

# Casimir Effect via a Generalized Matsubara Formalism

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**How to cite this paper:** Malbouisson, A.P.C. (2022) Casimir Effect via a Generalized Matsubara Formalism. *Journal of Applied Mathematics and Physics*, 10, 3601-3615.

<https://doi.org/10.4236/jamp.2022.1012239>

**Received:** October 17, 2022

**Accepted:** December 16, 2022

**Published:** December 19, 2022

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## Abstract

We investigate the Casimir effect in the context of a nontrivial topology by means of a generalized Matsubara formalism. This is performed in the context of a scalar field in  $D$  Euclidean spatial dimensions with  $d$  compactified dimensions. The procedure gives us the advantage of considering simultaneously spatial constraints and thermal effects. In this sense, the Casimir pressure in a heated system between two infinite planes is obtained and the results are compared with those found in the literature.

## Keywords

Cavities, Simple Systems, Time Evolution

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## 1. Introduction

The Casimir effect is a quantum phenomenon originally described as the attraction of two conducting, neutral, macroscopic objects in vacuum, induced by changes in the zero-point energy of the electromagnetic field [1]. This is not an exclusive feature of electromagnetic fields. It has been shown that any relativistic field under the influence of external conditions is able to exhibit an analogous kind of phenomenon [2]. This quantum vacuum effect is strongly dependent on the material properties of the medium where the macroscopic objects interact, on the nature of the quantum field, and on the boundary conditions under investigation. It has been related to many different physical systems ranging from cosmology, condensed matter, atomic and molecular physics to more recent developments in micro and nanoelectromechanical devices as discussed in the reviews found in Refs. [3]-[11]. It is a well-known fact that thermal fluctuations also produce Casimir forces. The pioneering works were devoted to explaining its thermodynamical behavior [12] [13]. General theoretical works [14]-[21], and controversial results in

realistic situations [22]-[29] were also explored. The first observation of the Casimir force was made by Sparnaay in 1956 [30].

On the other hand, the analysis of quantum field theory on toroidal spaces has been the focus of a large number of investigations in theoretical foundations and applications of the formalism: spontaneous symmetry breaking drive by both temperature and spatial boundaries as in recent work [31] [32] [33], second-order phase transitions in superconducting films, wires and grains red [34] [35] [36], finite-size effects in the presence of magnetic fields, finite chemical potential in first-order phase transitions [37], and also the Casimir effect [38]-[43]. It is well-known that one way to obtain thermal effects in quantum field theories is to consider the Matsubara formalism, in which a fourth dimension (mathematically analogous to imaginary time) has a finite extension equal to the inverse of temperature  $\beta$ , with a periodic boundary condition. The application of this procedure also to spatial dimensions has been introduced by Birrell and Ford [44] in order to describe field theories in spaces with finite geometries and has been generalized to what came to be known as quantum field theories on toroidal topologies ([31] [32] [33] [45] [46] [47] [48] [49]). This procedure can also be called a *generalized Matsubara formalism*. In general, this technique basically consists in considering quantum fields defined over spaces with topologies of the type  $(\mathbb{S}^1)^d \times \mathbb{R}^{D-d}$ , with  $1 \leq d \leq D$ , where  $D$  represents the total number of Euclidean dimensions and  $d$  the number of compactified ones through the imposition of periodic boundary conditions on the fields along them. One of these dimensions is compactified in a circumference of length  $\beta$ , whereas each of the spatial ones ( $i = 1, \dots, d-1$ ) in a circumference of length  $L_i$  and can be interpreted as boundaries of the Euclidean space [46] [48]. In short, this corresponds to imposing periodic (antiperiodic) boundary conditions for fields in  $D$  Euclidean dimensions with  $d$  compactified ones.

In the present paper, we revisit the Casimir effect, as an application of the above-mentioned generalized Matsubara formalism. We investigate the pressure experienced by the boundary in a compactified space when a scalar field is heated. The starting point is the so-called “local formulation”, introduced in [14], in which the pressure is associated with the 33 component of the energy-momentum tensor. Then, we follow the zeta-function regularization method originally employed by Elizalde and Romeo [47] for the computation of the Casimir energy. However, here we employ a general formalism of field theories on toroidal spaces as in Ref. [48], which allows applying the method for several simultaneously compactified dimensions. This is the case, for instance, of thermal field theories with a finite spatial extension, which needs the compactification of both the imaginary-time dimension and a spatial one for a unified approach for heated Casimir cavities.

We stress that in our computation with the toroidal formalism periodic boundary conditions are implemented both in imaginary time (circumference of length  $\beta$ ) and the third spatial coordinate (circumference of length  $L$ ), by construction. Moreover, as stated in [48], results for other boundary conditions may be

obtained from the periodic ones. For instance, the pressure for Dirichlet boundary conditions (much studied in the literature) can be determined by putting  $L = 2a$  in the expression from the toroidal computation, where  $a$  is the distance separating the parallel plates in Ref. [14].

The paper is organized as follows. In section 2 the Casimir pressure is linked to the vacuum expectation value of the energy-momentum tensor for a scalar field in  $D$  dimensions of the Euclidean space. The point-splitting technique is used to write it in terms of the free scalar propagator in Fourier space. In section 3 a corresponding expression for the pressure is obtained when one of the spatial dimensions is compactified with a finite extension. The computation of the Casimir pressure follows a path similar to that of the Elizalde-Romeo method [47], leading to a well-known result from the literature. In section 4, we compute the Casimir pressure in the configuration of a compact spatial dimension now in the presence of a thermal bath, which can also be compared with results found in the literature obtained from other techniques. In section 5 we present our final comments. Throughout this paper, we consider  $\hbar = c = k_B = 1$ .

## 2. Energy-Momentum Tensor for Scalar Fields

We start by writing the Euclidean Lagrangian of the free scalar field in a  $D$ -dimensional space,

$$\mathcal{L}_E = \frac{1}{2}(\partial_\mu \phi)^2 + \frac{1}{2}m^2 \phi^2, \quad (1)$$

where  $m$  is the mass of the quanta of the scalar field  $\phi$ . With the help of the point-splitting technique, the vacuum expectation value of the canonical energy-momentum tensor  $T_{\mu\nu}$  can be written as [48],

$$\mathcal{T}_{\mu\nu} = \langle 0 | T_{\mu\nu} | 0 \rangle = \lim_{x' \rightarrow x} \mathcal{O}_{\mu\nu}(x, x') \langle 0 | T \phi(x) \phi(x') | 0 \rangle, \quad (2)$$

where  $T$  denotes the time-ordered product of field operators and  $\mathcal{O}_{\mu\nu}(x, x')$  is a differential operator given by [48]

$$\mathcal{O}_{\mu\nu}(x, x') = \partial_\mu \partial'_\nu - \frac{1}{2} \delta_{\mu\nu} [\partial_\sigma \partial'_\sigma + m^2], \quad (3)$$

where  $\partial_\mu$  and  $\partial'_\mu$  are derivatives acting on  $x^\mu$  and  $x'^\mu$ , respectively, and  $\delta_{\mu\nu}$  represents the components of the metric tensor of the Euclidean space (Kronecker delta). Defining the Euclidean Green function of the scalar field as  $G(x-x') = i \langle 0 | T \{ \phi(x) \phi(x') \} | 0 \rangle$ , we obtain

$$\mathcal{T}_{\mu\nu} = \lim_{x' \rightarrow x} \mathcal{O}_{\mu\nu}(x, x') [G(x-x')]. \quad (4)$$

Considering the Fourier integral of the Euclidean Green function in momentum space,

$$G(x-x') = \int_{-\infty}^{\infty} \frac{d^D k}{(2\pi)^D} \frac{1}{k^2 + m^2} e^{ik(x-x')}, \quad (5)$$

where  $k$  and  $x$  are  $D$ -dimensional vectors, we are able to rewrite the vacuum expectation value of the energy-momentum tensor in Equation (4) as

$$T_{\mu\nu} = \int_{-\infty}^{\infty} \frac{d^D k}{(2\pi)^D} \left[ \frac{k_\mu k_\nu}{k^2 + m^2} - \frac{1}{2} \delta_{\mu\nu} \right]. \tag{6}$$

### 3. Casimir Pressure in a Compactified Space

In this section, we investigate the Casimir pressure for the particular case of just one compactified spatial dimension ( $d = 1$ ), along the lines of Ref. [47]. It is sufficient to consider the 33 component of the energy-momentum tensor to obtain the Casimir pressure resulting from a topological constraint imposed by periodic boundary conditions on the field at the parallel plates (taken as infinite planes) separated by a fixed distance  $L$  in the  $x_3$ -direction.

From Equation (6), it is straightforward to write the bulk expression

$$T_{33} = \frac{1}{2} \int_{-\infty}^{\infty} \frac{d^D k}{(2\pi)^D} \left[ \frac{k_3^2 - (k_\perp^2 + m^2)}{k_3^2 + k_\perp^2 + m^2} \right], \tag{7}$$

where  $k^2 = k_3^2 + k_\perp^2$ , and  $k_\perp$  refers to the  $(D - 1)$ -dimensional vector orthogonal to the 3-direction in Fourier space.

Let us call  $T_{33}^c$  the response of vacuum fluctuations in the object that plays the role of a topological constraint. We perform this by means of the compactification of just one spatial dimension. In order to obtain the Casimir pressure that acts on the boundary of the compactified space, we shall use the generalized Matsubara procedure, which is the original contribution of the present manuscript. Basically, in the general case, the technique consists in the replacement of integrals in momentum space by sums, namely,

$$\int \frac{dk_j}{2\pi} \rightarrow \frac{1}{L_j} \sum_{n_j=-\infty}^{+\infty}$$

where the index  $j$  assumes the values  $j = 1, 2, \dots, D - 1$ , the momentum coordinate  $k_j$  exhibits discrete values,

$$k_j = k_{n_j} = \frac{2\pi n_j}{L_j},$$

and  $L_j$  refer to the finite extension of each of the  $j$  spatial dimensions (compactification of  $D - 1$  spatial coordinates). For practical purposes, let us compactify just the  $x_3$ -component of the vector  $x$ . With these ideas in mind, the generalized Matsubara formalism enables us to substitute the bulk expression of Equation (7) by the following one:

$$T_{33}^c = \frac{1}{2L} \sum_{n=-\infty}^{+\infty} \int_{-\infty}^{\infty} \frac{d^{D-1} k_\perp}{(2\pi)^{D-1}} \left[ \frac{k_n^2 - (k_\perp^2 + m^2)}{k_n^2 + k_\perp^2 + m^2} \right]. \tag{8}$$

Using the well-known results provided by dimensional regularization,

$$\int_{-\infty}^{\infty} \frac{d^D k}{(2\pi)^D} \frac{1}{[k^2 + b^2]^s} = \frac{1}{(4\pi)^{\frac{D}{2}}} \frac{\Gamma\left(s - \frac{D}{2}\right)}{\Gamma(s)} \left(\frac{1}{b^2}\right)^{s - \frac{D}{2}}, \tag{9}$$

$$\int_{-\infty}^{\infty} \frac{d^D k}{(2\pi)^D} \frac{k^2}{[k^2 + b^2]^s} = \frac{D}{2} \frac{1}{(4\pi)^{\frac{D}{2}}} \frac{\Gamma\left(s - \frac{D}{2} - 1\right)}{\Gamma(s)} \left(\frac{1}{b^2}\right)^{s - \frac{D}{2} - 1}, \quad (10)$$

we obtain

$$\mathcal{T}_{33}^c = \left\{ f_s(\nu, L) \sum_{n=-\infty}^{+\infty} \left[ \frac{(an^2 - c^2)\Gamma(\nu)}{(an^2 + c^2)^\nu} - (s - \nu)\Gamma(\nu - 1) \frac{1}{(an^2 + c^2)^{\nu-1}} \right] \right\}_{s=1}, \quad (11)$$

where  $a = L^{-2}$ ,  $c = m/2\pi$ ,  $\nu = s - (D - 1)/2$ , and  $f_s(\nu, L)$  a function given by

$$f_s(\nu, L) = \frac{1}{2L} \frac{1}{(4\pi)^{s-\nu} (2\pi)^{2(\nu-1)} \Gamma(s)}. \quad (12)$$

Adding and subtracting the term  $c^2\Gamma(\nu)$  to the numerator of the first term on the right-hand side of Equation (11), we obtain

$$\mathcal{T}_{33}^c = \left\{ f_s(\nu, L) \left[ (2\nu - s - 1) \sum_{n=-\infty}^{+\infty} \frac{1}{(an^2 + c^2)^{\nu-1}} - 2c^2(\nu - 1) \sum_{n=-\infty}^{+\infty} \frac{1}{(an^2 + c^2)^\nu} \right] \right\}_{s=1}, \quad (13)$$

where we have used that  $\Gamma(\nu) = (\nu - 1)\Gamma(\nu - 1)$ . Recalling the general definition of the multidimensional Epstein-Hurwitz zeta function [47] [50] [51] [52],

$$Z_d^{c^2}(\nu; a_1, \dots, a_d) = \sum_{n_1, \dots, n_d = -\infty}^{+\infty} (a_1 n_1^2 + \dots + a_d n_d^2 + c^2)^{-\nu}, \quad (14)$$

in the particular case of one-dimensional compactification ( $d = 1$ ), it simplifies to

$$Z_1^{c^2}(\nu; a) = \sum_{n=-\infty}^{+\infty} (an^2 + c^2)^{-\nu}. \quad (15)$$

Substituting the previous expression into Equation (13), the pressure can then be rewritten as

$$\mathcal{T}_{33}^c = \left\{ f_s(\nu, L) \left[ (2\nu - s - 1) Z_1^{c^2}(\nu - 1; a) - 2c^2(\nu - 1) Z_1^{c^2}(\nu; a) \right] \right\}_{s=1}. \quad (16)$$

Following Ref. [33], these zeta functions can be evaluated on the whole complex plane by means of an analytic continuation described in the following manner [47] [50] [51] [52]:

$$\begin{aligned} & Z_d^{c^2}(\nu; a_1, \dots, a_d) \\ &= \frac{2\pi^{\frac{d}{2}}}{\sqrt{a_1 \dots a_d} \Gamma(\nu)} \left[ \frac{1}{2c^{2\nu-d}} \Gamma\left(\nu - \frac{d}{2}\right) \right. \\ & \quad \left. + 2 \sum_{j=1}^d \sum_{n_j=1}^{\infty} \left( \frac{\pi n_j}{c\sqrt{a_j}} \right)^{\nu - \frac{d}{2}} K_{\nu - \frac{d}{2}} \left( 2\pi c \frac{n_j}{\sqrt{a_j}} \right) \right] + \dots \\ & \quad + 2^d \sum_{n_1, \dots, n_d=1}^{\infty} \left( \frac{\pi}{c} \sqrt{\frac{n_1^2}{a_1} + \dots + \frac{n_d^2}{a_d}} \right)^{\nu - \frac{d}{2}} K_{\nu - \frac{d}{2}} \left( 2\pi c \sqrt{\frac{n_1^2}{a_1} + \dots + \frac{n_d^2}{a_d}} \right) \end{aligned} \quad (17)$$

where  $K_\nu(z)$  denotes modified Bessel functions of the second kind. For  $d = 1$ , the analytical continuation is reduced to

$$Z_1^{c^2}(\nu; a) = \frac{2\pi^{\frac{1}{2}}}{\sqrt{a}\Gamma(\nu)} \left[ \frac{1}{2c^{2\nu-1}} \Gamma\left(\nu - \frac{1}{2}\right) + 2 \sum_{n=1}^{\infty} \left(\frac{\pi n}{c\sqrt{a}}\right)^{\nu-\frac{1}{2}} K_{\nu-\frac{1}{2}}\left(2\pi c \frac{n}{\sqrt{a}}\right) \right]. \quad (18)$$

After some algebraic manipulations, we notice the presence of terms which are independent of the variable  $L$ , and for this reason are considered unphysical. Neglecting these terms, we can show that

$$\mathcal{T}_{33}^c = 2 \left(\frac{m}{2\pi L}\right)^{\frac{D}{2}} \left[ (1-D) \sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^{\frac{D}{2}} K_{\frac{D}{2}}(mnL) - mL \sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^{\frac{D}{2}-1} K_{\frac{D}{2}-1}(mnL) \right]. \quad (19)$$

The formula above corresponds to a general expression for the Casimir pressure exerted by the vacuum fluctuations that induce a topological effect due to the presence of the compactified manifold of length  $L$ . The result presented in Equation (19) is the Casimir vacuum radiation pressure for a massive scalar field submitted to periodic boundary conditions in  $D$  dimensions and is in agreement with Refs. [7] [53] [54].

For a 4-dimensional Euclidean space, we obtain [54]

$$\mathcal{T}_{33}^c(L, m) = -\frac{m^2}{2\pi^2 L^2} \left[ 3 \sum_{n=1}^{\infty} \frac{1}{n^2} K_2(mnL) + mL \sum_{n=1}^{\infty} \frac{1}{n} K_1(mnL) \right]. \quad (20)$$

From the following asymptotic formula of the Bessel function,

$$K_\nu(z) \approx 2^{\nu-1} z^{-\nu} \Gamma(\nu), \quad (21)$$

evaluated for small values of its argument ( $z \sim 0$ ) and  $\text{Re}(\nu) > 0$ , we obtain the small-mass limit Casimir pressure ( $mL \ll 1$ )

$$\mathcal{T}_{33}^c(L, 0) = -\frac{\pi^2}{30L^4}, \quad (22)$$

where we have neglected terms of  $\mathcal{O}(m^2)$ . The vacuum fluctuation Casimir force per unit area is a finite negative expression which suggests that the radiation pressure contracts the compactified space of circumference  $L$ .

A no-less important comment we present to finalize this section is that the corresponding negative Casimir pressure between two infinitely parallel planes, when one imposes to the massless scalar field Dirichlet boundary conditions, that is,  $\phi(x_3 = 0) = \phi(x_3 = L) = 0$ , is immediately recovered when the plane separation distance  $a$  is equal to the half circumference length  $L$  of the space dimension under compactification.

#### 4. Thermal Effects

In this section, thermal and boundary effects are treated simultaneously through the generalized Matsubara prescription. We then consider a  $D$ -dimensional space with a double compactification ( $d = 2$ ) of the Euclidean space corresponding to a compactified spatial dimension with length  $L$  and a compactification of the

imaginary-time dimension with length  $\beta$ . In other words, we take the simultaneous compactification of both the  $x_0$  and  $x_3$  coordinates of the vector  $x$ . Following the same steps as in the previous sections, the stress tensor component  $\mathcal{T}_{33}^c$  given by Equation (7) now becomes

$$\mathcal{T}_{33}^c = \frac{1}{2\beta L} \sum_{n_1, n_2 = -\infty}^{+\infty} \int_{-\infty}^{\infty} \frac{d^{D-2}k_{\perp}}{(2\pi)^{D-2}} \left[ \frac{k_{n_1}^2 - k_{n_2}^2 - (k_{\perp}^2 + m^2)}{k_{n_1}^2 + k_{n_2}^2 + k_{\perp}^2 + m^2} \right]. \tag{23}$$

Using dimensional regularization, Equations (9) and (10), the previous formula is rewritten as

$$\mathcal{T}_{33}^c = \left\{ f_s(\nu, \beta, L) \left[ \sum_{n_1, n_2 = -\infty}^{+\infty} \frac{[a_1 n_1^2 - a_2 n_2^2 - c^2] \Gamma(\nu)}{[a_1 n_1^2 + a_2 n_2^2 + c^2]^\nu} - \sum_{n_1, n_2 = -\infty}^{+\infty} \frac{(s-\nu) \Gamma(\nu-1)}{[a_1 n_1^2 + a_2 n_2^2 + c^2]^{\nu-1}} \right] \right\}_{s=1}, \tag{24}$$

where  $a_1 = L^{-2}$ ,  $a_2 = \beta^{-2}$ ,  $c = m/2\pi$ ,  $\nu = s - (D-2)/2$ , and  $f_s(\nu, \beta, L)$  is a function given by

$$f_s(\nu, \beta, L) = \frac{1}{2\beta L} \frac{1}{(4\pi)^{s-\nu} (2\pi)^{2(\nu-1)} \Gamma(s)}. \tag{25}$$

Adding and subtracting the term  $(a_2 n_2^2 + c^2) \Gamma(\nu)$  in the numerator of the first term on the right-hand side of Equation (24), we obtain

$$\mathcal{T}_{33}^c = \left\{ f_s(\nu, \beta, L) \Gamma(\nu-1) \left[ (2\nu-s-1) Z_2^{c^2}(\nu-1; a_1, a_2) - 2c^2(\nu-1) Z_2^{c^2}(\nu; a_1, a_2) + 2a_2 \frac{\partial}{\partial a_2} Z_2^{c^2}(\nu-1; a_1, a_2) \right] \right\}_{s=1}, \tag{26}$$

where we have used the definition of the two-dimensional Epstein-Hurwitz zeta function,  $Z_2^{c^2}(\nu; a_1, a_2)$ , obtained from Equation (17) for  $d = 2$ . From Equation (17), we get for  $d = 2$

$$\begin{aligned} & Z_2^{c^2}(\nu; a_1, a_2) \\ &= \frac{2\pi}{\sqrt{a_1 a_2} \Gamma(\nu)} \left[ \frac{1}{2c^{2(\nu-1)}} \Gamma(\nu-1) + 2 \sum_{n_1=1}^{\infty} \left( \frac{\pi n_1}{c\sqrt{a_1}} \right)^{\nu-1} K_{\nu-1} \left( 2\pi c \frac{n_1}{\sqrt{a_1}} \right) \right. \\ & \quad \left. + 2 \sum_{n_2=1}^{\infty} \left( \frac{\pi n_2}{c\sqrt{a_2}} \right)^{\nu-1} K_{\nu-1} \left( 2\pi c \frac{n_2}{\sqrt{a_2}} \right) \right. \\ & \quad \left. + 2^2 \sum_{n_1, n_2=1}^{\infty} \left( \frac{\pi}{c} \sqrt{\frac{n_1^2}{a_1} + \frac{n_2^2}{a_2}} \right)^{\nu-1} K_{\nu-1} \left( 2\pi c \sqrt{\frac{n_1^2}{a_1} + \frac{n_2^2}{a_2}} \right) \right]. \end{aligned} \tag{27}$$

Substituting Equation (27) in Equation (26), splitting  $\mathcal{T}_{33}^c$  into three terms,  $\mathcal{T}_{33}^c = \mathcal{T}_{n_1}^c + \mathcal{T}_{n_2}^c + \mathcal{T}_{n_1 n_2}^c$ , after removing removing nonphysical terms, we have

$$\begin{aligned} \mathcal{T}_{n_1}^c = & \frac{4\pi}{\sqrt{a_1 a_2}} f_s(\nu, \beta, L) (2\nu - s - 2) \sum_{n_1=1}^{\infty} \left( \frac{\pi n_1}{c\sqrt{a_1}} \right)^{\nu-2} K_{\nu-2} \left( 2\pi c \frac{n_1}{\sqrt{a_1}} \right) \\ & - 2c^2 \sum_{n_1=1}^{\infty} \left( \frac{\pi n_1}{c\sqrt{a_1}} \right)^{\nu-1} K_{\nu-1} \left( 2\pi c \frac{n_1}{\sqrt{a_1}} \right) \Bigg|_{s=1}, \end{aligned} \tag{28}$$

which corresponds to the contribution to the Casimir pressure due to vacuum fluctuations only. Using the definition (25), for  $a_1 = L^2$ ,  $a_2 = \beta^{-2}$ ,  $c = m/2\pi$ ,  $\nu = s - (D - 2)/2$ , Equation (19) shown in the previous section is recovered.

Also,

$$\begin{aligned} \mathcal{T}_{n_2}^c = & \frac{4\pi}{\sqrt{a_1 a_2}} f_s(\nu, \beta, L) \left[ (2\nu - s - 2) \sum_{n_2=1}^{\infty} \left( \frac{\pi n_2}{c\sqrt{a_2}} \right)^{\nu-2} K_{\nu-2} \left( 2\pi c \frac{n_2}{\sqrt{a_2}} \right) \right. \\ & - 2c^2 \sum_{n_2=1}^{\infty} \left( \frac{\pi n_2}{c\sqrt{a_2}} \right)^{\nu-1} K_{\nu-1} \left( 2\pi c \frac{n_2}{\sqrt{a_2}} \right) \\ & \left. + 2a_2 \frac{\partial}{\partial a_2} \sum_{n_2=1}^{\infty} \left( \frac{\pi n_2}{c\sqrt{a_2}} \right)^{\nu-2} K_{\nu-2} \left( 2\pi c \frac{n_2}{\sqrt{a_2}} \right) \right] \Bigg|_{s=1}, \end{aligned} \tag{29}$$

yields

$$\mathcal{T}_{n_2}^c(\beta, m) = 2 \left( \frac{m}{2\pi\beta} \right)^{\frac{D}{2}} \sum_{n_2=1}^{\infty} \left( \frac{1}{n_2} \right)^{\frac{D}{2}} K_{\frac{D}{2}}(m\beta n_2), \tag{30}$$

which is the Casimir force formula due exclusively to the thermal fluctuations. The final form of Equation (30) was obtained by means of the useful recurrence formula for Bessel functions,

$$K_{\alpha-1}(z) - K_{\alpha+1}(z) = -\frac{2\alpha}{z} K_{\alpha}(z). \tag{31}$$

For  $D = 4$ , we find

$$\mathcal{T}_{n_2}^c(\beta, m) = \left( \frac{m^2}{2\pi^2\beta^2} \right) \sum_{n_2=1}^{\infty} \left( \frac{1}{n_2} \right)^2 K_2(m\beta n_2). \tag{32}$$

Using Equation (21), we obtain the small-mass limit purely thermal Casimir pressure ( $m\beta \ll 1$ )

$$\mathcal{T}_{n_2}^c(\beta, 0) = \frac{\pi^2}{90\beta^4}, \tag{33}$$

which is in accordance with the well-known Stefan-Boltzmann thermal radiation pressure result. This is a finite positive force per unit area which is more intense than vacuum radiation Casimir pressure for low values of  $\beta$  (high-temperature or classical limit).

If we plot the ratio between the thermal radiation pressure for the massive scalar field (Equation (32)) and the massless one (Equation (33)), as a function of the dimensionless parameter  $m\beta$ , the normalized thermal Casimir force per



unit area  $\mathcal{T}_{n_2}^c(\beta, m)/\mathcal{T}_{n_2}^c(\beta, 0)$  presents the typical monotonically decreasing shape for increasing values of the parameter  $m\beta$ .

Finally, the formula

$$\begin{aligned} \mathcal{T}_{n_1 n_2}^c &= \frac{8\pi}{\sqrt{a_1 a_2}} f_s(\nu, \beta, L) \left\{ (2\nu - s - 2) \sum_{n_1, n_2=1}^{\infty} \left( \frac{\pi}{c} \sqrt{\frac{n_1^2}{a_1} + \frac{n_2^2}{a_2}} \right)^{\nu-2} \right. \\ &\times K_{\nu-2} \left( 2\pi c \sqrt{\frac{n_1^2}{a_1} + \frac{n_2^2}{a_2}} \right) - 2c^2 \sum_{n_1, n_2=1}^{\infty} \left( \frac{\pi}{c} \sqrt{\frac{n_1^2}{a_1} + \frac{n_2^2}{a_2}} \right)^{\nu-1} \\ &\times K_{\nu-1} \left( 2\pi c \sqrt{\frac{n_1^2}{a_1} + \frac{n_2^2}{a_2}} \right) + 2a_2 \frac{\partial}{\partial a_2} \sum_{n_1, n_2=1}^{\infty} \left[ \left( \frac{\pi}{c} \sqrt{\frac{n_1^2}{a_1} + \frac{n_2^2}{a_2}} \right)^{\nu-2} \right. \\ &\left. \left. \times K_{\nu-2} \left( 2\pi c \sqrt{\frac{n_1^2}{a_1} + \frac{n_2^2}{a_2}} \right) \right] \right\}_{s=1}, \end{aligned} \tag{34}$$

or

$$\begin{aligned} \mathcal{T}_{n_1 n_2}^c(L, \beta, m) &= 4 \left( \frac{m}{2\pi} \right)^{\frac{D}{2}} \left[ \sum_{n_1, n_2=1}^{\infty} \left( \frac{1}{\sqrt{n_1^2 L^2 + n_2^2 \beta^2}} \right)^{\frac{D}{2}} \right. \\ &\times \left( \frac{(1-D)n_1^2 L^2 + n_2^2 \beta^2}{n_1^2 L^2 + n_2^2 \beta^2} \right) K_{\frac{D}{2}} \left( m \sqrt{n_1^2 L^2 + n_2^2 \beta^2} \right) \\ &\left. - m \sum_{n_1, n_2=1}^{\infty} n_1^2 L^2 \left( \frac{1}{\sqrt{n_1^2 L^2 + n_2^2 \beta^2}} \right)^{\frac{D}{2}+1} K_{\frac{D}{2}-1} \left( m \sqrt{n_1^2 L^2 + n_2^2 \beta^2} \right) \right], \end{aligned} \tag{35}$$

gives the corrections to the Casimir pressure in a compact space in the presence of a massive scalar field heated at temperature  $1/\beta$ . In order to obtain the final form of the above expression, we have used the recurrence formula given by Equation (31). Considering  $D = 4$ , we get

$$\begin{aligned} \mathcal{T}_{n_1 n_2}^c(L, \beta, m) &= - \left( \frac{m}{\pi} \right)^2 \left[ \sum_{n_1, n_2=1}^{\infty} \frac{3n_1^2 L^2 - n_2^2 \beta^2}{(n_1^2 L^2 + n_2^2 \beta^2)^2} \times K_2 \left( m \sqrt{n_1^2 L^2 + n_2^2 \beta^2} \right) \right. \\ &\left. + m \sum_{n_1, n_2=1}^{\infty} \frac{n_1^2 L^2}{(n_1^2 L^2 + n_2^2 \beta^2)^{\frac{3}{2}}} K_1 \left( m \sqrt{n_1^2 L^2 + n_2^2 \beta^2} \right) \right], \end{aligned} \tag{36}$$

which is valid for arbitrary values of  $m$ ,  $L$  and  $\beta$ . Using Equation (21), we can show that in the small-mass case it reduces to

$$\mathcal{T}_{n_1 n_2}^c(L, \beta, 0) = - \frac{2}{\pi^2} \sum_{n_1, n_2=1}^{\infty} \frac{3n_1^2 L^2 - n_2^2 \beta^2}{(n_1^2 L^2 + n_2^2 \beta^2)^3}, \tag{37}$$

where we have disregarded terms of  $\mathcal{O}(m^2)$ . The corresponding expression for Dirichlet boundary conditions can be obtained by substituting  $L = 2a$ .

To clarify our results, we can show that the small-mass limit given by Equation (37) can be written as

$$T_{n_1 n_2}^c(L, \beta, 0) = \frac{1}{L^4} [3f(\xi) + \xi s(\xi)], \tag{38}$$

where  $\xi = L/\beta$  and

$$f(\xi) = -\frac{1}{8\pi^2} \sum_{n_1, n_2=1}^{\infty} \frac{(2\xi)^4}{[(\xi n_1)^2 + (n_2)^2]^2}, \tag{39}$$

$$s(\xi) = -f'(\xi) = \frac{1}{\pi^2} \sum_{n_1, n_2=1}^{\infty} \frac{(2\xi)^3 n_2^2}{[(\xi n_1)^2 + (n_2)^2]^3}. \tag{40}$$

The function  $f(\xi)$  obeys the inversion symmetry formula,

$$f(\xi) = \xi^4 f\left(\frac{1}{\xi}\right). \tag{41}$$

This is an intriguing expression, known as temperature inversion symmetry, that enables us to obtain the low and high-temperature limits after simple algebraic manipulations, (see Refs. [14] [18] [55]-[61] for more details). Following [14], the particular low-temperature limit ( $\beta \gg 1$ ) can be more easily performed after we compute the sum over index  $n_1$  in Equation (39),

$$f(\xi) = \frac{\xi^4}{\pi^2} \sum_{n_2=1}^{\infty} \frac{1}{n_2^4} - \frac{\xi^3}{2\pi} \sum_{n_2=1}^{\infty} \frac{\coth(\pi n_2/\xi)}{n_2^3} - \frac{\xi^2}{2} \sum_{n_2=1}^{\infty} \frac{1}{n_2^2} \frac{1}{\sinh^2(\pi n_2/\xi)}. \tag{42}$$

In the limit  $\xi \ll 1$ , the approximations

$$\coth(\pi n_2/\xi) \approx 1, \tag{43}$$

$$\sinh(\pi n_2/\xi) \approx \frac{1}{2} e^{\pi n_2/\xi}, \tag{44}$$

are valid. Substituting Equations (43) and (44) into Equation (42), and performing the sum over index  $n_2$ , we find, for  $\xi \ll 1$ ,

$$f(\xi) = \frac{\pi^2}{90} \xi^4 - \frac{\zeta(3)}{2\pi} \xi^3 - 2\xi^2 \left(1 + \frac{\xi}{\pi}\right) e^{-2\pi/\xi} + \mathcal{O}(e^{-4\pi/\xi}). \tag{45}$$

Inserting the above formula into Equation (38), we can show that

$$T_{n_1 n_2}^c(L, \beta, 0) = -\frac{\pi^2}{90\beta^4} + \frac{4\pi}{\beta L^3} \left(1 + \frac{L}{2\pi\beta}\right) e^{-2\pi\beta/L}. \tag{46}$$

In this sense, in the low-temperature limit ( $L \ll \beta$ ), collecting all the contributions, the final form of Casimir pressure in the massless case reads

$$T_{33}^c(L, \beta, 0) = -\frac{\pi^2}{30L^4} + \frac{4\pi}{\beta L^3} e^{-2\pi\beta/L}. \tag{47}$$

If we neglect the exponential factor, the Casimir pressure due exclusively to the vacuum fluctuations is dominant in this regime.

The high-temperature limit is also easily found by means of the inversion

symmetry relation given by Equation (41). Applying this formula in Equation (45), we get

$$f(\xi) = \frac{\pi^2}{90} - \frac{\zeta(3)}{2\pi} \xi - 2\xi^2 \left(1 + \frac{1}{\pi\xi}\right) e^{-2\pi\xi} + \mathcal{O}(e^{-4\pi\xi}). \quad (48)$$

Substituting Equation (48) into Equation (38), we find

$$\mathcal{T}_{n_1 n_2}^c(L, \beta, 0) = \frac{\pi^2}{30L^4} - \frac{\zeta(3)}{\pi\beta L^3} - \frac{1}{\beta L^3} \left( \frac{4\pi L^2}{\beta^2} + \frac{6L}{\beta} + \frac{4}{\pi} \right) e^{-2\pi L/\beta}. \quad (49)$$

Finally, in the high-temperature limit ( $L \gg \beta$ ), computing all terms, the final form of Casimir pressure is written as follows:

$$\mathcal{T}_{33}^c(L, \beta, 0) = \frac{\pi^2}{90\beta^4} - \frac{\zeta(3)}{\pi\beta L^3} - \frac{1}{\beta L^3} \left( \frac{4\pi L^2}{\beta^2} + \frac{6L}{\beta} + \frac{4}{\pi} \right) e^{-2\pi L/\beta}. \quad (50)$$

Notice that if we neglect the exponential factor, the Casimir pressure for large temperature is given by the classical thermal radiation pressure  $\pi^2/(90\beta^4)$  plus a negative linear correction factor proportional to  $\beta^{-1}$ .

## 5. Final Remarks

In the present work, we investigate some aspects of the Casimir effect in the context of nontrivial topologies. In particular, we revisited the Casimir effect for a massive scalar field in a heated compact space by means of the generalized Matsubara formalism. The usual attractive response of quantum and thermal fluctuations is obtained and our results are in accordance with those found in the literature. One may notice that all thermal contributions to the Casimir pressure, given by  $\mathcal{T}_{n_2}^c$  and  $\mathcal{T}_{n_1 n_2}^c$ , vanish in the zero-temperature ( $\beta \rightarrow \infty$ ) limit, remaining the pure dependence on the distance  $L$  between plates, which has a well-known  $L^{-4}$  dependence in the small- $L$  limit for a four-dimensional space. Also, the bulk limit  $L \rightarrow \infty$  reduces all expressions in  $D=4$  to the Stefan-Boltzmann law  $\beta^{-4}$ .

A rather peculiar aspect of the generalized Matsubara formalism is related to the renormalization of the expressions. Usually, in the Casimir context, the divergent terms are taken care of by subtraction of the bulk integral, without compactifications (see [48]). Here, there is no need to do so, as was also remarked by Elizalde and Romeo [47]. It is sufficient to obtain correct physical expressions to renormalize by subtraction the divergent term of the expansion of the Epstein-Hurwitz zeta functions  $Z_d^{c^2}$ , as it does not depend on the physical parameters  $L$  or  $\beta$ .

We also remark that the expression we obtain from the toroidal formalism, which conveys periodic boundary conditions in the compactified dimensions, leads to corresponding ones for the Dirichlet conditions, by substituting  $L = 2a$ . The  $D=4$ , small- $L$  limit of the Casimir pressure in the nonthermal case, given by Equation (19) becomes  $\mathcal{T}_{33} = -\pi^2/480a^4$  in the Dirichlet case for a quantum scalar field. For an electromagnetic field, we have then twice that value,  $\mathcal{T}_{33} = -\pi^2/240a^4$ , due to its two degrees of freedom. These are compatible with

the original Casimir results.

## Acknowledgments

This work was partially supported by the Brazilian agencies CNPq and FAPERJ.

## Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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