

# Planar Weak Form Quadrature Beam Elements Based on Absolute Nodal Coordinate Formulation: A Structural Mechanics Approach

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## Abstract

A planar nonlinear weak form quadrature beam element of arbitrary number of axial nodes is proposed on the basis of the absolute nodal coordinate formulation (ANCF). Elastic forces of the element are established through geometrically exact beam theory, resulting in good consistency with classical beam theory. Two examples with strong geometrical nonlinearity are presented to verify the effectiveness of the formulation.

## Keywords

Absolute Nodal Coordinate Formulation, Weak Form Quadrature Element, Geometrically Nonlinear Analysis

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## 1. Introduction

Various beam theories have found applications in geometrically nonlinear analysis of slender structures, among which the geometrically exact beam proposed by Reissner [1] and Simo [2] and the absolute nodal coordinate formulation (ANCF) proposed by Shabana [3] are frequently used.

Satisfactory results have been achieved in finite element analysis of beamlike structures based on geometrically exact theory [4]. However, due to the nonlinearity of finite rotation of the beam cross section, specific algorithms [5] are necessary in the dynamic analysis to ensure energy conservation. While in an ANCF beam element, position and slope vectors are chosen as nodal variables to describe the configuration of the beam. Consequently, there is no need to take the finite rotation theory into account in the expression of kinetic energy. In addition, a constant mass matrix is brought forth and conservation of energy is readily satisfied during dynamic analysis. Elements based on ANCF have been wide-

ly applied in the static and dynamic analysis of structures [6] [7].

ANCF enjoys good consistency with the nonlinear continuum mechanics theory and elastic forces are usually given through the Green strain tensor. As such, it is often termed continuum-mechanics-based approach [8]. Nonetheless, it has been noted that Poisson locking phenomenon will arise for nonzero Poisson ratio materials in continuum-mechanics-based analysis [9]. As an alternative, structural-mechanics-based approach is also commonly adopted where elastic forces are acquired using generalized strain measures in geometrically exact beam theory.

The weak form quadrature element method (QEM) [10] [11] is a numerical method setting out from performing numerical integration to the weak form description in the subdomain rather than constructing the shape function of element. The integration points are chosen as the element nodes and the predictive capability of a quadrature element is enhanced by raising the order of the numerical integration. The relevant quadrature elements based on geometrically exact beam theory [12] have been proposed which exhibited high efficiency and flexibility.

In this paper, planar structural-mechanics-based ANCF beam elements are developed. The order of numerical integration can be increased as desired, allowing geometrically nonlinear analysis of various beams with a minimum number of elements.

The rest of the paper is outlined as follows. The kinematics of the element and the strain measure in the geometrically exact beam are given in Section 2. In Section 3, the internal and external force vectors of the element are acquired via the principle of virtual work. Two numerical examples are provided in Section 4 to illustrate the effectiveness of the present formulation. At the end of this paper, a summary of the current work is presented.

## 2. ANCF and Geometrically Exact Beam Theory

### 2.1. Kinematics

With the Lagrangian description, the reference, initial and current configurations of an arbitrary planar beam are shown in **Figure 1**. The beam is straight in the reference configuration and its centroidal axis is aligned with the  $x$ -axis of the Cartesian coordinate system. The position vector of an arbitrary point in the reference beam is denoted as  $\mathbf{X}$ , while its counterparts in the initial and current configurations are denoted as  $\mathbf{r}^0$  and  $\mathbf{r}$ , respectively.

The rigid cross-sectional assumption is adopted in geometrically exact beam theory. The position of an arbitrary point in the beam can be determined by the centroidal position and direction of the cross section where the point is located. At an arbitrary point  $(x, y)$ , the position vector is given by

$$\mathbf{r}(x, y) = \mathbf{r}_0(x) + y\boldsymbol{\lambda}_y(x), \quad (1)$$

where the terms with subscript 0 are those at  $y = 0$ . The direction vectors of the cross section can be expressed by the slope vector as

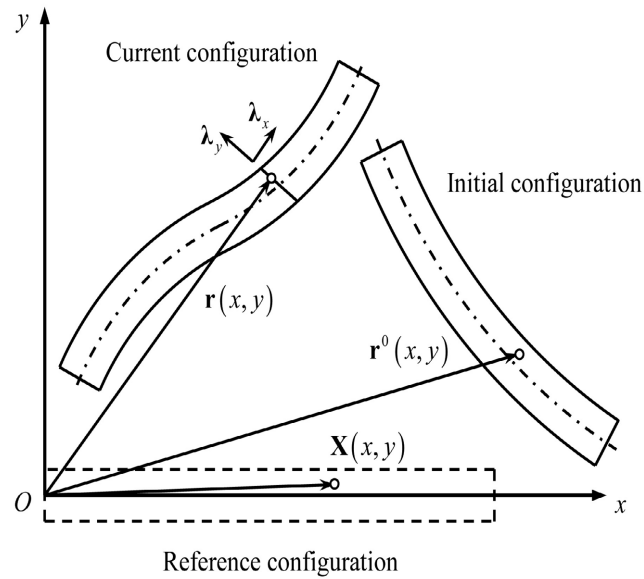


Figure 1. Beam configurations.

$$\lambda_y = \frac{\mathbf{r}_{0,y}}{\|\mathbf{r}_{0,y}\|}, \quad \lambda_x = \hat{\mathbf{E}}_3 \lambda_y, \quad \hat{\mathbf{E}}_3 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \quad (2)$$

In the weak form QEM, the integration points also serve as the element nodes, leading to the fact that only the parameters at element nodes are required in the subsequent computation. Considering the form of position description of the beam, the nodal variables can be chosen as

$$\mathbf{e} = [\mathbf{e}^{1T} \quad \dots \quad \mathbf{e}^{nT}]^T, \quad (3)$$

$$\mathbf{e}^k = [\mathbf{r}_0^T(x_k) \quad \mathbf{r}_{0,y}^T(x_k)]^T, \quad k = 1, \dots, n,$$

where  $n$  is the number of integration points in the element. The element employing such nodal variables is a gradient deficient element. Differing from most of the gradient deficient elements which are applied to the analysis of shear rigid beams, the proposed element dispenses with  $\mathbf{r}_{0,x}$  rather than  $\mathbf{r}_{0,y}$  in the nodal variables and is able to take the shear deformation into account.

## 2.2. Strains and Stresses

In geometrically exact beam theory, the deformation of a beam is characterized by axial strain  $\varepsilon$ , shear strain  $\gamma$  and curvature  $\kappa$ , which are defined as

$$\varepsilon = \lambda_x^T \mathbf{r}_{0,x} - 1, \quad \gamma = \lambda_y^T \mathbf{r}_{0,x}, \quad \kappa = \frac{\mathbf{r}_{0,y}^T \hat{\mathbf{E}}_3 \mathbf{r}_{0,yx}}{\|\mathbf{r}_{0,y}\|^2}. \quad (4)$$

If the beam has initial deformations such as bending, the initial strain should be excluded. Define the strain vector  $\boldsymbol{\varepsilon}$  as

$$\boldsymbol{\varepsilon} = [\varepsilon \quad \gamma \quad \kappa]^T. \quad (5)$$

The internal forces work-conjugated to the strains are the axial force  $N$ , shear

force  $V$  and bending moment  $M$  of the cross section. The internal force  $\mathbf{N}$  over the cross section can be obtained through generalized Hooke's law, *i.e.*

$$\mathbf{N} = \begin{bmatrix} N \\ V \\ M \end{bmatrix} = \begin{bmatrix} EA & & \\ & k_s GA & \\ & & EI \end{bmatrix} \begin{bmatrix} \varepsilon \\ \gamma \\ \kappa \end{bmatrix} = \mathbf{D}\boldsymbol{\varepsilon}. \quad (6)$$

where  $\mathbf{D}$  is the constitutive matrix determined by the parameters of material and cross section.

The variation of the strain components are

$$\begin{aligned} \delta\varepsilon &= \boldsymbol{\lambda}_x^T \delta\mathbf{r}_{0,x} + \mathbf{r}_{0,x}^T \delta\boldsymbol{\lambda}_x, \\ \delta\gamma &= \boldsymbol{\lambda}_y^T \delta\mathbf{r}_{0,x} + \mathbf{r}_{0,x}^T \delta\boldsymbol{\lambda}_y, \\ \delta\kappa &= -\frac{2\mathbf{r}_{0,y}^T \hat{\mathbf{E}}_3 \mathbf{r}_{0,yx} \mathbf{r}_{0,y}^T}{\|\mathbf{r}_{0,y}\|^4} \delta\mathbf{r}_{0,y} - \frac{\mathbf{r}_{0,yx}^T \hat{\mathbf{E}}_3}{\|\mathbf{r}_{0,y}\|^2} \delta\mathbf{r}_{0,y} + \frac{\mathbf{r}_{0,y}^T \hat{\mathbf{E}}_3}{\|\mathbf{r}_{0,y}\|^2} \delta\mathbf{r}_{0,yx}, \end{aligned} \quad (7)$$

with

$$\delta\boldsymbol{\lambda}_y = \frac{\|\mathbf{r}_{0,y}\|^2 \mathbf{I} - \mathbf{r}_{0,y} \mathbf{r}_{0,y}^T}{\|\mathbf{r}_{0,y}\|^3} \delta\mathbf{r}_{0,y}, \quad \delta\boldsymbol{\lambda}_x = \hat{\mathbf{E}}_3 \delta\boldsymbol{\lambda}_y. \quad (8)$$

Thus the variation of the strain vector  $\boldsymbol{\varepsilon}$  is

$$\delta\boldsymbol{\varepsilon} = \begin{bmatrix} -\frac{\mathbf{r}_{0,y}^T \hat{\mathbf{E}}_3}{\|\mathbf{r}_{0,y}\|} & \mathbf{0} & \mathbf{r}_{0,x}^T \hat{\mathbf{E}}_3 \frac{\|\mathbf{r}_{0,y}\|^2 \mathbf{I} - \mathbf{r}_{0,y} \mathbf{r}_{0,y}^T}{\|\mathbf{r}_{0,y}\|^3} \\ \frac{\mathbf{r}_{0,y}^T}{\|\mathbf{r}_{0,y}\|} & \mathbf{0} & \mathbf{r}_{0,x}^T \frac{\|\mathbf{r}_{0,y}\|^2 \mathbf{I} - \mathbf{r}_{0,y} \mathbf{r}_{0,y}^T}{\|\mathbf{r}_{0,y}\|^3} \\ \mathbf{0} & \frac{\mathbf{r}_{0,y}^T \hat{\mathbf{E}}_3}{\|\mathbf{r}_{0,y}\|^2} & -\frac{2\mathbf{r}_{0,y}^T \hat{\mathbf{E}}_3 \mathbf{r}_{0,yx} \mathbf{r}_{0,y}^T}{\|\mathbf{r}_{0,y}\|^4} - \frac{\mathbf{r}_{0,yx}^T \hat{\mathbf{E}}_3}{\|\mathbf{r}_{0,y}\|^2} \end{bmatrix} \begin{bmatrix} \delta\mathbf{r}_{0,x} \\ \delta\mathbf{r}_{0,yx} \\ \delta\mathbf{r}_{0,y} \end{bmatrix} = \boldsymbol{\Gamma} \delta\boldsymbol{\chi}, \quad (9)$$

where  $\delta\boldsymbol{\chi}$  is composed of the virtual changes of the slope vectors  $\mathbf{r}_{0,x}$  and  $\mathbf{r}_{0,y}$  and the derivative of  $\mathbf{r}_{0,y}$  along the axis. The approximation of derivatives will be provided in the subsequent section.

### 3. Element Formulation

Construction of the element formulation is based on the principle of virtual work symbolized as

$$\delta W_{int} - \delta W_{ext} = 0, \quad (10)$$

where  $\delta W_{int}$  and  $\delta W_{ext}$  are the internal and external virtual work, respectively.

#### 3.1. Internal Virtual Work

Define normalized coordinate along axis of the beam as  $\xi = 2x/L - 1$ , where  $L$  stands for the length of the beam. According to geometrically exact beam theory, the internal virtual work is expressed as

$$\begin{aligned}\delta W_{int}^{GE} &= \int_L \delta \boldsymbol{\varepsilon}^T \mathbf{N} dL = \int_L \delta \boldsymbol{\chi}^T \boldsymbol{\Gamma}^T \mathbf{N} dL \\ &= \int_{-1}^1 \delta \boldsymbol{\chi}^T \boldsymbol{\Gamma}^T \mathbf{N} \mu_1 d\xi,\end{aligned}\quad (11)$$

where  $\mu_1$  is the normalized lengthwise coordinate. Approximating Equation (11) with  $n$  point Lobatto quadrature gives

$$\delta W_{int}^{GE} = \sum_{k=1}^n w_k \mu_1 \delta \boldsymbol{\chi}_k^T \boldsymbol{\Gamma}_k^T \mathbf{N}_k, \quad (12)$$

where  $w_k$  is the pertinent weighting coefficients.

The differential quadrature technique [13] is used to evaluate the derivatives in Equation (12). The derivative of a function  $f$  at a certain point is approximates by a weighted linear summation of the function values at all grid points. For the first order derivative of a function there is

$$\left. \frac{df}{d\xi} \right|_{\xi=\xi_i} = \sum_{j=1}^n C_{ij}^{(1)} f(\xi_j), \quad (13)$$

where  $C_{ij}^{(1)}$ , the weighting coefficients for the first order derivatives, are given by

$$C_{ij}^{(1)} = \begin{cases} \frac{\Theta'(\xi_i)}{(\xi_i - \xi_j) \Theta'(\xi_j)}, & i \neq j; \\ -\sum_{k=1, k \neq i}^n C_{ik}^{(1)}, & i = j, \end{cases} \quad (14)$$

with

$$\Theta(\xi) = \prod_{j=1}^n (\xi - \xi_j), \quad \Theta'(\xi_i) = \prod_{j=1, j \neq i}^n (\xi_i - \xi_j). \quad (15)$$

Referring to Equation (13), the component  $\delta \boldsymbol{\chi}_k$  in Equation (12) becomes

$$\begin{aligned}\delta \boldsymbol{\chi}_k &= \mathbf{B}_k \delta \mathbf{e}, \quad \mathbf{B}_k = [\mathbf{b}_{k1} \quad \cdots \quad \mathbf{b}_{kt} \quad \cdots \quad \mathbf{b}_{kn}], \\ \mathbf{b}_{kt} &= \begin{bmatrix} \frac{C_{kt}^{(1)}}{\mu_1} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \frac{C_{kt}^{(1)}}{\mu_1} \mathbf{I} \\ \mathbf{0} & \delta_{kt} \mathbf{I} \end{bmatrix},\end{aligned}\quad (16)$$

where  $\mathbf{I}$  indicates the identity matrix. Substitution of Equation (16) into Equation (12) gives

$$\delta W_{int}^{GE} = \delta \mathbf{e}^T \mathbf{G}_{int}^{GE}, \quad \mathbf{G}_{int}^{GE} = \sum_{k=1}^n w_k \mu_1 \mathbf{B}_k^T \boldsymbol{\Gamma}_k^T \mathbf{N}_k. \quad (17)$$

Due to the rigid cross-sectional assumptions adopted in geometrically exact beam theory, the effect of cross-sectional deformation is not considered in the internal virtual work. However, there is no such restriction in ANCF and therefore the virtual work arising from thickness deformation should be introduced into the internal virtual work. The virtual work of thickness deformation is expressed in terms of the Green strain  $\boldsymbol{\varepsilon}_{yy}$  as

$$\begin{aligned}\delta W_{int}^{thickness} &= \int_L EA \varepsilon_{yy} \delta \varepsilon_{yy} dL, \\ \varepsilon_{yy} &= \frac{1}{2} (\mathbf{r}_{0,y}^T \mathbf{r}_{0,y} - 1).\end{aligned}\quad (18)$$

Analogously, the corresponding internal force vector is

$$\begin{aligned}\mathbf{G}_{int}^{thickness} &= \sum_{k=1}^n w_k \mu_1 EA \varepsilon_{yyk} \mathbf{C}_k^T \mathbf{r}_{0,yk}, \\ \mathbf{C}_k &= [\mathbf{c}_{k1} \quad \cdots \quad \mathbf{c}_{kt} \quad \cdots \quad \mathbf{c}_{kn}], \quad \mathbf{c}_{kt} = [\mathbf{0} \quad \delta_{kt} \mathbf{I}].\end{aligned}\quad (19)$$

The total internal force vector of the element is

$$\mathbf{G}_{int} = \mathbf{G}_{int}^{GE} + \mathbf{G}_{int}^{thickness}.\quad (20)$$

The tangential stiffness matrix is then given by

$$\begin{aligned}\mathbf{K}_{int} &= \frac{\partial \mathbf{G}_{int}}{\partial \mathbf{e}} = \sum_{k=1}^n w_k \mu_1 \mathbf{B}_k^T (\mathbf{\Gamma}_k^T \mathbf{D} \mathbf{\Gamma}_k + \mathbf{\Xi}_k) \mathbf{B}_k \\ &\quad + \sum_{k=1}^n w_k \mu_1 EA \mathbf{C}_k^T (\varepsilon_{yyk} \mathbf{I} + \mathbf{r}_{0,yk} \mathbf{r}_{0,yk}^T) \mathbf{C}_k, \\ \mathbf{\Xi}_k &= \begin{bmatrix} \mathbf{0} & \mathbf{0} & \hat{\mathbf{\Xi}}_k \\ \mathbf{0} & \mathbf{0} & \check{\mathbf{\Xi}}_k \\ \hat{\mathbf{\Xi}}_k^T & \check{\mathbf{\Xi}}_k^T & \tilde{\mathbf{\Xi}}_k \end{bmatrix}, \\ \hat{\mathbf{\Xi}}_k &= N \hat{\mathbf{E}}_3 \frac{\|\mathbf{r}_{0,y}\|^2 \mathbf{I} - \mathbf{r}_{0,y} \mathbf{r}_{0,y}^T}{\|\mathbf{r}_{0,y}\|^3} + V \frac{\|\mathbf{r}_{0,y}\|^2 \mathbf{I} - \mathbf{r}_{0,y} \mathbf{r}_{0,y}^T}{\|\mathbf{r}_{0,y}\|^3}, \\ \check{\mathbf{\Xi}}_k &= -M \hat{\mathbf{E}}_3 \frac{\|\mathbf{r}_{0,y}\|^2 \mathbf{I} - 2\mathbf{r}_{0,y} \mathbf{r}_{0,y}^T}{\|\mathbf{r}_{0,y}\|^4}, \\ \tilde{\mathbf{\Xi}}_k &= N \left( -\frac{\mathbf{r}_{0,y}^T \hat{\mathbf{E}}_3 \mathbf{r}_{0,x} \mathbf{I} + \hat{\mathbf{E}}_3 \mathbf{r}_{0,x} \mathbf{r}_{0,y}^T - \mathbf{r}_{0,y} \mathbf{r}_{0,x}^T \hat{\mathbf{E}}_3}{\|\mathbf{r}_{0,y}\|^3} + \frac{3\mathbf{r}_{0,y} \mathbf{r}_{0,y}^T \hat{\mathbf{E}}_3 \mathbf{r}_{0,x} \mathbf{r}_{0,y}^T}{\|\mathbf{r}_{0,y}\|^5} \right) \\ &\quad + V \left( -\frac{\mathbf{r}_{0,y}^T \mathbf{r}_{0,x} \mathbf{I} + \mathbf{r}_{0,x} \mathbf{r}_{0,y}^T + \mathbf{r}_{0,y} \mathbf{r}_{0,x}^T}{\|\mathbf{r}_{0,y}\|^3} + \frac{3\mathbf{r}_{0,y} \mathbf{r}_{0,y}^T \mathbf{r}_{0,x} \mathbf{r}_{0,y}^T}{\|\mathbf{r}_{0,y}\|^5} \right) \\ &\quad + M \left( 2 \frac{\mathbf{r}_{0,yx}^T \hat{\mathbf{E}}_3 \mathbf{r}_{0,y} \mathbf{I} + \mathbf{r}_{0,y} \mathbf{r}_{0,yx}^T \hat{\mathbf{E}}_3 - \hat{\mathbf{E}}_3 \mathbf{r}_{0,yx} \mathbf{r}_{0,y}^T}{\|\mathbf{r}_{0,y}\|^4} - 8 \frac{\mathbf{r}_{0,y} \mathbf{r}_{0,yx}^T \hat{\mathbf{E}}_3 \mathbf{r}_{0,y} \mathbf{r}_{0,y}^T}{\|\mathbf{r}_{0,y}\|^6} \right).\end{aligned}\quad (21)$$

### 3.2. External Virtual Work

The loads applied to the beam elements usually consist of concentrated, distributed forces and couples. Accounting for all possible loadings, the external virtual work is expressed as

$$\delta W_{ext} = \int_L (\delta \mathbf{r}_0^T \mathbf{p} + \delta \varphi m) dL + \sum_{k=1,n} (\delta \mathbf{r}_{0k}^T \mathbf{P}_k + \delta \varphi_k M_k).\quad (22)$$

Applying Lobatto quadrature to the integral term in Equation (22) results in

$$\delta W_{ext} = \sum_{k=1}^n w_k \mu_1 (\delta \mathbf{r}_k^T \mathbf{p}_k + \delta \varphi_k m_k) + \sum_{k=1,n} (\delta \mathbf{r}_{0k}^T \mathbf{P}_k + \delta \varphi_k M_k),\quad (23)$$

where

$$\delta\varphi = \lambda_y^T \delta\lambda_x = \frac{\mathbf{r}_{0,y}^T \hat{\mathbf{E}}_3}{\|\mathbf{r}_{0,y}\|^2} \delta\mathbf{r}_{0,y}. \quad (24)$$

Thus the external force vector  $\mathbf{G}_{ext}$  is given as

$$\begin{aligned} \mathbf{G}_{ext} &= \sum_{k=1}^n w_k \mu_1 \left( \mathbf{A}_k^T \mathbf{p}_k - \mathbf{C}_k^T \frac{m_k \hat{\mathbf{E}}_3 \mathbf{r}_{0,yk}}{\|\mathbf{r}_{0,yk}\|^2} \right) + \sum_{k=1,n} \left( \mathbf{A}_k^T \mathbf{P}_k - \mathbf{C}_k^T \frac{M_k \hat{\mathbf{E}}_3 \mathbf{r}_{0,yk}}{\|\mathbf{r}_{0,yk}\|^2} \right), \\ \mathbf{A}_k &= [\mathbf{a}_{k1} \quad \cdots \quad \mathbf{a}_{kt} \quad \cdots \quad \mathbf{a}_{kn}], \quad \mathbf{a}_{kt} = [\delta_{kt} \mathbf{I} \quad \mathbf{0}], \\ \mathbf{C}_k &= [\mathbf{c}_{k1} \quad \cdots \quad \mathbf{c}_{kt} \quad \cdots \quad \mathbf{c}_{kn}], \quad \mathbf{c}_{kt} = [\mathbf{0} \quad \delta_{kt} \mathbf{I}]. \end{aligned} \quad (25)$$

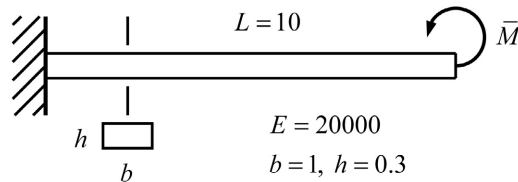
## 4. Numerical Examples

In this section, two examples are provided and comparisons are made between the present results and other solutions.

### 4.1. Pure Bending of a Uniform Cantilever Beam

A uniform cantilever beam is subject to a concentrated moment  $\bar{M}$  at its free end. The geometric and material properties of the beam are displayed in **Figure 2**. Following the analytical solution based on the classical beam theory, the beam will be deformed into a circular arc with constant curvature  $\kappa = \bar{M}/EI$ . The bending moment is assigned as  $\bar{M} = 2\pi EI/L$  and 4 loading increments are set.

Only one quadrature element is used for the entire beam during the analysis. The convergence of final displacement components at the free tip are listed in **Table 1**. It is seen that the displacements converge rapidly as the number of



**Figure 2.** Pure bending of a uniform cantilever beam.

**Table 1.** Convergence of displacement components at beam tip.

$n$	$\nu = 0$		$\nu = 0.3$	
	$u_x$	$u_y$	$u_x$	$u_y$
9	-10.00131	-6.249E-06	-10.00130	-5.892E-06
10	-10.00150	1.110E-06	-10.00146	1.064E-06
11	-10.00001	-5.468E-08	-10.00001	-4.995E-08
12	-10.00001	7.288E-09	-10.00001	6.887E-09
13	-10.00000	1.741E-09	-10.00000	1.707E-09
14	-10.00000	1.549E-09	-10.00000	1.560E-09
Analytical [14]	-10	0	-10	0

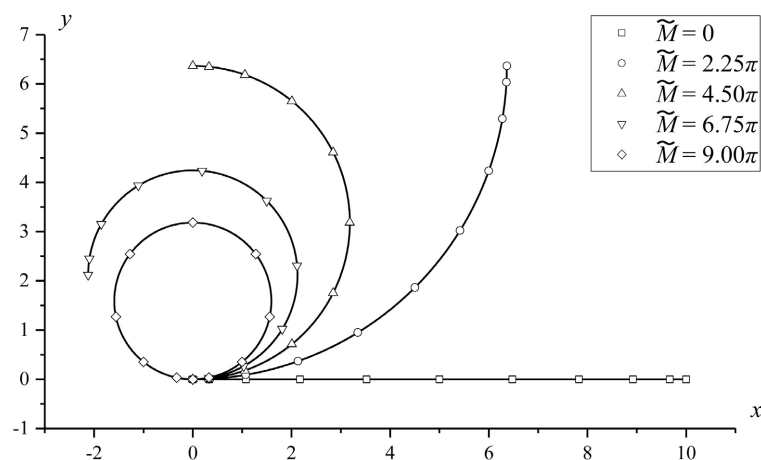
element nodes  $n$  increases. The final result of convergence agrees excellently with the analytical solution. The proposed element still gives accurate results when the Poisson's ratio is nonzero and Poisson locking does not turn up.

The deformed configurations of the beam during the loading process when  $n = 11$  is set as the number of element nodes are shown in **Figure 3**. It is seen that the beam has a constant curvature and eventually bends into a complete circle.

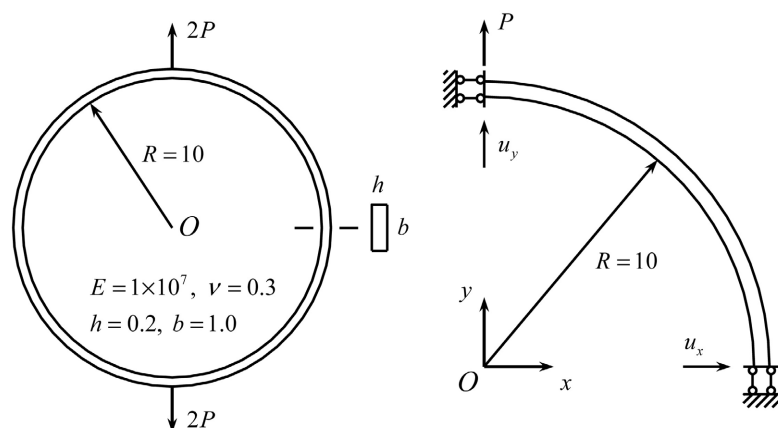
#### 4.2. A Circular Ring under a Pair of Concentrated Forces

A circular ring is subject to a pair of concentrated forces which are in opposite directions and pass through the center of the circle. The positive and negative loads represent the tension and compression of the ring, respectively. The geometric and material properties of the ring are given in **Figure 4**. Obviously, it is sufficient to take a quarter of the ring for analysis according to the symmetry.

The whole quarter circle is analyzed with only one quadrature element, and 12 nodes are used in the element to ensure the convergence of results. The following dimensionless quantities

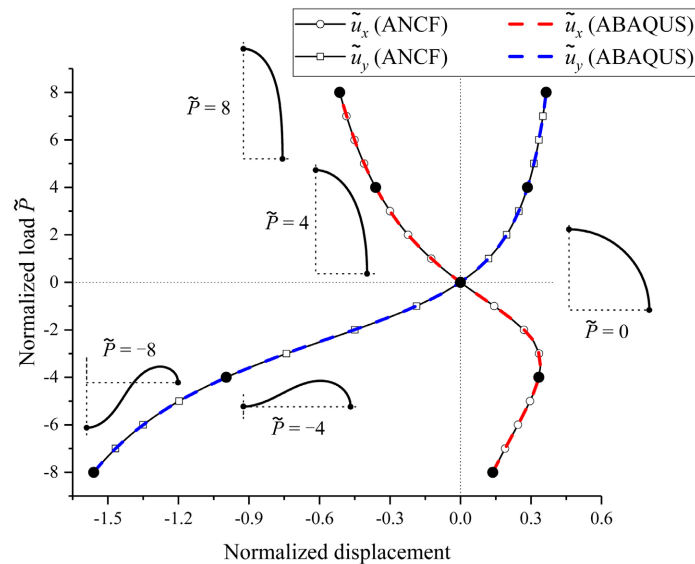


**Figure 3.** Bending configurations of the cantilever beam.



**Figure 4.** A circular ring under a pair of concentrated forces.





**Figure 5.** Load-displacement curves of the ring.

$$\tilde{P} = \frac{PR^2}{EI}, \quad \tilde{u}_x = \frac{u_x}{R}, \quad \tilde{u}_y = \frac{u_y}{R} \quad (26)$$

are introduced to represent the load and displacement components.

The load-displacement curves and several deformed configurations are shown in **Figure 5**. Those of ABAQUS model with 100 B31 elements are also shown for comparison. It is seen that the present predictions agree well with those of the ABAQUS model, demonstrating that the present formulation is capable of dealing with element with initial bending.

## 5. Conclusion

A planar weak form quadrature beam element based on ANCF has been constructed. The expression of the internal forces in the element is derived on the basis of geometrically exact beam theory, resulting in good consistency with classical beam theory. The number of element nodes along the axial direction of the beam can be chosen arbitrarily and a minimum number of elements is made likely in geometrically nonlinear beam analysis. The excellent agreement reached in the two examples indicates that the proposed element is competent in planar beamlike structural geometrically nonlinear analysis.

## Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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