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Existence of the Solutions for a Class of Quasilinear Schrödinger Equations with Nonlocal Term

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Abstract

In this paper, we deal with the existence of solution for a class of quasilinear Schrödinger equations with a nonlocal term

$$\begin{aligned}&-\operatorname{div}\left(g^2(u)\nabla u\right)+g(u)g'(u)|\nabla u|^2+V(x)u\\&=\left[|x|^{-\mu}*\left(KF(u)\right)\right]Kf(u),\ x\in\mathbb{R}^3,\end{aligned}$$

where $\mu\in(0,3)$, the function $K,V\in C\left(\mathbb{R}^3,\mathbb{R}^+\right)$ and $V(x)$ may be vanish at infinity, g is a C^1 even function with $g'(t)\le 0$ for all $t\ge 0$, $g(0)=1$, $\lim_{t\to+\infty} g(t)=a$, $0<a<1$, and F is the primitive function of f which is superlinear but subcritical at infinity in the sense of Hardy-littlewood-Sobolev inequality. By the mountain pass theorem, we prove that the above equation has a nontrivial solution.

Keywords

Quasilinear Schrödinger Equation, Nontrivial Solution, Variational Method

1. Introduction

In this paper, we consider investigating the existence of a nontrivial solution for the following generalized quasilinear Schrödinger equation with a nonlocal term

$$\begin{aligned}&-\operatorname{div}\left(g^2(u)\nabla u\right)+g(u)g'(u)|\nabla u|^2+V(x)u\\&=\left[|x|^{-\mu}*\left(KF(u)\right)\right]Kf(u),\ x\in\mathbb{R}^3,\end{aligned}\tag{1.1}$$

where $\mu\in(0,3)$, the function $V,K\in C\left(\mathbb{R}^3,\mathbb{R}^+\right)$ may be vanish at infinity,

$f:\mathbb{R}\to\mathbb{R}$, g is a C^1 even function with $g'(t)\le 0$ for all $t\ge 0$, $g(0)=1$, $\lim_{t\to+\infty} g(t)=a$, $0<a<1$, when $g(u)=1$, (1.1) boils down to the so called nonlinear Choquard or Choquard-Pekar equation

$$-\Delta u+V(x)u=\left[|x|^{-2}*\left(K(y)F(u(y))\right)\right]K(x)f(u(x)), \tag{1.2}$$

Such like equation has several physical origins. The problem

$$-\Delta u+u=\left[|x|^{-1}*|u|^2\right]u, \tag{1.3}$$

appeared at least as early as 1954, in a work by Pekar describing the quantum mechanics of a polaron at rest [1]. In 1976, Choquard used (1.3) to describe an electron trapped in its own hole and in a certain approximation to Hartree-Fock theory of one component plasma [2]. In 1996, Penrose proposed (1.3) as a model of self-gravitating matter, in a program in which quantum state reduction is understood as a gravitational phenomenon [3]. In this context, equation of type (1.3) is usually called the nonlinear Schrödinger-Newton equation. The first investigations for existence and symmetry of the solutions (1.3) go back to the works of Lieb [2] and Lions [4]. In [2], by using symmetric decreasing rearrangement inequalities, Lieb proved that the ground state solution of equation (3) is radial and unique up to translations. Lions [4] showed the existence of a sequence of radially symmetric solutions [5]. Wei and Winter consider strongly interacting bumps for the Schröding-Newton equation. Ma and Zhao [6] considered the generalized Choquard equation

$$-\Delta u+u=\left[|x|^{-\mu}*|u|^q\right]|u|^{q-2}u\,(q\ge 2), \tag{1.4}$$

and proved that every positive solution of it is radially symmetric and monotone decreasing about some fixed point, under the assumption that a certain set of real numbers, defined in terms of N, q, is nonempty. Under the same assumption, Cingolani, Clapp, and Secchi [7] gave some existence and multiplicity results in the electromagnetic case and established the regularity and some decay asymptotically at infinity of the ground states. In [8], Moroz and Van Schaftingen eliminated this restriction and showed the regularity, positivity, and radial symmetry of the ground states for the optimal range of parameters and derived decay asymptotically at infinity for them as well. Moreover, they [9] also obtained a similar conclusion under the assumption of Berestycki-Lions type nonlinearity. We point out that the existence, multiplicity, and concentration of such equations have been established by many authors. We refer the readers to [10] [11] for the existence of sign-changing solutions, [5] [12] for the existence and concentration behavior of the semiclassical solutions and [13] for the critical nonlocal part with respect to the Hardy-Littlewood-Sobolev inequality. For more details associated with the Choquard equation, please refer to [14] [15] [16] and the references therein. Li, Teng, Zhang, and Nie [17] investigate the existence of solutions for the following generalized quasilinear Schrödinger equation with a nonlocal term

$$
\begin{aligned}
&-\operatorname{div}\left(g^2(u)\nabla u\right)+g(u)g'(u)+|\nabla u|^2+V(x)u \\
&=\lambda\left[|x|^{-\mu}*|u|^p\right]|u|^{p-2}u, x\in\mathbb{R}^N,
\end{aligned}
\tag{1.5}
$$

and prove that the existence of solution. Li and Wu [18] considered the following generalized quasilinear Schrödinger equations with critical or supercritical growths

$$
-\operatorname{div}\left(g^2(u)\nabla u\right)+g(u)g'(u)+|\nabla u|^2+V(x)u=f(x,u)+\lambda|u|^{p-2}u, x\in\mathbb{R}^N. \tag{1.6}
$$

and prove the existence of nontrivial solutions. Recently, Chen, Zhang and Tang [19] considered following Kirchhoff-type equation with convolution term and prove the existence of ground state solutions. Li, Li and Ma [20] proved that (1.7) has a positive ground state solution by using a monotonicity trick introduced by Jeanjean [21] and a version of global compactness Lemma.

Inspired by the above in this paper, we will consider the existence of nontrivial solution for the generalized quasilinear Schrödinger equation when $V(x)\to 0$ as $|x|\to\infty$. The energy functional associated with (1.1)

$$
I(u)=\frac{1}{2}\int_{\mathbb{R}^3}\left(g^2(u)|\nabla u|^2+V(x)u^2\right)-\frac{1}{2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{K(y)K(x)F(u(y))F(u(x))}{|x-y|^{\mu}},
$$

where $F(u)=\int_0^u f(s)$, However, I is not well defined in $H^1(\mathbb{R}^3)$ since the term $\int_{\mathbb{R}^3} g^2(u)|\nabla u|^2$. To overcome this difficulty, we make a change of variable constructed by Shen and Wang in [22]: $v:=G(u):=\int_0^u g(t)$. Then we obtain

$$
\begin{aligned}
J(v)=&\frac{1}{2}\int_{\mathbb{R}^3}|\nabla v|^2+\frac{1}{2}\int_{\mathbb{R}^3}V(x)\left|G^{-1}(v)\right|^2 \\
&-\frac{1}{2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{K(y)K(x)F\left(G^{-1}(v(y))\right)F\left(G^{-1}(v(x))\right)}{|x-y|^{\mu}},
\end{aligned}
\tag{1.7}
$$

We say u is a solution of (1.1) if

$$
\begin{aligned}
\langle I'(u),\varphi\rangle=&\int_{\mathbb{R}^3}\left[g^2(u)\nabla u\nabla\varphi+g(u)g'(u)|\nabla u|^2\varphi+V(x)u\varphi\right] \\
&-\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{K(y)K(x)F(u(y))f(u(x))}{x-y}\varphi \\
=&\,0,
\end{aligned}
\tag{1.8}
$$

for all $\varphi\in C_0^\infty(\mathbb{R}^3)$. Let $\varphi=\frac{1}{g(u)}\psi$. By [21] we know that (1.8) is equivalent to

$$
\begin{aligned}
\langle J'(v),\psi\rangle=&\int_{\mathbb{R}^3}\left[\nabla v\nabla\psi+V(x)\frac{G^{-1}(v)}{g\left(G^{-1}(v)\right)}\psi\right] \\
&-\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{K(y)K(x)F\left(G^{-1}(v(y))\right)f\left(G^{-1}(v(x))\right)}{|x-y|^{\mu}g\left(G^{-1}(v(x))\right)}\psi \\
=&\,0,
\end{aligned}
\tag{1.9}
$$

for all $\psi \in C_0^\infty\left(\mathbb{R}^3\right)$. Therefore, in order to find the nontrivial solution of (1.1), it suffices to study the existence of the nontrivial of the following equations

$$-\Delta v+V(x)\frac{G^{-1}(v)}{g\left(G^{-1}(v)\right)}-\int_{\mathbb{R}^3}\frac{K(y)K(x)F\left(G^{-1}(v(y))\right)f\left(G^{-1}(v(x))\right)}{|x-y|^\mu g\left(G^{-1}(v(x))\right)}=0, \quad (1.10)$$

To describe our results, we firstly introduce the assumptions on V and K:

(VK_1) $V,K\in\left(\mathbb{R}^3,\mathbb{R}\right)$, $V(x),K(x)>0$, $\forall x\in\mathbb{R}^3, K\in L^\infty\left(\mathbb{R}^3\right)$;

(VK_2) If $\{A_n\}\subset\mathbb{R}^3$ is a sequence of borel sets with $|A_n|\le\delta$ for all n and some $\delta>0$, then

$$\lim_{r\to\infty}\int_{A_n\cap B_r^c(0)}K^{\frac{6}{6-\mu}}=0, \text{ uniformly in } n\in N,\ \mu\in(0,3);$$

(VK_3) $\dfrac{K^{\frac{6}{6-\mu}}}{V}\in L^\infty\left(\mathbb{R}^3\right)$;

(VK_4) there exists $p\in(2,6)$ such that

$$\frac{K^{\frac{6}{6-\mu}}(x)}{V^{\frac{6-p}{4}}(x)}\to 0 \text{ as } |x|\to+\infty,$$

For the nonlinearity f, and g, we have the following assumptions:

(f_1) $f\in C^1(\mathbb{R},\mathbb{R})$ and $f(t)=0, \forall t\le 0$;

(f_2) $\lim_{t\to 0}\dfrac{f(t)}{t^{\frac{3-\mu}{3}}}=0$ if (VK_3) holds; $\lim_{t\to 0}\dfrac{f(t)}{t^{\frac{p(6-\mu)}{6}-1}}<+\infty, p\in(2,6)$ if (VK_4) holds;

(f_3) $\lim_{t\to+\infty}\dfrac{f(t)}{|t|^{5-\mu}}=0$;

(f_4) $\lim_{t\to+\infty}\dfrac{f(t)}{t}=+\infty$;

(f_5) $\dfrac{f(t)}{t}$ is strictly increasing as $t>0$;

(f_6) there exist $T>0$ such that $tf(t)\ge\dfrac{2}{a}F(t)>0$, if $t>T$.

Then we have the following results.

Theorem 1.1. *Suppose that* (VK_1)-(VK_4) (f_1)-(f_5). *Then the problem* (1.1) *exists a nontrivial solution.*

Remark 1.1. *In this paper, we consider the potential function V is vanishing at infinity and the nonlocal term f is subcritical. By using mountain pass theorem and dominated theorem, we prove the theorem* 1.1. *At same time, we say lemma* 3.4 [23] *play a great role in this article. Moreover, if someone are interested in this case, they can consider nonlocal term f is critical and supercritical.*

In this paper, we will make use of the following notations:

- The characters $C, C_1, C_2, \cdots$ means to inexactly positive constants respectively;

- " $\to$ " denotes strong convergence and " $\rightharpoonup$ " denotes weak convergence;
- $L^p(\mathbb{R}^3), 1 \le p \le +\infty$, denotes the Lebesgue space with the norm $\|u\|_{L^p} = \left(\int_{\mathbb{R}^3} |u|^p\right)^{\frac{1}{p}}$.

2. Preliminary Results

Throughout the paper, we let

$$H := \left\{u \in H^1(\mathbb{R}^3) : \int_{\mathbb{R}^3} V(x) u^2 < \infty\right\}. \tag{2.1}$$

then H is a Hilbert space equipped with the inner product

$$(u,v) = \int_{\mathbb{R}^3} \nabla u \nabla v + V(x) uv,$$

and the norm

$$\|u\| = \left(\int_{\mathbb{R}^3} |\nabla u|^2 + V(x)|u|^2\right)^{\frac{1}{2}}.$$

We also define weighted Lebesgue space

$$L^{p+1}_{K^{\frac{6}{6-\mu}}}(\mathbb{R}^3) = \left\{u : \mathbb{R}^3 \to \mathbb{R} : u \text{ is measurable } \int_{\mathbb{R}^3} K^{\frac{6}{6-\mu}}(x)|u|^{p+1} < \infty\right\},$$

To begin with, we give some lemmas.

Lemma 2.1. [24] (*Hardy-Littlewood-Sobolev inequality*) *Let* $t, r > 1$, *and* $0 < \mu < N$ with $\frac{1}{t} + \frac{1}{r} + \frac{\mu}{N} = 2$, $f(t) \in L^t(\mathbb{R}^3)$ *and* $L^r(\mathbb{R}^3)$. *There exists a sharp comstant* $C = C(t, r, N, \mu) > 0$, *independent of* f, h, *such that*

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{f(x)h(y)}{|x-y|^{\mu}} \le C\|f\|_t \|h\|_r$$

Lemma 2.2. [25] T*he function* $g(t), G^{-1}(t), G(t)$ *enjoy the following properties.*

(g_1) the function $G(t)$ and $G^{-1}(t)$ are strictly increasing and odd;

(g_2) $|t| \le |G^{-1}(t)| \le |t|/a$ for all $t \in \mathbb{R}$;

(g_3) $G^{-1}(t)/t$ is nondecreasing for all $t \in \mathbb{R}$ and $\lim_{t\to 0} G^{-1}(t)/t = 1$, $\lim_{t\to\infty} G^{-1}(t)/t = 1/a$;

(g_4) $t^2 \le (t/g(t))G(t) \le t^2/a$ for all $t \in \mathbb{R}$.

Lemma 2.3. *Assume that* (f_1)-(f_5). *Then we have the following conditions:*

1) For every $\varepsilon > 0$, there exists $C_\varepsilon > 0$ satisfies that

$|f(t)| \le \varepsilon |t|^{\frac{3-\mu}{3}} + C_\varepsilon |t|^{5-\mu}$ and $|F(t)| \le \varepsilon |t|^{\frac{6-\mu}{3}} + C_\varepsilon |t|^{6-\mu}$ $\forall t \in \mathbb{R}$, if (VK_3) holds.

2) For every $\delta > 0$ and $p \in (2,6)$, there is $C_\delta > 0$ satisfies that

$|f(t)| \le \delta |t|^{p\frac{6-\mu}{6}-1} + C_\delta |t|^{5-\mu}$ and $|F(t)| \le \delta |t|^{p\frac{6-\mu}{6}} + C_\delta |t|^{6-\mu}$ $\forall t \in R$, if (VK_4) holds.

Proof: By the definition and straightforward calculus.

Lemma 2.4. [26] *Assume that there are* (VK_1)-(VK_2) *hold. Then, H is com-*

pactly embedded in $L^q_{K^{\frac{6}{6-\mu}}}\left(R^3\right)$ *for all* $q\in(2,6)$ *if* (VK_3) *holds. If* (VK_4) *holds, one has H is compactly embedded in* $L^q_{K^{\frac{6}{6-\mu}}}\left(\mathbb{R}^3\right)$, *for all* $q\in(2,6)$.

Proof: The proof will be made into two parts, firstly we consider the condition (VK_3), and after (VK_4). By assuming that (VK_3) is true, fixed $q\in(2,6)$ and given $\varepsilon>0$, there are $0<s_0<s_1$ and $C>0$ such that

$$K^{\frac{6}{6-\mu}}|s|^q\le\varepsilon C\left(V(x)|s|^2+|s|^6\right)+CK^{\frac{6}{6-\mu}}(x)\chi_{[s_0,s_1]}(|s|)|s|^6,\ s\in R. \tag{2.2}$$

Hence,

$$\int_{B_r^c(0)}K^{\frac{6}{6-\mu}}|v|^q\le\varepsilon CQ(v)+C\int_{B\cap B_r^c(0)}K^{\frac{6}{6-\mu}},\ v\in H, \tag{2.3}$$

where

$$Q(v)=\int_{\mathbb{R}^3}V(x)|v|^2+|v|^6,$$

and

$$B=\left\{x\in\mathbb{R}^3:s_0\le|v(x)|\le s_1\right\},$$

If (v_n) is a sequence such that $v_n\rightharpoonup v$ in *H*, there is $M_1>0$ such that

$$\int_{\mathbb{R}^3}\left(|\nabla v_n|^2+V(x)|v_n|^2\right)\le M_1\ \text{ and }\ \int_{\mathbb{R}^3}|v_n|^6\le M_1,\ \forall n\in\mathbb{N},$$

which imply that $(Q(v_n))$ is bounded. On the other hand, setting

$$B_n=\left\{x\in\mathbb{R}^3:s_0\le|v_n(x)|\le s_1\right\},$$

the last inequality implies that

$$s_0^6|B_n|\le\int_{B_n}|v_n(x)|^6\le M_1$$

showing that $\sup_{n\in\mathbb{N}}|A_n|<+\infty$. Therefore, from ($VK_2$), there is an $r>0$ such that

$$\int_{B_n\cap B_r^c(0)}K^{\frac{6}{6-\mu}}(x)<\frac{\varepsilon}{s_1^6},\ \forall n\in\mathbb{N}, \tag{2.4}$$

Now, (2.3) and (2.4) lead to

$$\int_{B_r^c(0)}K^{\frac{6}{6-\mu}}(x)|v_n|^q\le\varepsilon CM_1+s_1^6\int_{B_n\cap B_r^c(0)}K^{\frac{6}{6-\mu}}(x)<(CM_1+1)\varepsilon,\ \forall n\in\mathbb{N}, \tag{2.5}$$

Once that $q\in(2,6)$ and *K* is a continuous function, it follows from Sobolev embedding

$$\lim_{n\to+\infty}\int_{B_r(0)}K^{\frac{6}{6-\mu}}(x)|v_n|^q=\int_{B_r(0)}K^{\frac{6}{6-\mu}}(x)|v|^q, \tag{2.6}$$

Combining (2.5) and (2.6)

$$\lim_{n\to+\infty}\int_{\mathbb{R}^3}K^{\frac{6}{6-\mu}}(x)|v_n|^q=\int_{\mathbb{R}^3}K^{\frac{6}{6-\mu}}(x)|v|^q, \tag{2.7}$$

which yields

$$v_n \to v \text{ in } L^q_{K^{\frac{6}{6-\mu}}},\ \forall q \in (2,6).$$

Now, we will suppose that (VK_4) holds. First of all, it is important to observe that for each $x \in \mathbb{R}^3$ fixed, the function

$$g(s) = V(x)s^{2-p} + s^{6-p},\ \forall s > 0.$$

has $C_p V(x)^{\frac{6-p}{4}}$ as its minimum value, where

$$C_p = \left(\frac{6-p}{4}\right)^{\frac{p-6}{4}} \left(\frac{p-2}{4}\right)^{\frac{2-p}{4}},$$

Hence,

$$C_p V(x)^{\frac{6-p}{4}} \le V(x)s^{2-p} + s^{6-p},\ \forall x \in \mathbb{R}^3 \text{ and } s > 0.$$

Combining the last inequality with (VK_4), given $\varepsilon \in (0, C_p)$, there is $r > 0$ large enough, such that

$$K^{\frac{6}{6-\mu}}(x)|s|^p \le \varepsilon\left(V(x)|s|^2 + |s|^6\right),\ \forall s \in \mathbb{R} \text{ and } |x| \ge r,$$

leading to

$$\int_{B_r^c(0)} K^{\frac{6}{6-\mu}}(x)|u|^p \le \varepsilon \int_{B_r^c(0)} \left(V(x)|u|^2 + |u|^6\right),\ \forall u \in H.$$

If (v_n) is a sequence such that $v_n \rightharpoonup v$ in *H*, there is $M_1 > 0$ such that

$$\int_{\mathbb{R}^3} V(x)|v_n|^2 \le M_1 \text{ and } \int_{R^3} |v_n|^6 \le M_1,\ \forall n \in \mathbb{N},$$

and so

$$\int_{B_r^c(0)} K^{\frac{6}{6-\mu}}(x)|v_n|^p \le 2\varepsilon M_1,\ \forall n \in \mathbb{N}, \tag{2.8}$$

Once that $p \in (2,6)$ and $K^{\frac{6}{6-\mu}}$ is a continuous function, it follows from Sobolev embedding

$$\lim_{n\to+\infty} \int_{B_r(0)} K^{\frac{6}{6-\mu}}(x)|v_n|^p = \int_{B_r(0)} K^{\frac{6}{6-\mu}}(x)|v|^p, \tag{2.9}$$

From (2.9) and (2.10)

$$\lim_{n\to+\infty} \int_{\mathbb{R}^3} K^{\frac{6}{6-\mu}}(x)|v_n|^p = \int_{\mathbb{R}^3} K^{\frac{6}{6-\mu}}(x)|v|^p,$$

implying that

$$v_n \to v \text{ in } L^p_{K^{\frac{6}{6-\mu}}}(\mathbb{R}^3),$$

finishing the proof of the proposition.

Lemma 2.5. *Suppose that f satisfies* (f_1)-(f_5). *Let* (v_n) *be a sequence such that* $v_n \rightharpoonup v$ *in H. Then*

$$\lim_{n\to+\infty}\int_{\mathbb{R}^3}\left|K(x)F\left(G^{-1}(v_n)\right)\right|^{\frac{6}{6-\mu}}=\int_{\mathbb{R}^3}\left|K(x)F\left(G^{-1}(v)\right)\right|^{\frac{6}{6-\mu}},$$

$$\lim_{n\to+\infty}\int_{\mathbb{R}^3}\left|K(x)f\left(G^{-1}(v_n)\right)v_n\right|^{\frac{6}{6-\mu}}=\int_{\mathbb{R}^3}\left|K(x)F\left(G^{-1}(v)\right)v\right|^{\frac{6}{6-\mu}},$$

and

$$\lim_{n\to+\infty}\int_{\mathbb{R}^3}\left|\frac{K(x)f\left(G^{-1}(v(x))\right)v(x)}{g\left(G^{-1}(v_n(x))\right)}\right|^{\frac{6}{6-\mu}}=\int_{\mathbb{R}^3}\left|\frac{K(x)f\left(G^{-1}(v(x))\right)v(x)}{g\left(G^{-1}(v(x))\right)}\right|^{\frac{6}{6-\mu}}.$$

Proof: We will begin the proof by assuming that (VK_3) occurs. From Lemma 2.3, fixed $q\in(2,6)$ and given $\varepsilon>0$, there is $C>0$ such that

$$\left|K(x)F(s)\right|^{\frac{6}{6-\mu}}\le\varepsilon C\left(V(x)|s|^2+|s|^6\right)+CK^{\frac{6}{6-\mu}}(x)|s|^q,\ \forall s\in\mathbb{R},\tag{2.10}$$

From Lemma 2.4

$$\int_{\mathbb{R}^3}K^{\frac{6}{6-\mu}}(x)|v_n|^q\to\int_{\mathbb{R}^3}K^{\frac{6}{6-\mu}}(x)|v|^q,$$

then there is $r>0$ such that

$$\int_{B_r^c(0)}K^{\frac{6}{6-\mu}}(x)|v_n|^q<\varepsilon,\ \forall n\in\mathbb{N},\tag{2.11}$$

Since (v_n) is bounded in *H*, by lemma 2.2 there is $M_1>0$ such that

$$\int_{R^3}V(x)\left|G^{-1}(v_n)\right|^2\le\frac{M_1}{a^2}\text{ and }\int_{\mathbb{R}^3}\left|G^{-1}(v_n)\right|^6\le\frac{M_1}{a^6}.$$

Combining the last inequalities with (2.10) and (2.11)

$$\int_{B_r^c(0)}\left|K(x)F\left(G^{-1}(v_n)\right)\right|^{\frac{6}{6-\mu}}<\left(\frac{2}{a^6}C+1\right)\varepsilon,\ \forall n\in\mathbb{N},\tag{2.12}$$

Now, if (VK_4) holds, repeating the same arguments explored in the proof of Lemma 2.4, given $\varepsilon>0$ small enough, there is $r>0$ large enough such that

$$K^{\frac{6}{6-\mu}}(x)\le\varepsilon\left(V(x)|s|^{2-p}+|s|^{6-p}\right),\ \forall s\in R\setminus\{0\}\text{ and }|x|\ge r.$$

Hence

$$K^{\frac{6}{6-\mu}}(x)\left|F(s)\right|^{\frac{6}{6-\mu}}$$
$$\le\varepsilon\left(V(x)\left|F(s)\right|^{\frac{6}{6-\mu}}|s|^{2-p}+\left|F(s)\right|^{\frac{6}{6-\mu}}|s|^{6-p}\right),\ \forall s\in\mathbb{R}\text{ and }|x|\ge r.$$

From (f_2) and (f_3), there are $C,s_0,s_1>0$ verifying

$$K^{\frac{6}{6-\mu}}(x)\left|F(s)\right|^{\frac{6}{6-\mu}}\le\varepsilon\left(V(x)|s|^2+|s|^6\right),\ \forall s\in I\text{ and }|x|\ge r.$$

where $I=\{s\in R:|s|<s_0\text{ or }|s|>s_1\}$. Thereby, for any $v\in H$, we have the follow estimate

$$\int_{B_r^c(0)}K^{\frac{6}{6-\mu}}(x)\left|F\left(G^{-1}(v)\right)\right|^{\frac{6}{6-\mu}}\le\varepsilon CQ\left(G^{-1}(v)\right)+C\int_{A\cap B_r^c(0)}K^{\frac{6}{6-\mu}}(x),$$

with

$$Q\left(G^{-1}(v)\right)=\int_{\mathbb{R}^3}V(x)\left|G^{-1}(v)\right|^2+\int_{\mathbb{R}^3}\left|G^{-1}(v)\right|^6,$$

and

$$A=\left\{x\in R^3: s_0\le\left|G^{-1}(v)\right|\le s_1\right\}.$$

Once that (v_n) is bounded in *H*, there is $M_1>0$ such that

$$\int_{\mathbb{R}^3}V(x)\left|G^{-1}(v_n)\right|^2\le\frac{M_1}{a^2}\text{ and }\int_{\mathbb{R}^3}\left|G^{-1}(v_n)\right|^6\le\frac{M_1}{a^6},$$

Thus

$$\int_{B_r^c(0)}K^{\frac{6}{6-\mu}}(x)\left|F\left(G^{-1}(v_n)\right)\right|^{\frac{6}{6-\mu}}\le\frac{2M_1\varepsilon}{a^6}+C\int_{A_n\cap B_r^c(0)}K^{\frac{6}{6-\mu}}(x),$$

where

$$A_n=\left\{x\in\mathbb{R}^3: s_0\le\left|G^{-1}(v_n)\right|\le s_1\right\},$$

Repeating the same arguments used in the proof of Lemma 2.2, it follows that

$$\int_{A_n\cap B_r^c(0)}K^{\frac{6}{6-\mu}}(x)\to 0\text{ as }|x|\to+\infty,$$

and so, for *n* large enough

$$\int_{B_r^c(0)}\left|K^{\frac{6}{6-\mu}}(x)F\left(G^{-1}(v_n)\right)\right|^{\frac{6}{6-\mu}}\le\left(\frac{2M_1}{a^6}+1\right)\varepsilon.$$

Using compactness lemma of Strauss [27], Theorem A.I, p. 338, we have

$$\lim_{n\to+\infty}\int_{B_r(0)}\left|K(x)F\left(G^{-1}(v_n)\right)\right|^{\frac{6}{6-\mu}}=\int_{B_r(0)}\left|K(x)F\left(G^{-1}(v)\right)\right|^{\frac{6}{6-\mu}},$$

so

$$\lim_{n\to+\infty}\int_{\mathbb{R}^3}\left|K(x)F\left(G^{-1}(v_n)\right)\right|^{\frac{6}{6-\mu}}=\int_{\mathbb{R}^3}\left|K(x)F\left(G^{-1}(v)\right)\right|^{\frac{6}{6-\mu}},$$

Similarly, we can prove

$$\lim_{n\to+\infty}\int_{\mathbb{R}^3}\left|K(x)f\left(G^{-1}(v_n)\right)v_n\right|^{\frac{6}{6-\mu}}=\int_{\mathbb{R}^3}\left|K(x)f\left(G^{-1}(v)\right)v\right|^{\frac{6}{6-\mu}},$$

and

$$\lim_{n\to+\infty}\int_{\mathbb{R}^3}\left|\frac{K(x)f\left(G^{-1}(v(x))\right)v(x)}{g\left(G^{-1}(v_n(x))\right)}\right|^{\frac{6}{6-\mu}}=\int_{\mathbb{R}^3}\left|\frac{K(x)f\left(G^{-1}(v(x))\right)v(x)}{g\left(G^{-1}(v(x))\right)}\right|^{\frac{6}{6-\mu}}.$$

Lemma 2.6. *Assume the assumptions* (VK_1)-(VK_4) *and* (f_1)-(f_5) *hold. Then J satisfies the following conditions*:

i) There exist $\alpha,\rho>0$ $J(v)\ge\alpha$ if $\|v\|=\rho$.

ii) There exist an $e\in H$ with $\|e\|\geq\rho$ such that $J(e)<0$.

Proof: (i) If (VK_3) hold, by Lemma 2.3 and (f_3) and (f_4), we have

$$\begin{aligned}J(v)&=\frac{1}{2}\int_{\mathbb{R}^3}|\nabla v|^2+\frac{1}{2}\int_{\mathbb{R}^3}V(x)\left|G^{-1}(v)\right|^2\\&-\frac{1}{2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{K(x)K(y)F\left(G^{-1}(v(y))\right)F\left(G^{-1}(v(x))\right)}{|x-y|^{\mu}}\\&\geq\frac{1}{2}\|v\|^2-\frac{\varepsilon}{a^{\frac{12-2\mu}{3}}}\|v\|^{\frac{12-2\mu}{3}}-\frac{C_{\varepsilon}}{a^{12-2\mu}}\|v\|^{12-2\mu},\end{aligned}\tag{2.13}$$

Since $\mu\in(0,3)$, we can choose some $\rho>0,\alpha>0$ such that

$$J(v)\geq\alpha>0 \text{ with } \|v\|=\rho.$$

If (VK_4) holds, by the same way, we also have the same result.

(ii) First we note that $J(0)=0$. Furthermore, by Lemma 2.2, for fixed $v\in H\setminus\{0\}$ and $t>0$, we have

$$\begin{aligned}J(tv)&=\frac{1}{2}\int_{\mathbb{R}^3}|\nabla tv|^2+\frac{1}{2}\int_{\mathbb{R}^3}V(x)\left|G^{-1}(tv)\right|^2\\&-\frac{1}{2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{K(x)K(y)F\left(G^{-1}(tv(y))\right)F\left(G^{-1}(tv(x))\right)}{|x-y|^{\mu}}\\&\leq\frac{t^2}{2a^2}\|v\|^2-\frac{1}{2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{K(x)K(y)F\left(G^{-1}(tv(y))\right)F\left(G^{-1}(tv(x))\right)}{|x-y|^{\mu}},\end{aligned}\tag{2.14}$$

From Lemma 2.3 and (f_6), there exist $C_1,C_2>0$ such that

$$C_1|t|^{\frac{2}{\alpha}}-C_2\leq F(t).$$

By (f_1) and (f_5), we have $F(t)>0$, then

$$\begin{aligned}J(tv)&=\frac{t^2}{2a^2}\|v\|^2-\frac{1}{2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{K(x)K(y)F\left(G^{-1}(tv(y))\right)F\left(G^{-1}(tv(x))\right)}{|x-y|^{\mu}}\\&\leq\frac{t^2}{2a^2}\|v\|^2-\frac{1}{2}\int_{\mathbb{R}^3}\int_{R^3}\frac{K(x)K(y)\left(C_1\left|G^{-1}(tv(y))\right|^{\frac{2}{\alpha}}-C_2\right)\left(C_1\left|G^{-1}(tv(x))\right|^{\frac{2}{\alpha}}-C_2\right)}{|x-y|^{\mu}}\\&\leq\frac{t^2}{2a^2}\|v\|^2-\frac{1}{2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}K(x)K(y)\left(C_1|tv(y)|^{\frac{2}{\alpha}}-C_2\right)\left(C_1|tv(x)|^{\frac{2}{\alpha}}-C_2\right)\\&=\frac{t^2}{2a^2}\|v\|^2-\frac{1}{2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}K(x)K(y)\left(C_1^2|t|^{\frac{4}{a}}|v(x)|^{\frac{2}{a}}|v(y)|^{\frac{2}{a}}-C_1C_2t^{\frac{2}{a}}\left(|v(x)|^{\frac{2}{a}}+|v(y)|^{\frac{2}{a}}\right)+C_2^2\right)\\&=\frac{t^2}{2a^2}\|v\|^2-\frac{|t|^{\frac{3}{a}}}{2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}K(x)K(y)\left(C_1^2t|t|^{\frac{1}{a}}|v(x)|^{\frac{2}{a}}|v(y)|^{\frac{2}{a}}\right.\\&\left.-C_1C_2|t|^{-\frac{1}{a}}\left(|v(x)|^{\frac{2}{a}}+|v(y)|^{\frac{2}{a}}\right)+|t|^{-\frac{3}{a}}C_2^2\right)\to-\infty,\ t\to+\infty.\end{aligned}\tag{2.15}$$

Thus, we take $e=t_0v$ for some $t_0>0$, and (ii) holds.

By Lemma 2.6 and Ambrosetti-Rabinowitz mountain pass theorem [28], there exists a $(PS)_c$ sequence $v_n \subset H$

$$J(v_n) \to c \text{ and } J'(v_n) \to 0, \tag{2.16}$$

at the minimax level

$$c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J(\gamma(t)),$$

where $\Gamma = \gamma \in \{C([0,1],H) : \gamma(0) = 0, J(\gamma(1)) < 0\}$.

Lemma 2.7. *The sequence* (v_n) *given in* (2.16) *is bounded.*

Proof: By (2.16) and Lemma 2.3, we have

$$\begin{aligned}
&c+1+o_n(1)\|v_n\| \ge J(v_n) - \frac{a}{4}\langle J'(v_n), v_n\rangle \\
&\ge \left(\frac{1}{2}-\frac{a}{4}\right)\int_{\mathbb{R}^3}|\nabla v|^2 + \int_{\mathbb{R}^3}\frac{1}{4}V(x)\left|G^{-1}(v_n)\right|^2 \\
&+\frac{a}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{K(x)K(y)F\left(G^{-1}(v_n(y))\right)\left(f\left(G^{-1}(v_n(x))\right)G^{-1}(v_n(x)) - \frac{2}{a}F\left(G^{-1}(v_n(x))\right)\right)}{|x-y|^\mu} \\
&\ge \frac{1}{4}\|v_n\|^2,
\end{aligned} \tag{2.17}$$

which implies that $\{v_n\}$ is bounded in *H*.

Proof of Theorem 1.1: By Lemma 2.7, $\{v_n\}$ is bounded in *H*. Then, passing to a subsequence, $v_n \rightharpoonup v$ in *H*, $v_n \to v$ in $L^q_{loc}(\mathbb{R}^3)$ for $q \in [2,6)$, $v_n(x) \to v(x)$ a.e in $\mathbb{R}^3$.

By (1.9) and Fatou's Lemma, we obtain that

$$\begin{aligned}
o_n(1) &= \langle J'(v_n), v\rangle \\
&= \int_{\mathbb{R}^3}\nabla v_n \nabla v + \int_{\mathbb{R}^3} V(x)\frac{G^{-1}(v_n)v}{g\left(G^{-1}(v_n)\right)} \\
&\quad -\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{K(y)K(x)F\left(G^{-1}(v_n)\right)f\left(G^{-1}(v_n)\right)}{|x-y|^\mu g\left(G^{-1}(v_n)\right)}v \\
&\ge \|v\|^2 - \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{K(y)K(x)F\left(G^{-1}(v_n)\right)f\left(G^{-1}(v_n)\right)}{|x-y|^\mu g\left(G^{-1}(v_n)\right)}v.
\end{aligned} \tag{2.18}$$

and

$$\begin{aligned}
o_n(1) &= \langle J'(v_n), v_n\rangle \\
&= \int_{\mathbb{R}^3}\nabla v_n \nabla v_n + \int_{\mathbb{R}^3} V(x)\frac{G^{-1}(v_n)v_n}{g\left(G^{-1}(v_n)\right)} \\
&\quad -\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{K(y)K(x)F\left(G^{-1}(v_n)\right)f\left(G^{-1}(v_n(x))\right)v_n(x)}{|x-y|^\mu g\left(G^{-1}(v_n(x))\right)}.
\end{aligned} \tag{2.19}$$

Next, we prove that

$$\begin{aligned}
&\lim_{n\to\infty}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{K(y)K(x)F\left(G^{-1}(v_n(y))\right)f\left(G^{-1}(v_n(x))\right)v_n(x)}{|x-y|^\mu g\left(G^{-1}(v_n(x))\right)} \\
&= \lim_{n\to\infty}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{K(y)K(x)F\left(G^{-1}(v_n(y))\right)f\left(G^{-1}(v_n(x))\right)v(x)}{|x-y|^\mu g\left(G^{-1}(v_n(x))\right)}
\end{aligned}$$

$$= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{K(y)K(x)F\left(G^{-1}(v(y))\right) f\left(G^{-1}(v(x))\right)v(x)}{|x-y|^{\mu}\, g\left(G^{-1}(v(x))\right)}. \tag{2.20}$$

First we prove the first equality in (2.19),

$$\begin{aligned}
&\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{K(y)K(x)F\left(G^{-1}(v_n(y))\right) f\left(G^{-1}(v_n(x))\right)v_n(x)}{|x-y|^{\mu}\, g\left(G^{-1}(v_n(x))\right)} \\
&-\frac{K(y)K(x)F\left(G^{-1}(v(y))\right) f\left(G^{-1}(v(x))\right)v(x)}{|x-y|^{\mu}\, g\left(G^{-1}(v(x))\right)} \\
&= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{K(y)K(x)F\left(G^{-1}(v_n(y))\right)\left(f\left(G^{-1}(v_n(x))\right)v_n(x) - f\left(G^{-1}(v(x))\right)v(x)\right)}{|x-y|^{\mu}\, g\left(G^{-1}(v_n(x))\right)} \\
&\quad + \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{K(y)K(x)\left(F\left(G^{-1}(v_n(y))\right) - F\left(G^{-1}(v(y))\right)\right) f\left(G^{-1}(v(x))\right)v(x)}{|x-y|^{\mu}\, g\left(G^{-1}(v_n(x))\right)} \\
&\quad + \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{K(y)K(x)F\left(G^{-1}(v(y))\right) f\left(G^{-1}(v(x))\right)v(x)}{|x-y|^{\mu}} \left(\frac{1}{g\left(G^{-1}(v_n(x))\right)} - \frac{1}{g\left(G^{-1}(v(x))\right)}\right) \\
&= A_n + B_n + C_n.
\end{aligned} \tag{2.21}$$

where

$$A_n = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{K(y)K(x)F\left(G^{-1}(v_n(y))\right)\left(f\left(G^{-1}(v_n(x))\right)v_n(x) - f\left(G^{-1}(v(x))\right)v(x)\right)}{|x-y|^{\mu}\, g\left(G^{-1}(v_n(x))\right)},$$

$$B_n = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{K(y)K(x)\left(F\left(G^{-1}(v_n(y))\right) - F\left(G^{-1}(v(y))\right)\right) f\left(G^{-1}(v(x))\right)v(x)}{|x-y|^{\mu}\, g\left(G^{-1}(v_n(x))\right)},$$

and

$$C_n = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{K(y)K(x)F\left(G^{-1}(v(y))\right) f\left(G^{-1}(v(x))\right)v(x)}{|x-y|^{\mu}} \left(\frac{1}{g\left(G^{-1}(v_n(x))\right)} - \frac{1}{g\left(G^{-1}(v(x))\right)}\right).$$

Next, we prove $B_n \to 0$ as $n \to \infty$, since

$$\begin{aligned}
|B_n| &= \left|\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{K(y)K(x)\left(F\left(G^{-1}(v_n(y))\right) - F\left(G^{-1}(v(y))\right)\right) f\left(G^{-1}(v(x))\right)v(x)}{|x-y|^{\mu}\, g\left(G^{-1}(v_n(x))\right)}\right| \\
&\le C_1 \left(\int_{\mathbb{R}^3} \left|K(y)\left(F\left(G^{-1}(v_n(y))\right) - F\left(G^{-1}(v(y))\right)\right)\right|^{\frac{6}{6-\mu}}\right)^{\frac{6-\mu}{6}} \\
&\quad \times \left(\int_{\mathbb{R}^3} \left|K(x) f\left(G^{-1}(v(x))\right)v(x)\right|^{\frac{6}{6-\mu}}\right)^{\frac{6-\mu}{6}}
\end{aligned} \tag{2.22}$$

It follows from Lemma 2.5 that

$$\lim_{n\to\infty} \left(\int_{\mathbb{R}^3} \left|K(y)\left(F\left(G^{-1}(v_n(y))\right) - F\left(G^{-1}(v(y))\right)\right)\right|^{\frac{6}{6-\mu}}\right)^{\frac{6-\mu}{6}} = 0,$$

Thus

$$B_n \to 0, \text{ as } n \to \infty,$$

Similarly

$$A_n \to 0,\ C_n \to 0, \text{ as } n \to \infty,$$

Therefore, we have that

$$\begin{aligned}&\lim_{n\to\infty}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{K(y)K(x)F\left(G^{-1}(v_n(x))\right)f\left(G^{-1}(v_n(x))\right)v_n(x)}{|x-y|^{\mu}g\left(G^{-1}(v_n(x))\right)}\\&=\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{K(y)K(x)F\left(G^{-1}(v(x))\right)f\left(G^{-1}(v(x))\right)v(x)}{|x-y|^{\mu}g\left(G^{-1}(v(x))\right)}.\end{aligned}\tag{2.23}$$

Next, we say $v=0$ is impossible. We know $J(v_n)\to c>0$, then we have

$$\begin{aligned}J(v_n)&=\frac{1}{2}\int_{\mathbb{R}^3}|\nabla v_n|^2+\frac{1}{2}\int_{\mathbb{R}^3}V(x)\left|G^{-1}(v_n)\right|^2\\&\quad-\frac{1}{2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{K(y)K(x)F\left(G^{-1}(v_n(y))\right)F\left(G^{-1}(v_n(x))\right)}{|x-y|^{\mu}}\\&\to c\sim\frac{1}{2}\int_{\mathbb{R}^3}|\nabla v_n|^2+\frac{1}{2}\int_{\mathbb{R}^3}V(x)\left|G^{-1}(v_n)\right|^2\\&\to c+\frac{1}{2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{K(y)K(x)F\left(G^{-1}(v(y))\right)F\left(G^{-1}(v(x))\right)}{|x-y|^{\mu}},\end{aligned}$$

Let n large enough, we have

$$\begin{aligned}&\int_{\mathbb{R}^3}|\nabla v_n|^2+\int_{\mathbb{R}^3}V(x)\left|G^{-1}(v_n)\right|^2\\&>\frac{c}{2}+\frac{1}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{K(y)K(x)F\left(G^{-1}(v(y))\right)F\left(G^{-1}(v(x))\right)}{|x-y|^{\mu}},\end{aligned}$$

By (2.19) and Lemma 2.2

$$\begin{aligned}&o_n(1)+\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{K(y)K(x)F\left(G^{-1}(v)\right)f\left(G^{-1}(v(x))\right)v(x)}{|x-y|^{\mu}g\left(G^{-1}(v(x))\right)}\\&=\int_{\mathbb{R}^3}\nabla v_n\nabla v_n+\int_{\mathbb{R}^3}V(x)\frac{G^{-1}(v_n)v_n}{g\left(G^{-1}(v_n)\right)}\\&\ge\int_{\mathbb{R}^3}|\nabla v_n|^2+\int_{\mathbb{R}^3}V(x)G^{-1}(v_n)^2\\&>\frac{c}{2}+\frac{1}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{K(y)K(x)F\left(G^{-1}(v(y))\right)F\left(G^{-1}(v(x))\right)}{|x-y|^{\mu}},\end{aligned}$$

which contradict with $v=0$. Next, we prove $\langle J'(v),\varphi\rangle=0$, $\forall\varphi\in C_0^{\infty}(\mathbb{R}^3)$. since

$$\begin{aligned}&\left|\langle J'(v_n)-J'(v),\varphi\rangle\right|\\&\le\left|\int_{\mathbb{R}^3}\nabla(v_n-v)\nabla\varphi\right|+\left|\int_{\mathbb{R}^3}V(x)\left(\frac{G^{-1}(v_n)}{g\left(G^{-1}(v_n)\right)}-\frac{G^{-1}(v)}{g\left(G^{-1}(v)\right)}\right)\varphi\right|\end{aligned}$$

$$+\left|\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{K(y)K(x)}{|x-y|^{\mu}}\varphi(x)\left(\frac{F(G^{-1}(v_n))f(G^{-1}(v_n))}{g(G^{-1}(v_n))}-\frac{F(G^{-1}(v))f(G^{-1}(v))}{g(G^{-1}(v))}\right)\right|.$$

We prove it by two parts

$$\begin{aligned}&V(x)\left(\frac{G^{-1}(v_n)}{g(G^{-1}(v_n))}-\frac{G^{-1}(v)}{g(G^{-1}(v))}\right)\varphi\\&\le\left|V(x)G^{-1}(v_n)\right||\varphi|+\left|V(x)G^{-1}(v)\right||\varphi|\\&\le\left|V(x)\left(G^{-1}(v_n)^2+1\right)\right||\varphi|+\left|V(x)\left(G^{-1}(v)^2+1\right)\right||\varphi|\\&\le\left|V(x)\left(v_n^2+1\right)\right||\varphi|+\left|V(x)\left(v^2+1\right)\right||\varphi|;\end{aligned}\tag{2.24}$$

and

$$\begin{aligned}&\left|\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{K(y)K(x)}{|x-y|^{\mu}}\varphi(x)\left(\frac{F(G^{-1}(v_n))f(G^{-1}(v_n))}{g(G^{-1}(v_n))}-\frac{F(G^{-1}(v))f(G^{-1}(v))}{g(G^{-1}(v))}\right)\right|\\&\le\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\left|\frac{K(y)K(x)}{|x-y|^{\mu}}\varphi(x)F(G^{-1}(v_n))f(G^{-1}(v_n))\right|\\&+\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\left|\frac{K(y)K(x)}{|x-y|^{\mu}}\varphi(x)F(G^{-1}(v))f(G^{-1}(v))\right|.\end{aligned}\tag{2.25}$$

By Lemma 2.5 and Lemma 3.4 [23], we have

$$\begin{aligned}&\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\left|\frac{K(y)K(x)}{|x-y|^{\mu}}\varphi(x)F(G^{-1}(v_n))f(G^{-1}(v_n))\right|\\&\to\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\left|\frac{K(y)K(x)}{|x-y|^{\mu}}\varphi(x)F(G^{-1}(v))f(G^{-1}(v))\right|.\end{aligned}\tag{2.26}$$

By the (2.24)-(2.26), Lemma 2.5, and the Lemma 3.4 [28], we have

$$\left|\langle J'(v_n)-J'(v),\varphi\rangle\right|\to 0,\ \forall\varphi\in C_0^{\infty}(\mathbb{R}^3);$$

Then

$$J'(v)=0.\tag{2.27}$$

Hence, v is a nontrivial solution of Equation (1.1).

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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