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A Classification of Completely Positive Maps

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Abstract

This paper concerns classifying completely positive maps between certain C^*-algebras. Several invariants for classifying completely positive maps are constructed. It is proved that one of them is isomorphic to the Ext-group of C^*-algebra extensions in special circumstances. Furthermore, this invariant induces a functor from C^*-algebras to abelian groups which is split-exact.

Keywords

Completely Positive Map, Extension, Ext-Group

1. Introduction

The theory of completely positive maps plays an important part in operator algebras, operator spaces, and extensions of C^*-algebras. Many fundamental concepts and theorems are defined and proved via completely positive maps respectively, such as nuclearity, invertible extension, Stinespring's Theorem ([1] [2]), Voiculescu's Theorem ([3]), etc.

On the other hand, as an effective tool to study the structure of C^*-algebras and to classify C^*-algebras, the theory extensions of C^*-algebras originated from Busby's work in 1960's ([4]). Subsequently, Brown, Douglas and Fillmore established their famous BDF theory ([5] [6]) to study essentially normal operators on a separable infinite-dimensional Hilbert space and extensions of C^*-algebra $C(X)$ by compact operators, where $C(X)$ is the C^*-algebra of continuous functions on a compact metric space X. Since then, the theory of extensions of C^*-algebras has developed rapidly, and becomes an important invariant for classifying C^*-algebras together with K-theory and KK-theory (see [1] [7] [8] [9], etc.).

As we know, an extension of C^*-algebras is determined by its Busby invariant with respect to unitary equivalence, so to an extent classifying extensions of C^*-algebras is a sort of classifying homomorphisms between C^*-algebras. It should be pointed out that the KK-groups were defined via homomorphisms in this way at the beginning ([8]), and it was already used to classify homomorphisms (see [10] [11] [12], etc.). Completely positive maps can be seen as generalization of homomorphisms and what is particularly important is that Ext-groups were characterized by completely positive maps, so it is natural to consider classifying completely positive maps.

This note is engaging in classifying completely positive maps between certain C^*-algebras. Specifically, several invariants for classifying completely positive maps are introduced. As a main result, one of them is isomorphic to the Ext-group of C^*-algebra extensions. In addition, this invariant induces a functor from C^*-algebras to abelian groups which is split-exact.

2. Preliminaries

In this section, we need to recall some notations and definitions for C^*-algebras and extensions. One can also see [1] [7] [13] [14] [15] for more details.

Suppose that D is a C^*-algebra. Recall that $\theta_n : M_n(D) \to D$ is an inner isomorphism, if there are isometries $S_1, \cdots, S_n$ in D with $\sum_{i=1}^{n} S_i S_i^* = 1$ and $S_i^* S_j = 0$ for $i \neq j$, such that $\theta_n = Adv$, namely,

$$\theta_n([x_{ij}]) = v[x_{ij}]v^* = \sum_{i,j} S_i x_{ij} S_j^*,$$

for $[x_{ij}] \in M_n(D)$, where $v = (S_1, \cdots, S_n)$. Suppose that v_1 and v_2 are such elements. Then $v_1 v_2^* \in D$ and $v_1 v_2^* v_2 v_1^* = v_2 v_1^* v_1 v_2^* = 1$, and hence $v_1 v_2^*$ is a unitary in D.

Let A and B be C^*-algebras. An extension of A by B is a short exact sequence

$$e : 0 \to B \xrightarrow{\alpha} E \xrightarrow{\beta} A \to 0.$$

Denote this extension by e or (E, α, β).

The extension (E, α, β) is called trivial, if the above sequence splits, *i.e.* if there is a homomorphism $\gamma : A \to E$ such that $\beta \circ \gamma = id_A$.

For an extension (E, α, β), there is a unique homomorphism $\sigma : E \to M(B)$ such that $\sigma \circ \alpha = \iota$, where $M(B)$ is the multiplier algebra of B, and ι is the inclusion map from B into $M(B)$. The Busby invariant of (E, α, β) is a homomorphism τ from A into the corona algebra $Q(B) = M(B)/B$ defined by $\tau(a) = \pi(\sigma(b))$ for $a \in A$, where $\pi : M(B) \to Q(B)$ is the quotient map, and $b \in E$ such that $\beta(b) = a$.

Two extensions e_1 and e_2 are called (strongly) unitarily equivalent, denoted by $e_1 \overset{s}{\sim} e_2$, if there exists a unitary $u \in M(B)$ such that
$\tau_2(a) = \pi(u)\tau_1(a)\pi(u)^*$ for all $a \in A$. Denote by $\mathbf{Ext}(A, B)$ or $\mathbf{Ext}_s(A, B)$ the set of (strong) unitary equivalence classes of extensions of A by B.

Let H be a separable infinite-dimensional Hilbert space and $\mathcal{K}$ the ideal of compact operators in $B(H)$. If B is a stable C^*-algebra (*i.e.* $B \otimes \mathcal{K} \cong B$, where $\otimes$ is the tensor product operation), then the sum of two extensions τ_1 and τ_2

is defined to be the homomorphism $\tau_1 \oplus \tau_2$, where

$$\tau_1 \oplus \tau_2 : A \to \mathcal{Q}(B) \oplus \mathcal{Q}(B) \subseteq M_2(\mathcal{Q}(B)) \cong \mathcal{Q}(B)$$

and the isomorphism $M_2(\mathcal{Q}(B)) \cong \mathcal{Q}(B)$ is induced by an inner isomorphism from $M_2(M(B))$ onto $M(B)$, where $\oplus$ is the direct sum of C^*-algebras.

The above sets of equivalence classes of extensions are commutative semigroups with respect to this addition when B is stable. One can similarly define these semigroups replacing B by $B \otimes \mathcal{K}$ if B is not stable. Denote by $Ext(A, B)$ the quotient of $\mathbf{Ext}_s(A, B)$ by the subsemigroup of trivial extensions.

3. Main Result

Suppose that D is a unital properly infinite C^*-algebra, namely, there are two elements $S_1, S_2 \in D$ such that

$$S_i^* S_i = 1\,(i = 1,2),\ S_i^* S_j = 0\,(i \neq j),\ \sum_{i=1}^{2} S_i S_i^* = 1.$$

For every C^*-algebra A, we denote by $CP(A, D)$ the set of all completely positive maps from A into D.

Definition 3.1. Two elements $\varphi, \psi \in CP(A, D)$ are called (unitarily) equivalent, denoted by $\varphi \approx \psi$, if there is a unitary $u \in D$ such that $Adu \circ \varphi = \psi$.

It is easy to check that $\approx$ is an equivalence relation on $CP(A, D)$. Denote by $\{\varphi\}$ the equivalence class of φ.

Definition 3.2. $CP_1(A, D)$ is the equivalence classes in $CP(A, D)$ under the equivalence relation $\approx$, *i.e.* $CP_1(A, D) = CP(A, D)/\approx$.

Now we can define a diagonal addition in $CP_1(A, D)$ as follows:

$$\{\varphi\} + \{\psi\} = \left\{ Adv \circ \begin{pmatrix} \varphi & \\ & \psi \end{pmatrix} \right\} = \left\{ \begin{pmatrix} S_1 & S_2 \end{pmatrix} \begin{pmatrix} \varphi & \\ & \psi \end{pmatrix} \begin{pmatrix} S_1^* \\ S_2^* \end{pmatrix} \right\},$$

where $Adv : M_2(D) \to D$ is the inner isomorphism with $v = (S_1, S_2)$.

Proposition 3.3. Equipped with the above addition, $CP_1(A, D)$ is an abelian semigroup.

Proof. The following is similar to the proof of ([7], 3.2.3), and we give it here for the sake of completeness.

Suppose that φ, φ', ψ and ψ' are in $CP(A, D)$ such that $\varphi \approx \varphi'$ and $\psi \approx \psi'$. Then there are unitary elements $u_1, u_2 \in D$ such that $\varphi' = Adu_1 \circ \varphi$ and $\psi' = Adu_2 \circ \psi$. Thus

$$\begin{aligned} Adv \circ \begin{pmatrix} \varphi' & \\ & \psi' \end{pmatrix} &= v \begin{pmatrix} u_1 & \\ & u_2 \end{pmatrix} \begin{pmatrix} \varphi & \\ & \psi \end{pmatrix} \begin{pmatrix} u_1^* & \\ & u_2^* \end{pmatrix} v^* \\ &= v \begin{pmatrix} u_1 & \\ & u_2 \end{pmatrix} v^* v \begin{pmatrix} \varphi & \\ & \psi \end{pmatrix} v^* v \begin{pmatrix} u_1^* & \\ & u_2^* \end{pmatrix} v^*. \end{aligned}$$

Since

$$v \begin{pmatrix} u_1 & \\ & u_2 \end{pmatrix} v^*$$

is a unitary in D, we have

$$\left\{Adv\circ\begin{pmatrix}\varphi' & \\ & \psi'\end{pmatrix}\right\}=\left\{Adv\circ\begin{pmatrix}\varphi & \\ & \psi\end{pmatrix}\right\}.$$

It follows that the addition is well-defined.

Let θ_1 and θ_2 be two inner isomorphisms from $M_2(D)$ onto D with $\theta_1 = Adv_1$ and $\theta_2 = Adv_2$. Then

$$\begin{aligned}Adv_1\circ\begin{pmatrix}\varphi & \\ & \psi\end{pmatrix} &= v_1v_2^*v_2\begin{pmatrix}\varphi & \\ & \psi\end{pmatrix}v_2^*v_2v_1^* \\ &= Ad\left(v_1v_2^*\right)\circ Adv_2\circ\begin{pmatrix}\varphi & \\ & \psi\end{pmatrix},\end{aligned}$$

and hence,

$$\left\{Adv_1\circ\begin{pmatrix}\varphi & \\ & \psi\end{pmatrix}\right\}=\left\{Adv_2\circ\begin{pmatrix}\varphi & \\ & \psi\end{pmatrix}\right\}.$$

Therefore, the addition is independent of the choices of inner isomorphisms.

Suppose that $\varphi,\psi\in CP(A,D)$. Then

$$\begin{aligned}Adv\circ\begin{pmatrix}\varphi & \\ & \psi\end{pmatrix} &= v\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\varphi & \\ & \psi\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}v^* \\ &= v\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\psi & \\ & \varphi\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}v^*.\end{aligned}$$

Let

$$v' = v\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}.$$

Then Adv' is an inner isomorphism from $M_2(D)$ onto D and hence

$$\{\varphi\}+\{\psi\}=\{\psi\}+\{\varphi\}.$$

Suppose that $\varphi_1,\varphi_2,\varphi_3\in CP(A,D)$ and let S_1,S_2 be isometries with $S_1^*S_2=0$ and $S_1S_1^*+S_2S_2^*=1$. One can check the following computation:

$$\begin{aligned}\left(\{\varphi_1\}+\{\varphi_2\}\right)+\{\varphi_3\} &= \left\{S_1^2\varphi_1S_1^{*2}+S_1S_2\varphi_2S_2^*S_1^*+S_2\varphi_3S_2^*\right\} \\ &= \left\{\left(S_1^2,S_1S_2,S_2\right)\begin{pmatrix}\varphi_1 & & \\ & \varphi_2 & \\ & & \varphi_3\end{pmatrix}\begin{pmatrix}S_1^{*2}\\ S_2^*S_1^*\\ S_2^*\end{pmatrix}\right\}\end{aligned}$$

and

$$\begin{aligned}\{\varphi_1\}+\left\{\{\varphi_2\}+\{\varphi_3\}\right\} &= \left\{S_1\varphi_1S_1^*+S_2S_1\varphi_2S_1^*S_2^*+S_2^2\varphi_3S_2^{*2}\right\} \\ &= \left\{\left(S_1,S_2S_1,S_2^2\right)\begin{pmatrix}\varphi_1 & & \\ & \varphi_2 & \\ & & \varphi_3\end{pmatrix}\begin{pmatrix}S_1^*\\ S_1^*S_2^*\\ S_2^{*2}\end{pmatrix}\right\}.\end{aligned}$$

Put $v_1=\left(S_1^2,S_1S_2,S_2\right)$ and $v_2=\left(S_1,S_2S_1,S_2^2\right)$. Then Adv_1 and Adv_2 are two inner isomorphisms from $M_3(D)$ onto D. Note that

$$Adv_1 \circ \begin{pmatrix} \varphi_1 & & \\ & \varphi_2 & \\ & & \varphi_3 \end{pmatrix} = Ad\left(v_1 v_2^*\right) \circ Adv_2 \circ \begin{pmatrix} \varphi_1 & & \\ & \varphi_2 & \\ & & \varphi_3 \end{pmatrix}.$$

Since $v_1 v_2^*$ is a unitary in *D*, it follows that

$$\left(\{\varphi_1\} + \{\varphi_2\}\right) + \{\varphi_3\} = \{\varphi_1\} + \left(\{\varphi_2\} + \{\varphi_3\}\right).$$

This completes the proof of associativity.

Therefore, $CP_1(A, D)$ is an abelian semigroup.

Remark 3.4. Suppose that $\varphi, \psi \in CP(A, D)$. We write

$$\varphi \oplus \psi = \begin{pmatrix} S_1 & S_2 \end{pmatrix} \begin{pmatrix} \varphi & \\ & \psi \end{pmatrix} \begin{pmatrix} S_1^* \\ S_2^* \end{pmatrix},$$

or

$$\theta_2 \circ \begin{pmatrix} \varphi & \\ & \psi \end{pmatrix} = \begin{pmatrix} S_1 & S_2 \end{pmatrix} \begin{pmatrix} \varphi & \\ & \psi \end{pmatrix} \begin{pmatrix} S_1^* \\ S_2^* \end{pmatrix}.$$

Definition 3.5. Let $Hom(A, D)$ be the set of homomorphisms from *A* into *D*. An element is called degenerate in $CP(A, D)$ if it is also in $Hom(A, D)$.

Definition 3.6. Two elements $\{\varphi\}, \{\psi\} \in CP_1(A, D)$ are called equivalent, denoted by $\{\varphi\} \sim_0 \{\psi\}$, if there are $\varphi', \psi' \in Hom(A, D)$ such that $\{\varphi\} + \{\varphi'\} = \{\psi\} + \{\psi'\}$.

Then $\sim_0$ is an equivalence relation. The equivalence class of $\{\varphi\}$ is denoted by $\left[\{\varphi\}\right]_0$, or by $[\varphi]_0$ simply.

Definition 3.7. $CP_2(A, D)$ is the equivalence classes in $CP_1(A, D)$ under the equivalence relation $\sim_0$, *i.e.* $CP_2(A, D) = CP_1(A, D)/\sim_0$.

We define an addition + in $CP_2(A, D)$ by

$$[\varphi]_0 + [\psi]_0 = \left[\{\varphi\} + \{\psi\}\right]_0, \varphi, \psi \in CP(A, D).$$

To see the addition is well-defined, suppose that $\{\varphi'\} \sim_0 \{\varphi\}$ and $\{\psi'\} \sim_0 \{\psi\}$. Then there exist $\varphi_1, \varphi_1', \psi_1, \psi_1' \in Hom(A, D)$ such that

$$\{\varphi\} + \{\varphi_1\} = \{\varphi'\} + \{\varphi_1'\}, \{\psi\} + \{\psi_1\} = \{\psi'\} + \{\psi_1'\},$$

and hence

$$\{\varphi\} + \{\psi\} + \{\varphi_1\} + \{\psi_1\} = \{\varphi'\} + \{\psi'\} + \{\varphi_1'\} + \{\psi_1'\}.$$

Since

$$\{\varphi_1\} + \{\psi_1\} = \left\{\theta_2 \circ \begin{pmatrix} \varphi_1 & \\ & \psi_1 \end{pmatrix}\right\},$$

$$\theta_2 \circ \begin{pmatrix} \varphi_1 & \\ & \psi_1 \end{pmatrix} \in Hom(A, D).$$

Similarly,

$$\theta_2 \circ \begin{pmatrix} \varphi_1' & \\ & \psi_1' \end{pmatrix} \in Hom(A, D).$$

It follows that the addition is well-defined.

Remark 3.8. 1) Suppose that $\varphi_1, \varphi_2 \in CP(A, D)$. Then $[\varphi_1]_0 = [\varphi_2]_0 \in CP_2(A, D)$ if and only if there exist $\sigma_1, \sigma_2 \in Hom(A, D)$ such that $\varphi_1 \oplus \sigma_1$ is unitarily equivalent to $\varphi_2 \oplus \sigma_2$.

2) Suppose that $\eta \in CP(A, D)$. Then $[\eta]_0$ is the neutral element in $CP_2(A, D)$ if and only if for each $\varphi \in CP(A, D)$ there exist $\sigma_1, \sigma_2 \in Hom(A, D)$ such that $\varphi \oplus \eta \oplus \sigma_1$ is unitarily equivalent to $\varphi \oplus \sigma_2$.

Theorem 3.9. $CP_2(A, D)$ is a unital abelian semigroup. An element $[\varphi]_0$ is the unit of $CP_2(A, D)$ if and only if $\varphi \in Hom(A, D)$.

Proof. Suppose that $\varphi_1, \varphi_2, \varphi_3 \in CP(A, D)$. Then

$$\begin{aligned}[\varphi_1]_0 + ([\varphi_2]_0 + [\varphi_3]_0) &= [\varphi_1]_0 + [\{\varphi_2\} + \{\varphi_3\}]_0 \\ &= [\{\varphi_1\} + (\{\varphi_2\} + \{\varphi_3\})]_0 \\ &= [(\{\varphi_1\} + \{\varphi_2\}) + \{\varphi_3\}]_0 \\ &= ([\varphi_1]_0 + [\varphi_2]_0) + [\varphi_3]_0.\end{aligned}$$

It follows that $CP_2(A, D)$ is a semigroup. It is clear that $CP_2(A, D)$ is abelian.

Let $\eta \in Hom(A, D)$. For any $\varphi \in CP(A, D)$, take $\sigma_1 \in Hom(A, D)$ and set $\sigma_2 = \eta \oplus \sigma_1$. Then

$$(\varphi \oplus \eta) \oplus \sigma_1 \approx \varphi \oplus (\eta \oplus \sigma_1),$$

that is, $(\varphi \oplus \eta) \oplus \sigma_1 \approx \varphi \oplus \sigma_2$. Since $\sigma_1, \eta \oplus \sigma_1 \in Hom(A, D)$ and $\varphi \oplus \eta \sim_0 \varphi$, we have $[\varphi]_0 + [\eta]_0 = [\varphi]_0$ by Remark 3.8. Hence $[\eta]_0$ is the unit of $CP_2(A, D)$.

Suppose that $\psi \in CP(A, D)$ such that $[\psi]_0$ is the unit of $CP_2(A, D)$. For $\varphi \in Hom(A, D)$, $[\varphi]_0$ is also the unit of $CP_2(A, D)$, and hence $[\psi]_0 = [\varphi]_0$. Thus there exist $\varphi_1, \psi_1 \in Hom(A, D)$ such that $\{\psi\} + \{\psi_1\} = \{\varphi\} + \{\varphi_1\}$. Note that $\psi \oplus \psi_1$ is unitarily equivalent to $\varphi \oplus \varphi_1$. Since φ and φ_1 are both homomorphisms,

$$\theta_2 \circ \begin{pmatrix} \varphi & \\ & \varphi_1 \end{pmatrix}$$

is a homomorphism. Furthermore,

$$\begin{pmatrix} \psi & \\ & \psi_1 \end{pmatrix}$$

is in $Hom(A, M_2(D))$, and hence ψ is in $Hom(A, D)$.

Remark 3.10. The only invertible element in $CP_2(A, D)$ is the unit. In fact, suppose that $[\varphi]_0$ is an invertible element in $CP_2(A, D)$ with the inverse $[\psi]_0$. Then $[\varphi]_0 + [\psi]_0$ is the unit and

$$\theta_2 \circ \begin{pmatrix} \varphi & \\ & \psi \end{pmatrix}$$

is a homomorphism by Theorem 3.9. Thus,

$$\begin{pmatrix} \varphi & \\ & \psi \end{pmatrix}$$

is also a homomorphism. Therefore φ is in $Hom(A,D)$. It follows that $[\varphi]_0$ is the unit.

Definition 3.11. Let B be a closed ideal of D and $\pi: D \to D/B$ the quotient map. We define a relation $\sim$ on $CP_2(A,D)$ as follows: for $\varphi,\psi \in CP(A,D)$, we write $[\varphi]_0 \sim [\psi]_0$ if there exist $\varphi_1,\psi_1 \in CP(A,D)$ such that $[\varphi_1]_0 = [\varphi]_0$, $[\psi_1]_0 = [\psi]_0$, and $\pi \circ \varphi_1 = \pi \circ \psi_1$.

Suppose that $\varphi \sim \psi$, $\psi \sim \eta$. Then there exist $\varphi_1,\psi_1,\psi_2,\eta_2$ such that

$$[\varphi_1]_0 = [\varphi]_0, [\psi_1]_0 = [\psi]_0, \pi \circ \varphi_1 = \pi \circ \psi_1,$$

$$[\psi_2]_0 = [\psi]_0, [\eta_2]_0 = [\eta]_0, \pi \circ \psi_2 = \pi \circ \eta_2.$$

Since $[\psi_1]_0 = [\psi_2]_0$, there exist $\phi_1,\phi_2 \in Hom(A,D)$ such that $\{\psi_1\} + \{\phi_1\} = \{\psi_2\} + \{\phi_2\}$. Thus there is a unitary $u \in D$ such that

$$\theta_2 \circ \begin{pmatrix} \psi_1 & \\ & \phi_1 \end{pmatrix} = Adu \circ \theta_2 \circ \begin{pmatrix} \varphi_2 & \\ & \phi_2 \end{pmatrix}.$$

Put

$$\varphi_1' = \theta_2 \circ \begin{pmatrix} \varphi_1 & \\ & \phi_1 \end{pmatrix}, \eta_2' = Adu \circ \theta_2 \circ \begin{pmatrix} \eta_2 & \\ & \phi_2 \end{pmatrix}.$$

Then we have

$$[\varphi_1']_0 = [\varphi_1]_0 + [\phi_1]_0 = [\varphi_1]_0 = [\varphi]_0,$$

$$[\eta_2']_0 = [\eta_2]_0 + [\phi_2]_0 = [\eta_2]_0 = [\eta]_0,$$

and

$$\begin{aligned}
\pi \circ \varphi_1' &= \pi \circ \theta_2 \circ \begin{pmatrix} \varphi_1 & \\ & \phi_1 \end{pmatrix} = \theta_2' \circ \begin{pmatrix} \pi \circ \varphi_1 & \\ & \pi \circ \phi_1 \end{pmatrix} \\
&= \theta_2' \circ \begin{pmatrix} \pi \circ \psi_1 & \\ & \pi \circ \phi_1 \end{pmatrix} = \pi \circ \theta_2 \circ \begin{pmatrix} \psi_1 & \\ & \phi_1 \end{pmatrix} \\
&= \pi \circ Adu \circ \theta_2 \circ \begin{pmatrix} \psi_2 & \\ & \phi_2 \end{pmatrix} = Ad\pi(u) \circ \theta_2' \circ \begin{pmatrix} \pi \circ \psi_2 & \\ & \pi \circ \phi_2 \end{pmatrix} \\
&= Ad\pi(u) \circ \theta_2' \circ \begin{pmatrix} \pi \circ \eta_2 & \\ & \pi \circ \phi_2 \end{pmatrix} = \pi \circ Adu \circ \theta_2 \circ \begin{pmatrix} \eta_2 & \\ & \phi_2 \end{pmatrix} \\
&= \pi \circ \eta_2',
\end{aligned}$$

where θ_2' is the inner isomorphism from $M_2(D/B)$ onto D/B induced by θ.

It follows that $\sim$ is transitive, and hence $\sim$ is an equivalence relation on $CP_2(A,D)$. Denote the equivalence class of $[\varphi]_0$ by $\left[[\varphi]_0\right]$, or by $[\varphi]$ simply.

Let $CP_B(A,D) = CP_2(A,D)/\sim$. It is natural that we define an addition in $CP_B(A,D)$ as follows:

$$[\varphi] + [\psi] = \left[[\varphi]_0 + [\psi]_0\right].$$

Remark 3.12. The addition defined in Definition 3.11 is well-defined: for $[\varphi] = [\varphi']$ and $[\psi] = [\psi']$, there exist φ_1, φ_1', ψ_1, ψ_1' such that $[\varphi_1]_0 = [\varphi]_0$,

$[\varphi_1']_0 = [\varphi']_0$, $[\psi_1]_0 = [\psi]_0$, $[\psi_1']_0 = [\psi']_0$, $\pi\circ\varphi_1' = \pi\circ\varphi_1$, and $\pi\circ\psi_1' = \pi\circ\psi_1$. Then

$$\pi\left(S_1\varphi_1'S_1^* + S_2\psi_1'S_2^*\right) = \pi\left(S_1\varphi_1S_1^* + S_2\psi_1S_2^*\right),$$

and hence

$$\left[[\varphi]_0 + [\psi]_0\right] = \left[[\varphi_1]_0 + [\psi_1]_0\right] = \left[[\varphi_1']_0 + [\psi_1']_0\right] = \left[[\varphi']_0 + [\psi']_0\right].$$

It is easy to see that $[0]$ is the unit of $CP_B(A,D)$. Thus $(CP_B(A,D),+)$ is a unital abelian semigroup. In particular, for $B = \{0\}$, we have $(CP_B(A,D),+) = (CP_2(A,D),+)$; and for $B = D$, we have $CP_D(A,D) = \{0\}$.

Definition 3.13. Let $CP_B^{-1}(A,D)$ be the set of invertible elements in $CP_B(A,D)$. Then $CP_B^{-1}(A,D)$ is an abelian group.

Theorem 3.14. Let φ be in $CP(A,D)$. Then $[\varphi] = 0$ in $CP_B^{-1}(A,D)$ if and only if there exist $\varphi', \phi_1, \phi_2 \in Hom(A,D)$ and a unitary $u \in M_2(D)$ such that

$$\begin{pmatrix}\varphi & \\ & \phi_1\end{pmatrix} = Adu\circ\begin{pmatrix}\varphi' & \\ & \phi_2\end{pmatrix}.$$

Proof. Suppose that $[\varphi] = 0$ in $CP_B^{-1}(A,D)$. Since $[\varphi] = [0] = 0$, there exist $\varphi', \phi' \in CP(A,D)$ such that $[\varphi']_0 = [\varphi]_0$, $[\phi']_0 = 0$ and $\pi\circ\varphi' = \pi\circ\phi'$. Hence, by Theorem 3.9, we have $\phi' \in Hom(A,D)$. Since $[\varphi']_0 = [\varphi]_0$, there exist $\phi_1, \phi_2 \in Hom(A,D)$ such that $\{\varphi\} + \{\phi_1\} = \{\varphi'\} + \{\phi_2\}$. Then there is a unitary $u \in M_2(D)$ such that

$$\begin{pmatrix}\varphi & \\ & \phi_1\end{pmatrix} = Adu\circ\begin{pmatrix}\varphi' & \\ & \phi_2\end{pmatrix}.$$

Conversely, suppose that there exist $\varphi', \phi_1, \phi_2 \in Hom(A,D)$ and a unitary $u \in M_2(D)$ such that

$$\begin{pmatrix}\varphi & \\ & \phi_1\end{pmatrix} = Adu\circ\begin{pmatrix}\varphi' & \\ & \phi_2\end{pmatrix}.$$

Set $v_1 = (S_1, S_2)$ and $v_2 = vu$. Then Adv_1, Adv_2 are both inner isomorphisms from $M_2(D)$ onto *D*. Therefore

$$Adv_1\circ\begin{pmatrix}\varphi & \\ & \phi_1\end{pmatrix} = Adv_2\circ\begin{pmatrix}\varphi' & \\ & \phi_2\end{pmatrix}.$$

Note that $[\varphi'] = [\phi_1] = [\phi_2] = 0$. Thus $[\varphi] = [\varphi] + [\phi_1] = [\varphi'] + [\phi_2] = 0$.

Remark 3.15. Suppose that $[\varphi] = 0$ in $CP_B^{-1}(A,D)$. By Theorem 3.14, we have

$$\begin{aligned}
\pi\circ\begin{pmatrix}\varphi & \\ & \phi_1\end{pmatrix} &= \pi\circ Adu\circ\begin{pmatrix}\varphi' & \\ & \phi_2\end{pmatrix} \\
&= Ad\pi(u)\circ\begin{pmatrix}\pi\circ\varphi' & \\ & \pi\circ\phi_2\end{pmatrix} \\
&= Ad\pi(u)\circ\begin{pmatrix}\pi\circ\phi' & \\ & \pi\circ\phi_2\end{pmatrix} \\
&= \pi\circ Adu\circ\begin{pmatrix}\phi' & \\ & \phi_2\end{pmatrix} \\
&= \pi\circ\phi,
\end{aligned}$$

where

$$\phi = Adu \circ \begin{pmatrix} \phi' & \\ & \phi_2 \end{pmatrix} \in Hom\left(A, M_2\left(D\right)\right),$$

and $\pi : M_2(D) \to M_2(D/B)$ is induced by the quotient map $\pi : D \to D/B$.

Set $\phi = (\phi_{i,j})$, we have $\pi \circ \phi_{i,j} = 0\,(i \neq j)$ and $\pi \circ \varphi = \pi \circ \phi_{1,1}$.

Theorem 3.16. Let $\varphi \in CP(A,D)$. Then $[\varphi] \in CP_B^{-1}(A,D)$ if and only if there is $\phi = (\phi_{i,j}) \in Hom(A, M_2(D))$ and $\psi \in CP(A,D)$ such that

$$\pi \circ \begin{pmatrix} \varphi & \\ & \psi \end{pmatrix} = \pi \circ \phi.$$

Proof. Suppose that $[\varphi] \in CP_B^{-1}(A,D)$ with the inverse $[\varphi']$. Let $v = (S_1, S_2)$. Since $[\varphi] + [\varphi'] = 0$, there exist $\phi' = (\phi'_{i,j})$ in $Hom(A, M_2(D))$ and a unitary $u = (u_{i,j})$ in $M_2(D)$ such that

$$\begin{aligned} \pi \circ \begin{pmatrix} \varphi & \\ & \varphi' \end{pmatrix} &= Ad\pi\left(v^*\right) \circ Ad\pi\left(v\right) \circ \begin{pmatrix} \varphi & \\ & \varphi' \end{pmatrix} \\ &= Ad\pi\left(v^*\right) \circ \phi'_{1,1} \\ &= \pi \circ \begin{pmatrix} S_1^*\phi'_{1,1}S_1 & S_1^*\phi'_{1,1}S_2 \\ S_2^*\phi'_{1,1}S_1 & S_2^*\phi'_{1,1}S_2 \end{pmatrix}. \end{aligned}$$

Set

$$v_1 = \begin{pmatrix} S_1^* & 0 \\ S_2^* & 0 \\ 0 & I \end{pmatrix} \text{ and } v_2 = \begin{pmatrix} I & 0 & 0 \\ 0 & S_1 & S_2 \end{pmatrix}.$$

Then Adv_1 is an inner isomorphism from $M_2(D)$ onto $M_3(D)$ and Adv_2 is an inner isomorphism from $M_3(D)$ onto $M_2(D)$. It follows that

$$\pi \circ Adv_1 \circ \phi' = \pi \circ \begin{pmatrix} S_1^*\phi'_{1,1}S_1 & S_1^*\phi'_{1,1}S_2 & S_1^*\phi'_{1,2} \\ S_2^*\phi'_{1,1}S_1 & S_2^*\phi'_{1,1}S_2 & S_2^*\phi'_{1,2} \\ \phi'_{2,1}S_1 & \phi'_{2,1}S_2 & \phi'_{2,2} \end{pmatrix} = \pi \circ \begin{pmatrix} \varphi & & \\ & \varphi' & \\ & & \phi'_{2,2} \end{pmatrix}.$$

Set

$$\psi = \begin{pmatrix} S_1 & S_2 \end{pmatrix} \begin{pmatrix} \varphi' & \\ & \phi'_{2,2} \end{pmatrix} \begin{pmatrix} S_1^* \\ S_2^* \end{pmatrix}$$

and

$$\phi = Adv_2 \circ Adv_1 \circ \phi' \in Hom\left(A, M_2\right)\left(D\right).$$

Then we have

$$\pi \circ \phi = \pi \circ Adv_2 \circ \begin{pmatrix} \varphi & & \\ & \varphi' & \\ & & \phi'_{2,2} \end{pmatrix} = \pi \circ \begin{pmatrix} \varphi & \\ & \psi \end{pmatrix}.$$

Conversely, since

$$\pi \circ \begin{pmatrix} \varphi & \\ & \psi \end{pmatrix} = \pi \circ \phi,$$

$$Adv \circ \pi \circ \begin{pmatrix} \varphi & \\ & \psi \end{pmatrix} = Adv \circ \pi \circ \phi.$$

Then

$$\pi \circ \theta_2 \circ \begin{pmatrix} \varphi & \\ & \psi \end{pmatrix} = \pi \circ \theta_2 \circ \phi.$$

Thus

$$[\varphi] + [\psi] = \left[\theta_2 \circ \begin{pmatrix} \varphi & \\ & \psi \end{pmatrix} \right] = [\theta_2 \circ \phi] = 0.$$

Proposition 3.17. Suppose that $\varphi \in CP(A, D)$ such that $[\varphi] \in CP_B^{-1}(A, D)$. Then $\pi \circ \varphi$ is a homomorphism.

Proof. Suppose that $\varphi_1 \in CP_B^{-1}(A, D)$ and $[\varphi_2]$ is the inverse of $[\varphi_1]$. Set

$$\psi = \theta_2 \circ \begin{pmatrix} \varphi_1 & \\ & \varphi_2 \end{pmatrix} = s_1 \varphi_1 s_1^* + s_2 \varphi_2 s_2^*.$$

By Theorem 3.14, there exist $\phi \in Hom(A, M_2(D))$ and $\phi_1 \in Hom(A, D)$ such that

$$\pi \circ \begin{pmatrix} \psi & \\ & \phi_1 \end{pmatrix} = \pi \circ \phi.$$

Hence $\pi \circ \psi$ is a homomorphism, and thus

$$\pi(S_1)\big(\pi \circ \varphi_1(ab) - \pi(\varphi_1(a)\varphi_1(b))\big)\pi(S_1^*)$$
$$+ \pi(S_2)\big(\pi \circ \varphi_2(ab) - \pi(\varphi_2(a)\varphi_2(b))\big)\pi(S_2^*) = 0.$$

Set $x = \pi \circ \varphi_1(ab) - \pi(\varphi_1(a)\varphi_1(b))$ and $y = \pi \circ \varphi_2(ab) - \pi(\varphi_2(a)\varphi_2(b))$. Then

$$\pi(S_1) x \pi(S_1^*) + \pi(S_2) y \pi(S_2^*) = 0,$$

that is,

$$(\pi(S_1), \pi(S_2)) \begin{pmatrix} x & \\ & y \end{pmatrix} \begin{pmatrix} \pi(S_1^*) \\ \pi(S_2^*) \end{pmatrix} = 0.$$

Put $v' = (\pi(S_1), \pi(S_2))$. Then $v'^* v' = I \in M_2(D)$. Hence

$$\begin{pmatrix} x & \\ & y \end{pmatrix} = 0.$$

This implies that $x = y = 0$, and furthermore $\pi \circ \varphi_1(ab) = \pi(\varphi_1(a)\varphi_1(b))$. It follows that $\pi \circ \varphi_1$ is a homomorphism.

Lemma 3.18. ([7], 3.2.9) Suppose that A is a separable C^*-algebra and B is a stable C^*-algebra. Let $\phi \in Hom(A, \mathcal{Q}(B))$. Then the following three statements are equivalent:

1) $[\phi]$ is invertible in $Ext(A, B)$.

2) There exists $\psi \in CP(A, M(B))$ such that $\phi = \pi \circ \psi$.

3) There exists $\varphi \in Hom(A, M_2(M(B)))$ such that

$$\begin{pmatrix} \phi & 0 \\ 0 & 0 \end{pmatrix} = \pi\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\varphi\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right).$$

It is well known that $M_2(M(B))$ and $M(B)$ are innerly isomorphic if B is a stable C^*-algebra. Then we have the following result.

Theorem 3.19. Let A and B be C^*-algebras with B stable. Then

$$CP_B^{-1}(A, M(B)) \cong Ext^{-1}(A, B).$$

Proof. Note that the condition that A is separable is not necessary in the proof of (1) $\Rightarrow$ (2) in Lemma 3.18 ([7], 3.2.9). Suppose that $\phi \in Hom(A, \mathcal{Q}(B))$ such that $[\phi]$ is invertible in $Ext(A, B)$. Then there exists $\varphi \in CP(A, M(B))$ such that $\phi = \pi \circ \varphi$. We define a map

$$\Phi : Ext^{-1}(A, B) \to CP_B(A, M(B))$$

by $[\phi] \mapsto [\varphi]$, where $\pi \circ \varphi = \phi$.

1) Prove that Φ is well-defined.

Suppose that $\phi_1, \phi_2 \in Hom(A, M(B)/B)$ such that $[\phi_1], [\phi_2] \in Ext^{-1}(A, B)$. Then there exist $\varphi_1, \varphi_2 \in CP_B(A, M(B))$ such that $\phi_1 = \pi \circ \varphi_1$ and $\phi_2 = \pi \circ \varphi_2$. If $[\phi_1] = [\phi_2]$, there exist $\varphi_1', \varphi_2' \in Hom(A, M(B))$ and $u \in M_2(M(B))$ such that

$$\tilde{\theta}_B \circ \begin{pmatrix} \phi_1 & \\ & \pi \circ \varphi_1' \end{pmatrix} = Ad\pi(u) \circ \tilde{\theta}_B \circ \begin{pmatrix} \phi_2 & \\ & \pi \circ \varphi_2' \end{pmatrix}.$$

Hence,

$$\pi\left(\theta_B \circ \begin{pmatrix} \varphi_1 & \\ & \varphi_1' \end{pmatrix}\right) = \pi\left(Adu \circ \theta_B \circ \begin{pmatrix} \varphi_2 & \\ & \varphi_2' \end{pmatrix}\right).$$

Since θ_B is an inner isomorphism,

$$[\varphi_1]_0 = \left[\theta_B \circ \begin{pmatrix} \varphi_1 & \\ & \varphi_1' \end{pmatrix}\right]_0 \text{ and } [\varphi_2]_0 = \left[Adu \circ \theta_B \circ \begin{pmatrix} \varphi_2 & \\ & \varphi_2' \end{pmatrix}\right]_0.$$

Then $[\varphi_1] = [\varphi_2]$, and hence Φ is well-defined.

2) Prove that Φ is a homomorphism.

Note that

$$\Phi([\phi_1]) + \Phi([\phi_2]) = [\varphi_1] + [\varphi_2] = \left[[\varphi_1]_0 + [\varphi_2]_0\right].$$

Since

$$\pi \circ \theta_B \circ \begin{pmatrix} \varphi_1 & \\ & \varphi_2 \end{pmatrix} = \tilde{\theta}_B \circ \begin{pmatrix} \pi \circ \varphi_1 & \\ & \pi \circ \varphi_2 \end{pmatrix} = \tilde{\theta}_B \circ \begin{pmatrix} \phi_1 & \\ & \phi_2 \end{pmatrix},$$

we have

$$\Phi([\phi_1] + [\phi_2]) = \Phi([\phi_1]) + \Phi([\phi_2]).$$

It follows that Φ is a homomorphism.

3) Prove that $\Phi(Ext^{-1}(A, B)) \subseteq CP_B^{-1}(A, M(B))$.

Suppose that $[\phi_1]$ is an invertible element with the inverse $[\phi_2]$. Then we have

$$\Phi([\phi_1])+\Phi([\phi_2])=\Phi([\phi_1]+[\phi_2])=\Phi(0).$$

Therefore, $\Phi([\phi_1])$ is invertible.

4) Prove that $\Phi: Ext^{-1}(A,B)\to CP_B^{-1}(A,M(B))$ is injective.

Suppose that $\Phi([\phi_1])=[\varphi_1]$ and $\Phi([\phi_2])=[\varphi_2]$, where $\phi_1=\pi\circ\varphi_1$ and $\phi_2=\pi\circ\varphi_2$.

If $[\varphi_1]=[\varphi_2]$ in $CP_B^{-1}(A,M(B))$, then there exist $\varphi_1',\varphi_2'\in CP(A,M(B))$ such that $[\varphi_1]_0=[\varphi_1']_0$, $[\varphi_2]_0=[\varphi_2']_0$ and $\pi\circ\varphi_1'=\pi\circ\varphi_2'$. Therefore there exist $\sigma_1,\sigma_2,\tau_1,\tau_2\in Hom(A,M(B))$ and unitary elements $u_1,u_2\in M_2(M(B))$ such that

$$\begin{pmatrix}\varphi_1 & \\ & \sigma_1\end{pmatrix}=Adu_1\circ\begin{pmatrix}\varphi_1' & \\ & \sigma_2\end{pmatrix},\begin{pmatrix}\varphi_2 & \\ & \tau_1\end{pmatrix}=Adu_2\circ\begin{pmatrix}\varphi_2' & \\ & \tau_2\end{pmatrix}.$$

Put

$$X=\begin{pmatrix}\varphi_1' & \\ & \sigma_2\end{pmatrix}, Y=\begin{pmatrix}\varphi_2' & \\ & \tau_2\end{pmatrix}, E=\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}.$$

Then we have

$$\begin{pmatrix}\varphi_1 & & \\ & \sigma_1 & \\ & & \tau_2\end{pmatrix}=\begin{pmatrix}u_1Xu_1^* & 0\\ 0 & \tau_2\end{pmatrix}=\begin{pmatrix}u_1 & \\ & 1\end{pmatrix}\begin{pmatrix}X & 0\\ 0 & \tau_2\end{pmatrix}\begin{pmatrix}u_1^* & \\ & 1\end{pmatrix},$$

$$\begin{pmatrix}\varphi_2 & & \\ & \tau_1 & \\ & & \sigma_2\end{pmatrix}=\begin{pmatrix}u_2Yu_2^* & 0\\ 0 & \sigma_2\end{pmatrix}=\begin{pmatrix}u_2 & \\ & 1\end{pmatrix}\begin{pmatrix}Y & 0\\ 0 & \tau_2\end{pmatrix}\begin{pmatrix}u_2^* & \\ & 1\end{pmatrix},$$

and

$$\begin{aligned}&\pi\left(\begin{pmatrix}u_1 & \\ & 1\end{pmatrix}\begin{pmatrix}X & 0\\ 0 & \tau_2\end{pmatrix}\begin{pmatrix}u_1^* & \\ & 1\end{pmatrix}\right)\\
&=\begin{pmatrix}\pi(u_1) & \\ & \pi(1)\end{pmatrix}\begin{pmatrix}\pi\circ X & 0\\ 0 & \pi\circ\tau_2\end{pmatrix}\begin{pmatrix}\pi(u_1^*) & \\ & \pi(1)\end{pmatrix}\\
&=\begin{pmatrix}\pi(u_1) & \\ & \pi(1)\end{pmatrix}\pi(E)\begin{pmatrix}\pi\circ Y & 0\\ 0 & \pi\circ\sigma_2\end{pmatrix}\pi(E)\begin{pmatrix}\pi(u_1^*) & \\ & \pi(1)\end{pmatrix}\\
&=\pi\left(\begin{pmatrix}u_1 & \\ & 1\end{pmatrix}E\begin{pmatrix}Y & 0\\ 0 & \sigma_2\end{pmatrix}E\begin{pmatrix}u_1^* & \\ & 1\end{pmatrix}\right).\end{aligned}$$

Thus,

$$\pi\left(\begin{pmatrix}\varphi_1 & & \\ & \sigma_1 & \\ & & \tau_2\end{pmatrix}\right)=\pi\left(\begin{pmatrix}u_1 & \\ & 1\end{pmatrix}E\begin{pmatrix}u_2^* & \\ & 1\end{pmatrix}\begin{pmatrix}\varphi_2 & & \\ & \tau_1 & \\ & & \sigma_2\end{pmatrix}\begin{pmatrix}u_2 & \\ & 1\end{pmatrix}E\begin{pmatrix}u_1^* & \\ & 1\end{pmatrix}\right).$$

Set

$$u_3=\begin{pmatrix}u_1 & \\ & 1\end{pmatrix}E\begin{pmatrix}u_2^* & \\ & 1\end{pmatrix}.$$

One can check that u_3 is a unitary in $M_3(M(B))$. Then we have

$$\begin{pmatrix} \phi_1 & & \\ & \pi\circ\sigma_1 & \\ & & \pi\circ\tau_2 \end{pmatrix} = Ad\pi(u_3)\circ\begin{pmatrix} \phi_2 & & \\ & \pi\circ\tau_1 & \\ & & \pi\circ\sigma_2 \end{pmatrix}.$$

It follows that

$$[\phi_1]=[\phi_1]+[\pi\circ\sigma_1]+[\pi\circ\tau_2]=[\phi_2]+[\pi\circ\tau_1]+[\pi\circ\sigma_2]=[\phi_2].$$

Therefore, Φ is injective.

5) Prove that $\Phi: Ext^{-1}(A,B)\to CP_B^{-1}(A,M(B))$ is surjective.

Suppose that $[\varphi_1]\in CP_B^{-1}(A,M(B))$. Then by Theorem 3.16 there exist $[\varphi_2]$ and an inner isomorphism $\phi\in Hom(A,M_2(M(B)))$ with $\phi=Adv$ and $v=(S_1,S_2)$, such that $[\varphi_1]+[\varphi_2]=[Adv\circ\phi]$. Since $\pi\circ\varphi_1=\phi_1$ and $\pi\circ\varphi_2=\phi_2$, by Theorem 3.17, ϕ_1 and ϕ_2 are homomorphisms and

$$[\phi_1]+[\phi_2]=[\pi\circ Adv\circ\phi]=[0].$$

Thus $[\phi_1]\in Ext^{-1}(A,B)$ and $\Phi([\phi_1])=[\varphi]$. This implies that Φ is surjective.

Similar to Lemma 3.18, we have the following result.

Corollary 3.20. Let A and B be C^*-algebras with B stable and let $\phi\in Hom(A,\mathcal{Q}(B))$. Consider the following three statements:

1) $[\phi]$ is invertible in $Ext(A,B)$.

2) There exists $\psi\in CP(A,M(B))$ such that $\phi=\pi\circ\psi$.

3) There exist $\varphi\in Hom(A,M_2(M(B)))$ and $\phi'\in Hom(A,M(B))$ such that

$$\begin{pmatrix} \phi & \\ & \phi' \end{pmatrix}=\pi\circ\varphi.$$

Then (1) $\Leftrightarrow$ (3) $\Rightarrow$ (2).

Proposition 3.21. Let A and C be C^*-algebras and $h,\phi\in Hom(A,C)$. Then

1) The map $h_*: CP_2(C,D)\to CP_2(A,D)$ defined by $[\varphi]_0\mapsto[\varphi\circ h]_0$ is a semigroup homomorphism.

2) The map $\phi_*: CP_B(C,D)\to CP_B(A,D)$ defined by $[\varphi]\mapsto[\varphi\circ\phi]$ is a unital semigroup homomorphism. Furthermore, it is a group homomorphism from $CP_B^{-1}(C,D)$ into $CP_B^{-1}(A,D)$.

Theorem 3.22. Let $\mathcal{C}$ be the category of C^*-algebras and $\mathcal{SG}$ the category of abelian semigroups. Define $CP_B(\cdot,D):\mathcal{C}\to\mathcal{SG}$ by $A\mapsto CP_B(A,D)$ and $\phi\mapsto\phi_*$ for any $A\in\mathcal{C}$ and $\phi\in Hom(A,C)$. Then $CP_B(\cdot,D)$ is a contravariant functor from $\mathcal{C}$ to $\mathcal{SG}$.

Proof. 1) For a C^*-algebra A and $[\varphi]\in CP_B(A,D)$, we have $I_*([\varphi])=[\varphi\circ I]=[\varphi]$. Then I_* is the unit of $CP_B(A,D)$.

2) Let $\varphi_1\in Hom(A,E)$ and $\varphi_2\in Hom(E,C)$. Set $F=CP_B(\cdot,D)$. Then

$$F(\varphi_2\circ\varphi_1)[\varphi]=[\varphi\circ\varphi_2\circ\varphi_1]=F(\varphi_1)[\varphi\circ\varphi_2]=F(\varphi_1)\circ F(\varphi_2)[\varphi].$$

Thus $CP_B(\cdot,D)$ is a contravariant functor.

Corollary 3.23. Let $\mathcal{G}$ be the category of abelian groups. Then $CP_B(\cdot,D)$

induces a contravariant functor $CP_B^{-1}(\cdot,D)$ from $\mathcal{C}$ into $\mathcal{G}$ by $A \mapsto CP_B^{-1}(A,D)$, and from $Hom(A,C)$ into $Hom\left(CP_B^{-1}(C,D),CP_B^{-1}(A,D)\right)$ by $\phi \mapsto \phi_*$.

For a short exact sequence of C^*-algebras $0 \to C \xrightarrow{\varphi_1} E \xrightarrow{\varphi_2} A \to 0$, the functor $CP_B(\cdot,D)$ from $\mathcal{C}$ to $\mathcal{SG}$ is not exact, and it is even not split-exact. The following is a counterexample.

Example 3.24. Suppose that H is an infinite dimensional separable Hilbert space. Let $A = C = K(H)$, $E = A \oplus C$, $D = B(H)$ and $B = 0$. Then $CP_B(A,D) = CP_2(A,D)$. Let $f_1 : C \to E$ be the inclusion map and let $f_2 : E \to A$ be the quotient map. Then the exact sequence

$$0 \to C \xrightarrow{f_1} E \xrightarrow{f_2} A \to 0$$

is split.

Take a nonzero element $\eta \in CP(A, \mathbb{C}I_D)$. We define a map $\varphi : E \to D$ by $\varphi|_C = I_C$ and $\varphi|_A = \eta$. Then $\varphi \in CP(E,D)$ and $[\varphi \circ f_1] = 0$. If $[\psi \circ f_2]_0 = [\varphi]_0$ for some $\psi \in CP_2(A,D)$, then there exist $\phi_1, \phi_2 \in Hom(E,D)$ and a unitary $u \in U(M_2(D))$ such that

$$\begin{pmatrix} \psi \circ f_2 & \\ & \phi_1 \end{pmatrix} = u \begin{pmatrix} \varphi & \\ & \phi_2 \end{pmatrix} u^*.$$

Put

$$u = \begin{pmatrix} u_1 & u_2 \\ u_3 & u_4 \end{pmatrix}.$$

Since $(\psi \circ f_2)(E) = 0$, $u_1\varphi(E)u_1^* + u_2\phi_2(E)u_2^* = 0$. Note that $u_1\varphi(e)u_1^*, u_2\phi_2(e)u_2^*$ are positive if e is positive in E. It follows that

$$u_1\varphi(e)u_1^* = u_2\phi_2(e)u_2^* = 0.$$

Therefore $u_1K(H)u_1^* = 0$. Furthermore, $u_1u_1^* = 0$ since there is a sequence in $K(H)$ which is convergent to I in the strong operator topology on $B(H)$. Then $u_1 = 0$. Hence $u_4 = 0$ and $u_2, u_3 \in U(D)$. Therefore,

$$u \begin{pmatrix} \varphi & \\ & \phi_2 \end{pmatrix} u^* = \begin{pmatrix} u_2\phi_2u_2^* & \\ & u_3\varphi u_3^* \end{pmatrix}.$$

Since $u_3\varphi u_3^* = \phi_1$ is a homomorphism, φ is also a homomorphism. However, $\varphi|_A$ is not a homomorphism by the definition of φ. Otherwise, if $\varphi|_A$ is a homomorphism from $K(H)$ to $\mathbb{C}$, then it follows that $\varphi|_A = 0$ since a completely positive map preserves self-adjoint elements. This is in contradiction to the fact that $\varphi|_A \neq 0$.

Theorem 3.25. Suppose that

$$0 \to C \xrightarrow{f_1} E \xrightarrow{f_2} A \to 0$$

is a split short exact sequence, then

$$0 \to CP_B^{-1}(A,D) \xrightarrow{(f_2)_*} CP_B^{-1}(E,D) \xrightarrow{(f_1)_*} CP_B^{-1}(C,D) \to 0$$

is also a split short exact sequence.

Proof. Since $\left(f_1\right)_* \circ \left(f_2\right)_* \left(\left[\varphi_A\right]\right) = \left[\varphi_A \circ f_2 \circ f_1\right] = 0$, we have $Im\left(f_2\right)_* \subset Ker\left(f_1\right)_*$.

Assume that $E = A \oplus C$. For any $[\varphi] \in Ker\left(\left(f_1\right)_*\right)$, let $\varphi_A = \varphi|_A$ and $\varphi_C = \varphi|_C$. Then $\varphi = \varphi_A \oplus \varphi_C$. Note that $[\varphi]$ is invertible and

$$\left[\varphi_C\right] = \left[\varphi \circ f_1\right] = \left(f_1\right)_*\left([\varphi]\right).$$

Hence, $\left[\varphi_C\right] \in CP_B^{-1}(C, D)$. Similarly, $\left[\varphi_A\right]$ is also invertible.

Suppose that the inverses of $\left[\varphi_A\right]$ and $\left[\varphi_C\right]$ are $\left[\varphi_A'\right]$ and $\left[\varphi_C'\right]$ respectively. Let $\left[\varphi'\right] = \left[\varphi_A' \oplus \varphi_C'\right]$. Now we show that $[\varphi] + \left[\varphi'\right]$ is the unit. Suppose that $\left[\varphi_A''\right]_0 = \left[\varphi_A\right]_0 + \left[\varphi_A'\right]_0$ and $\left[\varphi_C''\right]_0 = \left[\varphi_C\right]_0 + \left[\varphi_C'\right]_0$ such that $\pi \circ \varphi_A'' = \pi \circ \phi_A$ and $\pi \circ \varphi_C'' = \pi \circ \phi_C$, where ϕ_A and ϕ_C are homomorphisms. Then

$$[\varphi]_0 + \left[\varphi'\right]_0 = \left[\varphi_A'' \oplus \varphi_C''\right]_0.$$

Since $\pi\left(\varphi_A'' \oplus \varphi_C''\right)$ is a homomorphism, $[\varphi] + \left[\varphi'\right]$ is the unit of $CP_B^{-1}(E, D)$. Since $[\varphi] \in Ker\left(\left(f_1\right)_*\right)$, $\left[\varphi \circ f_1\right] = 0$. Then $\pi \circ \varphi_C = \pi \circ \phi_C$ and hence $\left[\varphi_C'\right]$ is the inverse of $\left[\varphi_C\right]$. Therefore, $\left[\varphi'\right] = \left[\varphi_A' \oplus 0\right]$ is the inverse of $[\varphi]$. Since $\left(f_2\right)_*\left(\left[\varphi_A'\right]\right) = \left[\varphi'\right]$, $[\varphi] \in Im\left(\left(f_2\right)_*\right)$. Thus,

$$Im\left(\left(f_2\right)_*\right) = Ker\left(\left(f_1\right)_*\right).$$

Suppose that $\left(f_2\right)_*\left(\left[\varphi_A\right]\right) = 0$. Then $\left[\varphi_A \oplus 0\right] = 0$, and there exist $\psi \in CP(E, D)$ and $\phi \in Hom(E, D)$ such that $[\psi]_0 = \left[\varphi_A + 0\right]_0$ and $\pi \circ \psi = \pi \circ \phi$. Hence, $\pi \circ \psi|_A = \pi \circ \phi|_A$. Note that $\phi|_A \in Hom(A, D)$ and $\left[\phi|_A\right]_0 = \left[\varphi_A\right]_0$. It follows that $\left[\varphi_A\right] = 0$ and $\left(f_2\right)_*$ is an injective homomorphism.

Suppose that $\left[\varphi_C\right] \in CP_B^{-1}(C, D)$. Then we have

$$\left(f_1\right)_*\left(\left[0 \oplus \varphi_C\right]\right) = \left[\left(0 \oplus \varphi_C\right) \circ f_1\right] = \left[\varphi_C\right].$$

Therefore, $\left(f_1\right)_*$ is surjective.

Define

$$f_* : CP_B^{-1}(C, D) \to CP_B^{-1}(E, D), \left[\varphi_C\right] \mapsto \left[0 \oplus \varphi_C\right].$$

Then $\left(f_1\right)_* \circ f_* = I$. Finally,

$$0 \to CP_B^{-1}(A, D) \xrightarrow{\left(f_2\right)_*} CP_B^{-1}(E, D) \xrightarrow{\left(f_1\right)_*} CP_B^{-1}(C, D) \to 0$$

is a split short exact sequence.

Remark 3.26. For any C^*-algebra *B*, we can define $CP_2(A, B)$, $CP_I^{-1}(A, B)$, etc., to be $CP_2\left(A, M\left(\mathcal{K} \otimes B\right)\right)$, $CP_I^{-1}\left(A, M\left(\mathcal{K} \otimes B\right)\right)$, respectively. Since for any stable C^*-algebra its multiplier algebra is properly infinite, these invariants are well-defined.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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