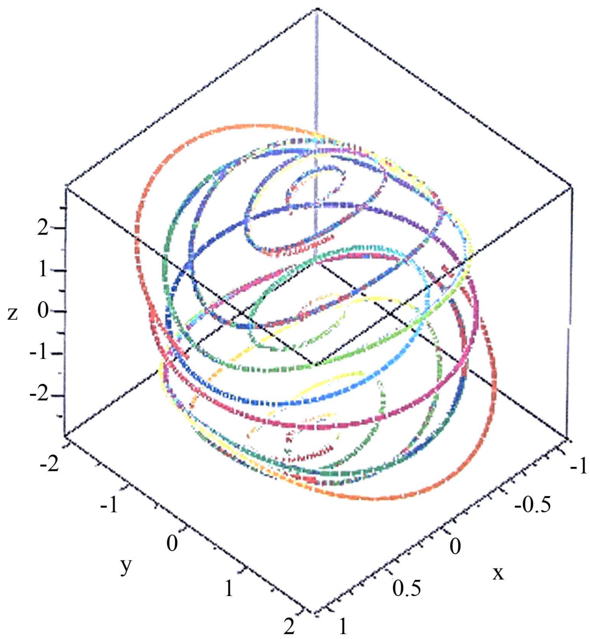


**Figure 10.** An ellipsoid as attractor.

When $\frac{x^2}{g^2}+\frac{y^2}{h^2}-\frac{z^2}{l^2}=1$, all parts containing $a$ in the solutions $x(t), y(t)$ and $z(t)$ are gone.

See **Figure 11**, where $a=b=g=1, h=2, l=3$. We can see 5 LC on the surface of the hyperboloid, one for each initial value.

### 4.3. Bifurcation

In some textbooks we can find examples of bifurcation. Blanchard, Devaney and Hall [3] define bifurcation as a drastic change in the long-term behavior of the solutions, as a parameter changes. One example is Hopf' bifurcation [5]:

$$\frac{\mathrm{d}x}{\mathrm{d}t}=ax+by-\frac{a}{f^2}x\left(x^2+y^2\right) \tag{85}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t}=ay-bx-\frac{a}{f^2}y\left(x^2+y^2\right) \tag{86}$$

I have written the system in the way that I am used to.

If $a<0$, we will get an inwards rotating spiral. If $a=0$, we will see a circle. If $a>0$, we will get an outwards rotating spiral ending up at a circular LC, if we choose the initial values inside a circle with radius *f*. And if we choose initial values outside the circle with radius *f*, the solution curves will rotate inwards ending up at the same circular LC.

Similar examples of bifurcations are easy to make. As an example, we can use system (129), (20) and (21):

$$\frac{\mathrm{d}x}{\mathrm{d}t}=ax+by-x\left[a\left(\frac{x^2}{g^2}+\beta\frac{xy}{gh}+\frac{y^2}{h^2}\right)+bxy\left(\frac{1}{g^2}-\frac{1}{h^2}\right)+\beta\frac{b}{2gh}\left(y^2-x^2\right)\right] \tag{20}$$

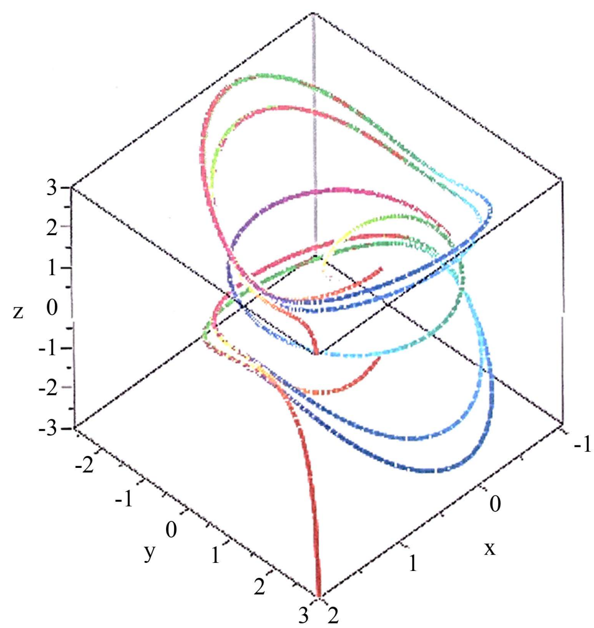


**Figure 11.** A hyperboloid of one sheet as attractor.

$$\frac{\mathrm{d}y}{\mathrm{d}t} = ay - bx - y\left[a\left(\frac{x^2}{g^2} + \beta\frac{xy}{gh} + \frac{y^2}{h^2}\right) + bxy\left(\frac{1}{g^2} - \frac{1}{h^2}\right) + \beta\frac{b}{2gh}\left(y^2 - x^2\right)\right] \quad (21)$$

If $a > 0$, we will get an elliptic LC, where (0, 0) is spiral source. If $a = 0$, we will see an ellipse, where (0, 0) is center. If $a < 0$, we will get an inwards rotating spiral, where (0, 0) is spiral sink. The same ellipse is now an unstable LC.

We can give this elliptic LC a three-dimensional form, for example project it on a paraboloid or hyperbolic paraboloid.

$$\begin{aligned}\frac{\mathrm{d}z}{\mathrm{d}t} &= kz + (2a - k)\left(cx^2 + dy^2\right) + 2bxy(c - d) \\ &\quad - \left(cx^2 + dy^2\right)\left[2a\left(\frac{x^2}{g^2} + \beta\frac{xy}{gh} + \frac{y^2}{h^2}\right) + 2bxy\left(\frac{1}{g^2} - \frac{1}{h^2}\right) + \beta\frac{b}{gh}\left(y^2 - x^2\right)\right]\end{aligned} \quad (87)$$

Together with (20) and (21) we get system (502). If $a > 0$, we will see a three-dimensional LC, where (0, 0) is spiral source. If $a = 0$, we see a closed three-dimensional curve, where (0, 0) is center. If $a < 0$, we will get a three-dimensional unstable LC, where (0, 0) is spiral sink.

## 5. Conclusions

It is possible to make systems of nonlinear ODEs that are exhibiting limit cycles by making the general solutions first. Provided that we know the formula to the closed curve in two dimensions, and that all parts of the formula are of same degree.

It is also possible to give this two-dimensional LC a three-dimensional shape by projecting this closed curve onto a surface that we know the formula to, no matter how complicated this formula is.

The solutions $x(t)$ and $y(t)$ must contain the formula to the closed curve as a constant term under the fraction line in the solutions. The solution $z(t)$ must contain the formula to the surface that we want to project the LC on. By building up the solutions according to the constant terms and differentiating the solutions, we will get a system of ODEs with the desired behavior. In this paper, we have used only the exponential function and sine and cosine to build up the solutions.

Many more LC and attractors in two and three dimensions are easy to make.

In Part Two, we are going to use an additional variable to make limit cycles where not all parts in the formula of the closed curve are of same degree. We are getting an extra ODE. Keeping this extra variable constant, we will achieve almost the desired result. We will try to make a system of ODEs that are exhibiting a limit cycle as near as possible to the closed curve $\left(x^2 + y^2\right)^4 - P^2\left(x^3 - 3xy^2\right)^2 = f^8$ and many other funny limit cycles. $P, f$ are parameters.

To be continued.

## Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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