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Numerical Methods for Solving Logarithmic Nonlinear Schrödinger's Equation

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Abstract

In this study, we will construct numerical techniques for tackling the logarithmic Schrödinger's nonlinear equation utilizing the explicit scheme and the Crank-Nicolson scheme of the finite difference method. These schemes will be subjected to accuracy and stability tests before being used. Efficacy and robustness of the techniques under consideration will be demonstrated using an exact solution, one-Gausson, as well as conserved quantities. Interaction of two-soliton will be conducted. The numerical findings revealed, the interplay behavior is flexible.

Keywords

Explicit Scheme, Implicit Scheme, Exact Solutions, Bounded Domain, Stability, One Soliton, Soliton Interaction, Gaussons

1. Introduction

One of the most significant fields of nonlinear optical study recently has been the dynamics of solitons in optical communication [1]-[14]. In the last ten years, there has been a surge in interest in the study of optical solitons with log-law nonlinearity, commonly known as optical Gaussons. Several analytical findings have been made public [15]. The non-linear Schrödinger's (NLS) equation is employed in different domains, such as mathematical biology, fluid dynamics, plasma physics, nonlinear optics, and many more. The topic of integrability is one of the most critical issues that arise in this context. The NLS equation is rendered unintegrable by the majority of these nonlinear forms. Much research is being conducted all around the world to address the issue of integrability. One of the nonlinearity forms is log-law nonlinearity. As a result of the logarithmic nonlinearity blowing up, there are significant difficulties in designing numerical methods, as well as setting error boundaries for the logarithmic NLS (log NLS) equation. Because of the logarithmic non-linearity and the physical domain's unboundedness, it's challenging to build numerical algorithms to solve the logarithmic Schrödinger's equation over unbounded fields [16].

Biswas and Aceves presented a perturbation approach for studying optical solitons defined by a perturbed nonlinear Schrödinger's equation in the year 2000 [1]. Power law, parabolic law, dual-power law, and Kerr law are examples of nonlinearities explored by Kohl *et al.* in 2008 [2]. The optical soliton cooling in the presence of perturbation terms is discussed by Biswas in the same year. Saturable nonlinearity is the sort of nonlinearity that was examined. Gaussian, Sech, super-sech, and super-Gaussian optical pulses are the four kinds studied [3]. In 2010, Biswas *et al.* found an accurate one-soliton answer to the equation of Schrödinger's involving the log-law nonlinearity in the existence of transient disturbances [4]. Khalique and Biswas in 2010 developed a method for integrating Schrödinger's equation having log-law nonlinearity into a single equation. This is accomplished through the use of the Lie symmetry technique. As a result, stationary solutions were discovered [5]. In the same year, Biswas and Milovic used the solitary wave ansatz approach to characterize the one soliton solution of Schrödinger's equation having log law nonlinearity [6].

Biswas *et al.* in 2011 it was described how Schrödinger's nonlinear equation works, which governs optical solitons, is explored using non-Kerr law nonlinearity in the existence of perturbation components with complete nonlinearity [7]. Biswas *et al.* in 2012 discussed the non-linear optical soliton solution. His variational principle is used to derive Schrödinger's equation with log law nonlinearity [8]. Hongwei *et al.* in 2019 focused on the approximate solution of the log NLS equation problem on unbounded domains. Unbounded domains make it challenging to create numerical approaches for the logarithmic Schrödinger's equation, which has nonlinearity and is unbounded [16]. For the log NLS equation, BAO *et al.* in 2019 proposed and established an error bound for a regularized finite difference technique. The blowup of logarithmic nonlinearity is a key difficulty in creating numerical algorithms and defining error boundaries for the log NLS equation [17].

Wazwaz used the variation iteration approach to conduct research on the logarithmic Schrodinger equation, examining it both with and without a detuning term in 2019 [18]. Salman *et al.* discussed about the multi-photon absorption and bandpass filters in the same year. He also explained the mean free velocity of optical Gaussons that move with stochastic perturbation [19]. According to Gaxiola *et al.* in 2020, the Laplace-Adomian decomposition approach is useful in the study of optical Gaussons. The numerical simulations were shown both with and without the detuning term. The scheme's error assessment was also presented [15]. Darvishiat *et al.* presented a total of three novel logarithmic nonlinear amplitude equations in 2022. In this study, these logarithmic equations were analyzed in order to locate the Gaussian solitary waves that they produce [20].

2. Problem Statement

In this study, we shall provide numerical solutions to Schrödinger's nonlinear logarithmic equation in limited domains.

Now we'll consider, the log NLS equation

$$i\frac{\partial u}{\partial t}+a\frac{\partial^2 u}{\partial x^2}+b\log\left(|u|^2\right)u=0, \tag{1}$$

in the finite domain $x_L \le x \le x_R$, $0 \le t \le T$, using the following initial and boundary conditions:

$$u(x,0)=g(x), x_L \le x \le x_R,$$

$$u(x_L,t)=0=u(x_R,t), 0 \le t \le T,$$

where a denotes the dispersion coefficient of the group velocity and b is the log law nonlinearity coefficient. Group velocity dispersion is a dispersive medium characteristic used to determine how it affects pulse duration. The wave profile is the dependent variable $u(x,t)$, whereas the spatial and temporal variables are independent variables x and t, respectively [21].

To sidestep complex calculations, we suppose

$$u(x,t)=v(x,t)+iw(x,t), \tag{2}$$

$v(x,t)$ and $w(x,t)$ are both real-valued functions. By replacing (2) into (1), we can get the coupled system shown below by segregating the real and imaginary portions

$$\frac{\partial v}{\partial t}+a\frac{\partial^2 w}{\partial x^2}+bzw=0, \tag{3}$$

$$\frac{\partial w}{\partial t}-a\frac{\partial^2 v}{\partial x^2}-bzv=0, \tag{4}$$

where $z=\log\left(v^2+w^2\right)$. The system contained in Equations (3) and (4), may be expressed as a matrix-vector as

$$\frac{\partial \mathbf{u}}{\partial t}+a\psi\frac{\partial^2 \mathbf{u}}{\partial x^2}+b\psi z\mathbf{u}=\mathbf{0}. \tag{5}$$

where

$$\mathbf{u}=\begin{bmatrix} v \\ w \end{bmatrix}, \psi=\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

The log NLS equation's exact soliton solution is provided by [6] [8] [15], [19] and [21].

$$u(x,t)=A\mathrm{e}^{-B^2(x-st)^2}\mathrm{e}^{i(-kx+\zeta t+\theta)}, \tag{6}$$

where $s=-2ak$, $B=\sqrt{b/2a}$, $\zeta=2b\log(A)-ak^2-b$, and $ab>0$.

Here A denotes the amplitude of Gaussons and θ represents the center of Gausson's phase. The wave number from the phase component is expressed by ζ, s is the Gausson velocity which is related to its frequency and B indicates the inverse width.

The three preserved quantities in the log NLS Equation (1) are, [6] [19] [21]

$$I_1 = \int_{-\infty}^{\infty} |u|^2 \, \mathrm{d}x, \tag{7}$$

$$I_2 = i\int_{-\infty}^{\infty} \left[u^* u_x - u u_x^* \right] \mathrm{d}x, \tag{8}$$

$$I_3 = \int_{-\infty}^{\infty} \left[a|u_x|^2 - b|u|^2 \log|u|^2 + b|u|^2 \right] \mathrm{d}x, \tag{9}$$

where u^* represents *u*'s complex conjugate.

3. Numerical Methods

To derive numerical methods for solving Equation (1), numerically, the problem domain is specified as $x_L \le x \le x_R$, $0 \le t \le T$, these coordinates are covered by a rectangular grid of points, $x = x_m = x_L + mh$, $m = 0,1,2,\cdots,N$, $t = t_n = nk$, $n = 0,1,2,\cdots$, $h = \frac{x_R - x_L}{N}$, where h and k denote the spatial and temporal increases.

3.1. Explicit Method

To construct an explicit technique for resolving the log NLS equation, we assume $\mathbf{U}_m^n$ to be the numerical solutions, and $\mathbf{u}_m^n$ to be accurate solutions, at the point (x_m, t_n), where $\mathbf{U}_m^n = \begin{bmatrix} V_m^n \\ W_m^n \end{bmatrix}$. To solve Equations (3) and (4), by using the basic explicit finite difference method, which uses forward approximation for time and central difference approximation for space, as follows

$$\frac{V_m^{n+1} - V_m^n}{k} + a\frac{1}{h^2}\delta_x^2 W_m^n + bz_m^n W_m^n = 0 \tag{10}$$

$$\frac{W_m^{n+1} - W_m^n}{k} - a\frac{1}{h^2}\delta_x^2 V_m^n - bz_m^n V_m^n = 0 \tag{11}$$

Equations (10) and (11) can be written as

$$V_m^{n+1} = V_m^n - ar\left[W_{m-1}^n - 2W_m^n + W_{m+1}^n \right] - bkz_m^n W_m^n \tag{12}$$

$$W_m^{n+1} = W_m^n + ar\left[V_{m-1}^n - 2V_m^n + V_{m+1}^n \right] + bkz_m^n V_m^n, \tag{13}$$

where $r = \frac{k}{h^2}$, $m = 1,2,\cdots,N$, $n = 0,1,2,\cdots$, the resulting system in Equations (12) and (13) is two level explicit scheme $\mathbf{U}_m^{n+1}, \mathbf{U}_m^n$, we can solve this system for V_m^{n+1}, W_m^{n+1} directly, where we enforce the boundary conditions $\mathbf{U}_0^n = 0 = \mathbf{U}_M^n$.

3.1.1. Accuracy of the Scheme

To test the effectiveness of the suggested method, we examine the rigor of (10). To achieve this, we substitute the numerical solution V_m^n, W_m^n with the accurate solution v_m^n, w_m^n, and by applying Taylor's series expansion of all terms in Equation (10), around the point (x_m, t_n), we get the following expansions.

$$\frac{v_m^{n+1} - v_m^n}{k} = \frac{\partial v}{\partial t} + \frac{k}{2}\frac{\partial^2 v}{\partial t^2} + O(k^2),$$

$$\frac{1}{h^2}\delta_x^2 w_m^n = \frac{\partial^2 w}{\partial x^2} + \frac{h^2}{12}\frac{\partial^4 w}{\partial x^4} + O\left(h^4\right), \tag{14}$$

$$bz_m^n w_m^n = bzw,$$

Equations (14) are substituted into Equation (10) to get

$$\left[\frac{\partial v}{\partial t} + a\frac{\partial^2 w}{\partial x^2} + bzw\right] + \frac{k}{2}\frac{\partial^2 v}{\partial t^2} + \frac{h^2}{12}\frac{\partial^4 w}{\partial x^4} + \cdots$$

The value of the first bracket will be zero, if differential Equation (3) is used, As a result, the local truncation error will be

$$LTE = \frac{k}{2}\frac{\partial^2 v}{\partial t^2} + \frac{h^2}{12}\frac{\partial^4 w}{\partial x^4} + \cdots \tag{15}$$

The schema is second order in space and first order in time. It is possible to research the accuracy of the scheme (11) the same way.

3.1.2. Stability of the Scheme

This section looks at the numerical method's stability, or how sensitive the numerical solution, to examine scheme (1) stability, using the explicit scheme, we write it as follows

$$i\left(\mathbf{U}_m^{n+1} - \mathbf{U}_m^n\right) + ar\delta_x^2\mathbf{U}_m^n + bkz\mathbf{U}_m^n = 0, \tag{16}$$

where $r = \frac{k}{h^2}$. We note that schema (16) is non-linear and the von-Neumann stability analysis can only be used for linear schemas. We can achieve the linear version by freezing the phrase that makes the scheme nonlinear.

$$i\left(\mathbf{U}_m^{n+1} - \mathbf{U}_m^n\right) + \left(ar\delta_x^2 + bk\lambda\right)\mathbf{U}_m^n = 0, \tag{17}$$

where $\lambda = \max\left(z\right)$. Now we'll do a von-Neumann stability analysis, as shown below, let

$$\mathbf{U}_m^n = \mathrm{e}^{\alpha nk}\mathrm{e}^{i\beta mh}, \tag{18}$$

where α and β are constant, $i = \sqrt{-1}$, then

$$\delta_x^2\mathbf{U}_m^n = -4\sin^2\left(\frac{\beta h}{2}\right)\mathbf{U}_m^n. \tag{19}$$

Substitute Equations (18) and (19) into Equation (17) we will get

$$i\left(\mathrm{e}^{\alpha k} - 1\right) - 4ar\sin^2\left(\frac{\beta h}{2}\right) + bk\lambda = 0,$$

it can also be expressed as

$$\left|\mathrm{e}^{\alpha k}\right|^2 = 1 + \left[4ar\sin^2\left(\frac{\beta h}{2}\right) - bk\lambda\right]^2,$$

which shows that $\mathrm{e}^{\alpha k} > 1$, and hence the scheme is unconditionally unstable.

3.2. The Implicit Scheme of Second Order

The explicit scheme developed in the preceding part is second order in space and first order in time, but it has an issue with stability since it is unstable. In this part, we will overcome this problem by designing a Crank-Nicolson-like schema that is unconditionally stable and has second-order precision of space and time. We assume that $v(x_m,t_n), w(x_m,t_n)$ and $V(x_m,t_n), W(x_m,t_n)$ are the exact and approximate solutions respectively at the point (x_m,t_n).

The numerical solution to Equations (3) and (4) are obtained by employing a Crank-Nicolson-like technique,

$$\frac{V_m^{n+1}-V_m^n}{k}+\frac{a}{2h^2}\delta_x^2\left(W_m^{n+1}+W_m^n\right)+b\left(\frac{z_m^{n+1}+z_m^n}{2}\right)\left(\frac{W_m^{n+1}+W_m^n}{2}\right)=0, \tag{20}$$

$$\frac{W_m^{n+1}-W_m^n}{k}-\frac{a}{2h^2}\delta_x^2\left(V_m^{n+1}+V_m^n\right)-b\left(\frac{z_m^{n+1}+z_m^n}{2}\right)\left(\frac{V_m^{n+1}+V_m^n}{2}\right)=0. \tag{21}$$

Equations (20) and (21) can be written as

$$\begin{aligned}&V_m^{n+1}-V_m^n+r_1\left[\left(W_{m-1}^{n+1}-2W_m^{n+1}+W_{m+1}^{n+1}\right)+\left(W_{m-1}^n-2W_m^n+W_{m+1}^n\right)\right]\\&+r_2\left(z_m^{n+1}+z_m^n\right)\left(W_m^{n+1}+W_m^n\right)=0,\end{aligned} \tag{22}$$

$$\begin{aligned}&W_m^{n+1}-W_m^n-r_1\left[\left(V_{m-1}^{n+1}-2V_m^{n+1}+V_{m+1}^{n+1}\right)+\left(V_{m-1}^n-2V_m^n+V_{m+1}^n\right)\right]\\&-r_2\left(z_m^{n+1}+z_m^n\right)\left(V_m^{n+1}+V_m^n\right)=0,\end{aligned} \tag{23}$$

where $r_1=\dfrac{ak}{2h^2}$, $r_2=\dfrac{bk}{4}$.

The resultant scheme is an implicit nonlinear scheme that generates a block nonlinear tridiagonal system. The nonlinear system obtained may be solved using a variety of approaches. We will, in this paper, utilize the fixed point method to solve Equations (22) and (23). As follows

$$\begin{aligned}&V_m^{n+1,(s+1)}+r_1\left[W_{m-1}^{n+1,(s+1)}-2W_m^{n+1,(s+1)}+W_{m+1}^{n+1,(s+1)}\right]+r_2\left[z_m^{n+1,(s)}+z_m^n\right]W_m^{n+1,(s+1)}\\&=V_m^n-r_1\left[W_{m-1}^n-2W_m^n+W_{m+1}^n\right]-r_2\left[z_m^{n+1,(s)}+z_m^n\right]W_m^n,\end{aligned} \tag{24}$$

$$\begin{aligned}&W_m^{n+1,(s+1)}-r_1\left[V_{m-1}^{n+1,(s+1)}-2V_m^{n+1,(s+1)}+V_{m+1}^{n+1,(s+1)}\right]-r_2\left[z_m^{n+1,(s)}+z_m^n\right]V_m^{n+1,(s+1)}\\&=W_m^n+r_1\left[V_{m-1}^n-2V_m^n+V_{m+1}^n\right]+r_2\left[z_m^{n+1,(s)}+z_m^n\right]V_m^n.\end{aligned} \tag{25}$$

where $s=0,1,2,\text{etc.}$, The resulting system in Equations (24) and (25) may be expressed as a matrix-vector,

$$J\mathbf{U}^{n+1,(s+1)}=\mathbf{F}\left(\mathbf{U}^{n+1,(s)},\mathbf{U}^n\right) \tag{26}$$

The block tridiagonal matrix J may be expressed in the following way:

$$J(u)=\begin{bmatrix}A_1 & C_1 & 0 & \cdots & 0\\ B_2 & A_2 & C_2 & \ddots & \vdots\\ 0 & \ddots & \ddots & \ddots & 0\\ \vdots & \ddots & B_{n-1} & A_{n-1} & C_{n-1}\\ 0 & \cdots & 0 & B_n & A_n\end{bmatrix},$$

hence the elements of the matrix J may be described in detail as follows:

$$B_m = \begin{bmatrix} 0 & r_1 \\ -r_1 & 0 \end{bmatrix}_m, A_m = \begin{bmatrix} 1 & a_{12} \\ a_{21} & 1 \end{bmatrix}_m, C_m = \begin{bmatrix} 0 & r_1 \\ -r_1 & 0 \end{bmatrix}_m,$$

where

$$a_{12} = -2r_1 + r_2\left[z_m^{n+1,(s)} + z_m^n\right],$$

$$a_{21} = 2r_1 - r_2\left[z_m^{n+1,(s)} + z_m^n\right],$$

$$m = 1, 2, \cdots, N.$$

Crout's method may be used to solve the system in Equation (26) and we take the initial guess as $\underline{U}^{n+1,(0)} = \underline{U}^n$. The iteration process is repeated until the following condition is met:

$$\left\| \left(U_m^{n+1}\right)^{(s+1)} - \left(U_m^{n+1}\right)^{(s)} \right\|_\infty < tol,$$

where "*tol*" is a small number used to quantify error.

3.2.1. Accuracy of the Scheme

To investigate the system's accuracy in Equations (20) and (21), we replace the numerical solution V_m^n, W_m^n by the exact solution v_m^n and w_m^n to get

$$\frac{v_m^{n+1} - v_m^n}{k} + a\delta_x^2\left(\frac{w_m^{n+1} + w_m^n}{2h^2}\right) + b\left(\frac{z_m^{n+1} + z_m^n}{2}\right)\left(\frac{w_m^{n+1} + w_m^n}{2}\right) = 0, \quad (27)$$

$$\frac{w_m^{n+1} - w_m^n}{k} - a\delta_x^2\left(\frac{v_m^{n+1} + v_m^n}{2h^2}\right) - b\left(\frac{z_m^{n+1} + z_m^n}{2}\right)\left(\frac{v_m^{n+1} + v_m^n}{2}\right) = 0. \quad (28)$$

By expanding the terms in Equation (27), using Taylor's series, we get,

$$\frac{v_m^{n+1} - v_m^n}{k} = \frac{\partial v}{\partial t} + \frac{k}{2}\frac{\partial^2 v}{\partial t^2} + \frac{k^2}{6}\frac{\partial^3 v}{\partial t^3} + \cdots,$$

$$\frac{1}{2h^2}\delta_x^2\left(w_m^{n+1} + w_m^n\right) = \frac{\partial^2 w}{\partial x^2} + \frac{k}{2}\frac{\partial^3 w}{\partial x^2 \partial t} + \frac{h^2}{12}\frac{\partial^4 w}{\partial x^4} + \frac{k^2}{4}\frac{\partial^4 w}{\partial x^2 \partial t^2} + \cdots,$$

$$\left(\frac{z_m^{n+1} + z_m^n}{2}\right)\left(\frac{w_m^{n+1} + w_m^n}{2}\right) = zw + \frac{k}{2}\frac{\partial}{\partial t}(zw) + \frac{k^2}{4}\left[\frac{\partial}{\partial t}\left(z\frac{\partial w}{\partial t}\right) + \frac{\partial^2 z}{\partial t^2}w\right] + \cdots, \quad (29)$$

Then, if we put Equation (29) into Equation (27) we'll get

$$\left(\frac{\partial v}{\partial t} + \frac{k}{2}\frac{\partial^2 v}{\partial t^2} + \frac{k^2}{6}\frac{\partial^3 v}{\partial t^3} + \cdots\right) + a\left(\frac{\partial^2 w}{\partial x^2} + \frac{k}{2}\frac{\partial^3 w}{\partial x^2 \partial t} + \frac{h^2}{12}\frac{\partial^4 w}{\partial x^4} + \frac{k^2}{4}\frac{\partial^4 w}{\partial x^2 \partial t^2} + \cdots\right)$$
$$+ b\left\{zw + \frac{k}{2}\frac{\partial}{\partial t}(zw) + \frac{k^2}{4}\left[\frac{\partial}{\partial t}\left(z\frac{\partial w}{\partial t}\right) + \frac{\partial^2 z}{\partial t^2}w\right] + \cdots\right\}$$

and it's also possible to write it as

$$\left[\frac{\partial v}{\partial t} + a\frac{\partial^2 w}{\partial x^2} + bzw\right] + \frac{k}{2}\frac{\partial}{\partial t}\left[\frac{\partial v}{\partial t} + a\frac{\partial^2 w}{\partial x^2} + bzw\right]$$
$$+ \frac{k^2}{2}\left\{\frac{1}{3}\frac{\partial^3 v}{\partial t^3} + \frac{1}{2}\frac{\partial^4 w}{\partial x^2 \partial t^2} + \frac{1}{2}\left[\frac{\partial}{\partial t}\left(z\frac{\partial w}{\partial t}\right) + \frac{\partial^2 z}{\partial t^2}w\right]\right\} + a\frac{h^2}{12}\frac{\partial^4 W}{\partial x^4} + \cdots$$

When differential Equation (3) is used, all terms within the first and second brackets both have a value of zero, resulting in the local truncation error (LTE) as

$$LTE = \frac{k^2}{2}\left\{\frac{1}{3}\frac{\partial^3 v}{\partial t^3} + \frac{1}{2}\frac{\partial^4 w}{\partial x^2 \partial t^2} + \frac{1}{2}\left[\frac{\partial}{\partial t}\left(z\frac{\partial w}{\partial t}\right) + \frac{\partial^2 z}{\partial t^2} w\right]\right\} + a\frac{h^2}{12}\frac{\partial^4 W}{\partial x^4} + \cdots, \quad (30)$$

this shows that in both time and space, the scheme is second-order, It is possible to research the accuracy of the scheme (21) the same way.

3.2.2. Scheme's Stability

The linear form of schema (5), is obtained by freezing the term that makes the scheme nonlinear,

$$\mathbf{U}_m^{n+1} - \mathbf{U}_m^n + \frac{k}{2}\left[\frac{a}{h^2}\psi\delta_x^2 + b\eta\psi\right]\left(\mathbf{U}_m^{n+1} + \mathbf{U}_m^n\right) = 0, \quad (31)$$

where $\eta = \max\{z\}$. Now we'll do a von-Neumann stability analysis, as shown below.

We assume

$$\mathbf{U}_m^n = \mathbf{U}_0^n \mathrm{e}^{i\beta mh}, \mathbf{U}_m^{n+1} = G\mathbf{U}_0^n \mathrm{e}^{i\beta mh}, \quad (32)$$

where $\mathbf{U}_0^n$ is referred to as the constant vector, while G is referred to as the amplification matrix and

$$\delta_x^2 U_m^n = -4\sin^2\left(\frac{\beta h}{2}\right) U_m^n \quad (33)$$

Substitute Equations (32) and (33) into Equation (31) we will get

$$[I + \psi\gamma]G = [I - \psi\gamma] \quad (34)$$

where $\gamma = \frac{k}{2}\left[b\eta - \frac{4a}{h^2}\sin^2\left(\frac{\beta h}{2}\right)\right]$, Equation (34) may be expressed in the following way:

$$G = [I + \psi\gamma]^{-1}[I - \psi\gamma] \quad (35)$$

The G matrix's eigenvalues are

$$\lambda_1 = \frac{1-\gamma^2 - 2i\gamma}{1+\gamma^2} \text{ and } \lambda_2 = \frac{1-\gamma^2 + 2i\gamma}{1+\gamma^2} \quad (36)$$

The $\max|\lambda_m| \le 1, m = 1,2$ criterion is required for stability when utilizing von-Neumann. It is clear from (36) that, $|\lambda_m| = 1$, and hence the scheme is unconditionally stable.

4. Numerical Tests

We will try to solve the cases listed below to see how well the suggested strategies work.

4.1. One Soliton

To use the exact solution in Equation (6), we use the initial condition at $t = 0$, $u(x,0) = A\mathrm{e}^{-B^2x^2}\mathrm{e}^{i(-kx+\theta)}$, this test makes use of the following parameters: $h = 0.1$,

$k = 0.0001$, $s = 0.4$, $A = 0.4$, $a = 0.5$, $b = 1$, $\theta = 0.5$. We investigate the suggested method's accuracy by computing the L_∞-error norm, as defined by

$$L_\infty = \left\| U^n - u^n \right\|_\infty = \max \left| U\left(x_m, t_m\right) - u\left(x_m, t_m\right) \right|,$$

The error analysis of the explicit technique and the Crank-Nicolson method employing L_∞ norms is presented in Table 1 and Table 2, respectively. For both the real and imaginary parts of the equation, the error of the L_∞ norm increases as time goes on. Table 3 and Table 4 exhibit the conserved quantities of one soliton using the explicit technique and the Crank-Nicolson method, respectively. Clearly, given the two tables, Crank Nicolson provides far better-conserved quantities than the other. Figure 1 illustrates a single soliton solution using the Explicit approach, with parameters $h = 0.1$, $k = 0.0001$ and $s = 0.4$. A single soliton solution for the identical parameters as Figure 1, is shown in Figure 2 using the Crank Nicolson technique.

Table 1. Error analysis by varying time using explicit method.

Time	$L_\infty(V)$	$L_\infty(W)$
0.00	0.000000	0.000000
1.00	0.000911	0.001779
2.00	0.001946	0.001852
3.00	0.004109	0.002869
4.00	0.005308	0.004220
5.00	0.007105	0.006551

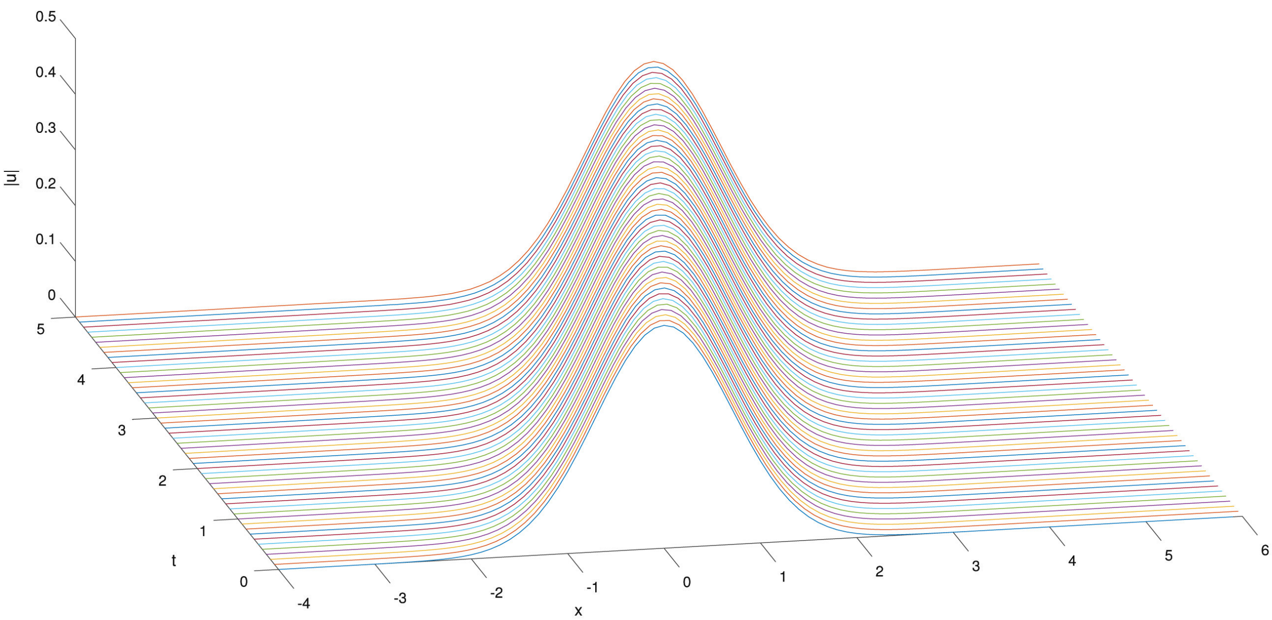

Figure 1. Simulation shows a single solution using the explicit scheme with the parameters as $h = 0.1$, $k = 0.0001$, $s = 0.4$, $0 \le t \le 5$.

Table 2. Error analysis by using crank-nicolson method.

Iteration	Time	$L_\infty(V)$	$L_\infty(W)$
0	0.00	0.000000	0.000000
2	1.00	0.000833	0.001875
2	2.00	0.001344	0.001595
2	3.00	0.002503	0.003054
2	4.00	0.002844	0.003667
2	5.00	0.004200	0.003933

Table 3. Quantities preserved using explicit scheme.

Time	I_1	I_2	I_3
0.00	0.200530	−0.159582	0.783269
1.00	0.200703	−0.159802	0.783770
2.00	0.200877	−0.160041	0.784309
3.00	0.201050	−0.160266	0.784816
4.00	0.201223	−0.160503	0.785346
5.00	0.201397	−0.160733	0.785867

Table 4. Quantities preserved by using crank-nicolson method.

Iteration	Time	I_1	I_2	I_3
0	0.00	0.200530	−0.159582	0.783269
2	1.00	0.200530	−0.159571	0.783248
2	2.00	0.200530	−0.159579	0.783266
2	3.00	0.200530	−0.159572	0.783251
2	4.00	0.200530	−0.159578	0.783260
2	5.00	0.200530	−0.159577	0.783257

4.2. Collision of Two Solitons

In this exam, we will make a collision between two solitons, by using the following form

$$u(x,t) = A\sum_{m=1}^{2} \mathrm{e}^{-B^2(x-x_m-s_m t)^2}\,\mathrm{e}^{i(-k_m(x-x_m)+\zeta_m t+\theta)},$$

where

$$s_m = -2ak_m,\ B = \sqrt{b/2a},\ \zeta_m = 2b\log(A) - ak_m^2 - b,\ m = 1,2 \ \text{ and } \ ab > 0.$$

In this test, the parameters below are utilized.

$$h = 0.1, k = 0.0001, s_1 = 0.8, s_2 = -0.4, A = 0.4,$$
$$a = 0.5, b = 1, \theta = 0.5, x_1 = 0, x_2 = 4.5$$

Detailed numerical study of the interaction between two solitons is reported in Table 5 and Table 6. In this case, the Crank Nicolson scheme is evaluated highly in terms of conservation of quantities since the results obtained by the Explicit technique are inconsistent. Two solitons appear to revert to their original shape after interacting with one another, from left to right, velocities are 0.8 and −0.4 in Figure 3 obtained from the Explicit approach. Figure 4 shows the simulation of interaction of two solitons using Crank-Nicolson scheme, for the same velocities for $0 \le t \le 5$. In our observations, the two waves clash and exit the interaction area without causing any change in their identities.

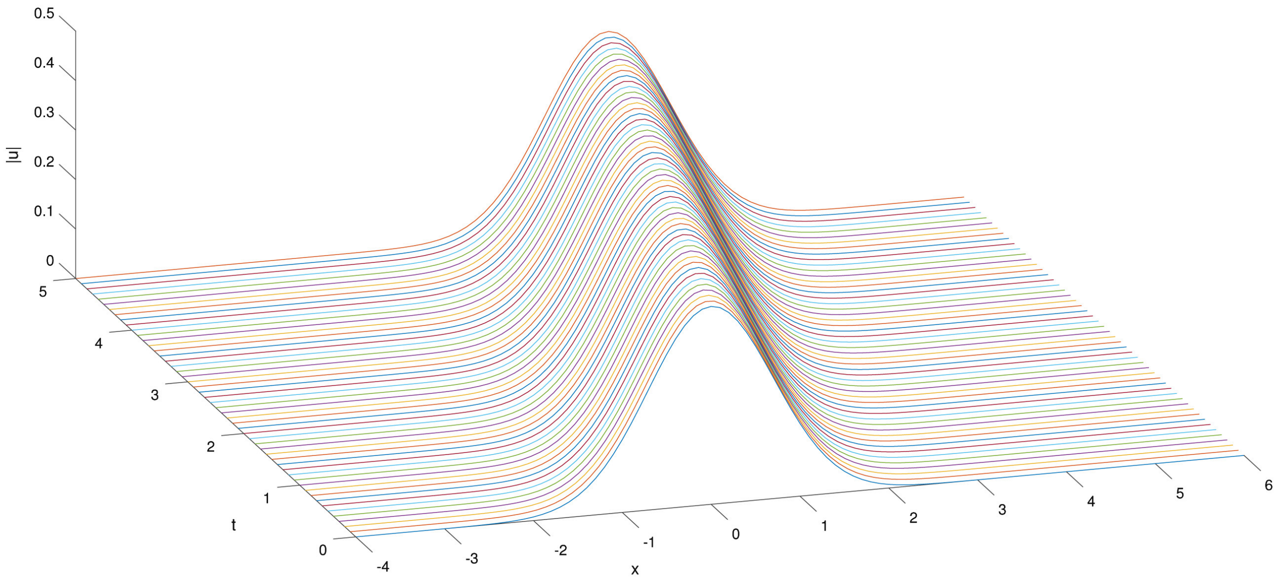

Figure 2. The simulation demonstrates a single solution using the Crank-Nicolson method and parameters $h = 0.1$, $k = 0.0001$, $s = 0.4$, $0 \le t \le 5$.

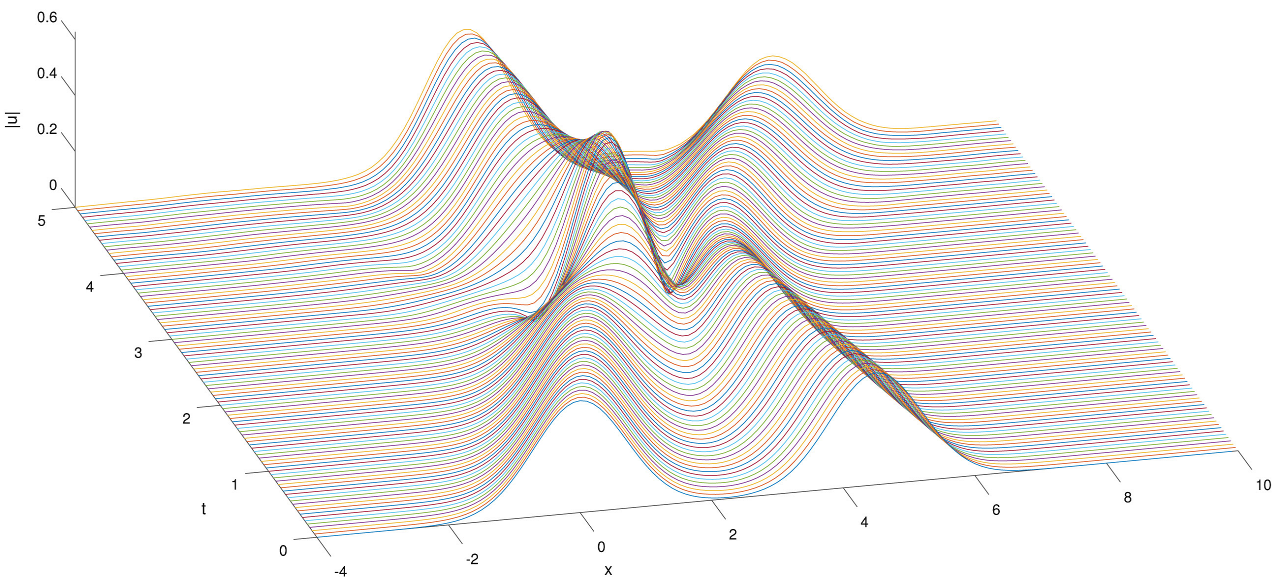

Figure 3. Simulation of the interaction of two solitons by using the explicit scheme.

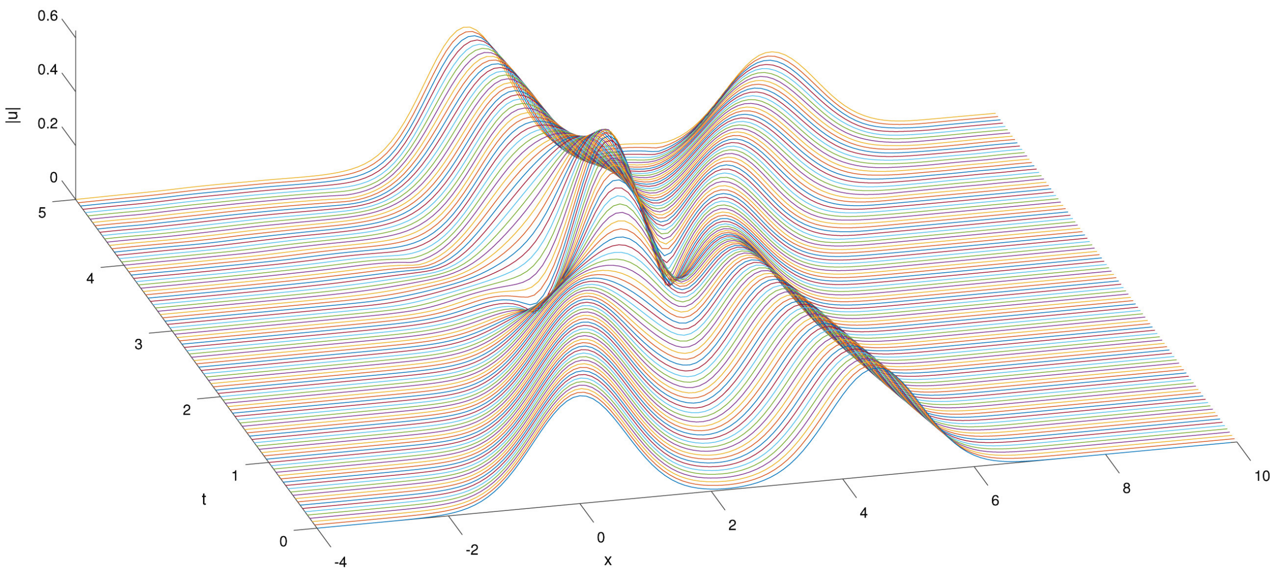

Figure 4. Simulation of the interaction of two solitons by using the Crank-Nicolson scheme for the same parameters in the explicit scheme.

Table 5. Preserved quantities for the collision of two solitons by using the explicit scheme.

Time	I_1	I_2	I_3
0.00	0.401069	−0.159232	1.613564
1.00	0.401454	−0.159432	1.614715
2.00	0.401843	−0.158895	1.615874
3.00	0.402258	−0.158897	1.614267
4.00	0.402680	−0.160935	1.619449
5.00	0.403103	−0.161372	1.619203

Table 6. Preserved quantities for the collision of two solitons by using the Crank-Nicolson scheme.

Iteration	Time	I_1	I_2	I_3
0	0.00	0.401069	−0.159233	1.613564
2	1.00	0.401069	−0.159125	1.613450
2	2.00	0.401069	−0.158250	1.613315
2	3.00	0.401068	−0.157637	1.610117
2	4.00	0.401068	−0.158945	1.613603
2	5.00	0.401068	−0.158667	1.611674

5. Conclusion

In this paper, we numerically investigated the log NLS equation using both the explicit and implicit scheme of finite difference method. In comparison to the

Explicit approach, we found that the Crank-Nicolson method yields findings that are almost as accurate as possible while still conserving the conserved values. The explicit scheme's precision is of the second order in space and the first order in time, and it is unstable. The Crank-Nicolson scheme is a second-order spatial and temporal accuracy scheme that is also stable under all conditions. Two solitons are seen interacting with each other. We have observed that the collision between the two solitons is flexible.

Conflicts of Interest

The authors state that they have no conflicts of interest to disclose in relation to the current study.

Data Availability

The article contains the supporting data for the study's findings.

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