Scientific Research Publishing

Journal of Applied Mathematics and Physics, 2022, 10, 3218-3239
https://www.scirp.org/journal/jamp
ISSN Online: 2327-4379
ISSN Print: 2327-4352

A New Approach to Solving Two-Dimensional Unsteady Incompressible Navier-Stokes Equations

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How to cite this paper: Hasan, Z.A. and Al-Saif, A.-S.J. (2022) A New Approach to Solving Two-Dimensional Unsteady Incompressible Navier-Stokes Equations. *Journal of Applied Mathematics and Physics*, **10**, 3218-3239.
https://doi.org/10.4236/jamp.2022.1010214

Received: September 19, 2022
Accepted: October 25, 2022
Published: October 28, 2022

Abstract

This paper proposes a new approach that combines the reduced differential transform method (RDTM), a resummation method based on the Yang transform, and a Padé approximant to the kinetically reduced local Navier-Stokes equation to find approximate solutions to the problem of lid-driven square cavity flow. The new approach, called PYRDM, considerably improves the convergence rate of the truncated series solution of RDTM and also is based on a simple process that yields highly precise estimates. The numerical results achieved by this method are compared to earlier studies' results. Our results indicate that this method is more efficient and precise in generating analytic solutions. Furthermore, it provides highly precise solutions with good convergence that is simple to apply for great Reynolds and low Mach numbers. Moreover, the new solution' graphs demonstrate the new approach's validity, usefulness, and necessity.

Keywords

Navier-Stokes Equations, RDTM, Yang Transform, Padé Approximation, Accuracy, Convergence Analysis

1. Introduction

Fluid flow is an important engineering phenomenon that has piqued the interest of both theoretical and practical scientists. Several of these studies concentrate on mathematical models that simulate these phenomena. In order to obtain analytical and numerical solutions for the Navier-Stokes equations, which are the fundamental model for describing fluid motion, researchers devote the substantial effort. This paper examines two-dimensional Navier-Stokes equations describing unstable viscous incompressible flows. The form of the non-dimensional momentum and continuity equations is as follows:

$$\begin{aligned} u_t &= -\left(uu_x + vu_y + p_x\right) + \frac{1}{Re}\left(u_{xx} + u_{yy}\right), \\ v_t &= -\left(uv_x + vv_y + p_y\right) + \frac{1}{Re}\left(v_{xx} + v_{yy}\right), \end{aligned} \tag{1}$$

and

$$u_x + v_y = 0. \tag{2}$$

where t is the physical time, $u(x,y,t)$ and $v(x,y,t)$ are the fluid velocity components, $\mathcal{P}(x,y,t)$ is the pressure, and Re is the Reynolds number. Because the Navier-Stokes equations are certain partial differential equations that describe the motion of viscous fluid substances, named after the French engineer and physicist Claude-Louis Navier and Anglo-Lish physicist and mathematician George Gabriel Stoke. Because there is no explicit equation for estimating pressure, and these equations are difficult to solve, various studies have proposed the thermal model of incompressible fluid flows as a substitute. The kinetically reduced local Navier-Stokes (KRLNS) equations are one of these alternative equations. [1] [2] [3] [4] [5], which are derived by substituting the pressure by

$$\mathcal{P} = g + \frac{u^2 + v^2}{2} \tag{3}$$

and the continuity equation by

$$g_t = -\frac{1}{(Ma)^2}\left(u_x + v_y\right) + \frac{1}{Re}\left(g_{xx} + g_{yy}\right), \tag{4}$$

where Ma is the Mach number, and $g(x,y,t)$ is the grand potential. The time scale in INS equations is related to that of KRLNS equations; $t_{KRLNS}(\tau) = Ma \times t_{NS}$. The KRLNS equation system thus takes the following form.

$$\begin{aligned} u_t &= -\left(2uu_x + vv_x + vu_y + g_x\right) + \frac{1}{Re}\left(u_{xx} + u_{yy}\right), \\ v_t &= -\left(uv_x + uu_y + 2vv_y + g_y\right) + \frac{1}{Re}\left(v_{xx} + v_{yy}\right), \\ g_t &= -\frac{1}{(Ma)^2}\left(u_x + v_y\right) + \frac{1}{Re}\left(g_{xx} + g_{yy}\right). \end{aligned} \tag{5}$$

All papers that provided the KRLNS equations for modeling unsteady incompressible viscous flow problems employed numerical ways to solve them. The KRLNS equations are proposed in [6] for modeling low Mach number flows, and the numerical solution of the three-dimensional Taylor Green vortex flow was found using the spectral element technique. In [3], the four-stage Runge-Kutta method and the central difference scheme are used to solve the KRLNS equations in order to simulate two-dimensional shear layers and to decay homogeneous isotropic turbulence; the numerical results are then compared to those from other methods, such as the artificial compressibility method and the lattice Boltzmann method. In [5], higher-order difference approximations are employed to determine the solutions of the two-dimensional KRLNS equations simulations of the Womersley equation with doubly periodic shear layers. In [2] simplifies and compares the two-dimensional KRLNS system to Chorin's artificial compressibility approach for steady-state computation of flow in a two-dimensional lid-driven cavity and Taylor-Green vortex flow. The lid-driven cavity problem concerns the flow in a box cavity with no wall slippage and one or more moving walls that move continuously. It has been widely utilized as a benchmark case for the research of computing techniques for solving Navier-Stokes equations because of the geometry's and boundary conditions' simplicity. Through the use of various numerical techniques in cavities that are rectangular or square, numerous research papers have offered answers to this problem. In [7], for instance, the Chebyshev-collocation approach in space is used with the Adams-Bash forth backward-Euler scheme to determine the solution of a three-dimensional lid-driven cavity. In [8] describes how to solve both the stable and unstable two-dimensional lid-driven cavity issue at a high Reynolds number using the implicit cell-vertex finite volume method. In [9] employs the finite volume approach with numerical approximations of second-order accuracy and successive Richardson extrapolations to solve the issue of flow within a square cavity with constant velocity. The stable two-dimensional incompressible lid-driven square cavity flow problem is solved using a stream function-velocity formulation in [10], which provides a compact finite difference approximation for non-uniform orthogonal Cartesian grids. Numerous analytical techniques can be used without a perturbation parameter. First, in 2009, the Turkisch mathematician Keskin [11] proposed for the first time the reduced differential transform method (RDTM). Since it has been used by numerous authors to address a wide range of issues, it has attracted a lot of attention [12] [13] [14] [15] [16]. Second, a novel integral transform is the Yang transform (YT), suggested by Xiao-Jun [17] in 2016. It was initially used in the equation for steady-state heat transfer. Notably, this technique is accurate and successful in determining analytical solutions for partial equations, and it is employed by a variety of researchers to address various issues. [18] [19]. In 2018, Dattu [20] presented the essential properties of the Yang transform and used Yang transform to solve differential issues with constant coefficients. Third, Henri Padé (1863-1953) presented an approximation technique in his doctoral thesis in 1892 which is called Padé approximation. Since it has been used by numerous authors to address a wide range of issues, it has attracted a lot of attention [21] [22] [23] [24] [25]. Therefore, in this study, we present a new approach that combines (RDTM, Padé approximation, and YT) for finding analytical solutions for KRLNS. The solution to KRLNS is obtained in convergent series forms using YRDTM. After that, we create its Pad'e approximant of an order $[L/M]$ to convert the power series solution obtained by YRDTM into a meromorphic function. The values for L and M are selected at random. In this stage, the Pad's approximant improves the accuracy and convergence of the truncated series solution by expanding the domain of that solution. This is an approach that we name PYRDTM. The main goal of this work is to describe a new approach to solving KRLNS. According to the calculations shown in the tables and figures, The PYRDTM procedures are very effective and more accurate in resolving unsteady viscous incompressible flow problems at low Mach numbers and for various Reynolds numbers. Additionally, by solving this problem, we presented the PYRDTM as a handy tool with great potential to solve nonlinear PDE.

2. Reduced Differential Transform Method

In 2009, the Turkish mathematician Keskin [11] suggested the reduced differential transform method (RDTM) to study the analytical solutions of linear and non-linear wave equations. RDTM reduces the size of computational work and is easily applicable to many non-linear physical problems. The following are the basic definitions and operations of the two-dimensional reduced differential transform method [26] [27] [28] [29] (**Table 1**).

Definition 2.1.

If function $u(x,y,t)$ is analytic and differentiated continuously with respect to time and space in the domain of interest, then let

$$U_k(x,y)=\frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}u(x,y,t)\right]_{t=0}, \tag{6}$$

where the t-dimensional spectrum function $U_k(x,y)$ is the transformed function in this work, the lower case $u(x,y,t)$ represents the original function while the upper case $U_k(x,y)$ represents the transformed function.

Definition 2.2.

The differential inverse transform of $U_k(x,y)$ is defined as

$$u(x,y,t)=\sum_{k=0}^{\infty}U_k(x,y)(t-t_0)^k, \tag{7}$$

Table 1. The fundamental operations of RDTM.

Functional Form	Transformed Form
$u(x,y,t)$	$U_k(x,y)=\frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}u(x,y,t)\right]_{t=0}$
$w(x,y,t)=au(x,y,t)\pm bv(x,y,t)$	$W_k(x,y)=aU_k(x,y)\pm bV_k(x,y)$
$w(x,y,t)=u(x,y,t)v(x,y,t)$	$W_k(x,y)=\sum_{s=0}^{\infty}V_s(x,y)U_{k-s}(x,y)=\sum_{s=0}^{\infty}U_s(x,y)V_{k-s}(x,y)$
$w(x,y,t)=\frac{\partial^s}{\partial t^s}u(x,y,t)$	$W_k(x,y)=\frac{(k+s)!}{k!}U_{k+s}(x,y)$
$w(x,y,t)=\frac{\partial^2}{\partial y^2}u(x,y,t)$	$W_k(x,y)=\frac{\partial^2}{\partial y^2}U_k(x,y)$
$w(x,y,t)=\frac{\partial^2}{\partial x^2}u(x,y,t)$	$W_k(x,y)=\frac{\partial^2}{\partial x^2}U_k(x,y)$

then by combining Equations (6) and (7) we obtain

$$u(x,y,t)=\sum_{k=0}^{\infty}\frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}u(x,y,t)\right]_{t=0} t^k . \quad (8)$$

It is worth noting that the function $u(x,y,t)$ can be expressed as a finite series as follows:

$$u_n(x,y,t)=\sum_{k=0}^{\infty}U_k(x,y)(t-t_0)^k+R_n(x,y,t), \quad (9)$$

when the tail function $R_n(x,y,t)$ is insignificant. As a result, the exact solution to the issues is given

$$u(x,y,t)=\lim_{n\to\infty}u_n(x,y,t). \quad (10)$$

3. Yang Transform

The integral transforms play a significant role in a variety of scientific disciplines and works of literature; they are used extensively in mathematical physics, optics, mathematical engineering, and other disciplines to solve differential equations such as those of Laplace, Fourier, Mellin, Hankel, and Sumudu. Recently, Xiao-Jun Yang [17] suggested a novel integral transform named the Yang Transform. It was initially used for the equation for steady-state heat transfer. Yang transform of function $u(t)$ is denoted by $Y\{u(t)\}$ or $T(s)$ and is defined as [17],

$$Y[u(t)]=T(s)=\int_0^{\infty}\mathrm{e}^{\frac{-t}{s}}u(t)\mathrm{d}t, t>0 \quad (11)$$

Provided the integral exists for some *s*, where $s\in(-t_1,t_2)$.

If we substitute $\frac{-t}{s}=x$ then Equation (11) becomes,

$$Y[u(t)]=T(s)=s\int_0^{\infty}\mathrm{e}^{-x}u(sx)\mathrm{d}x, x>0 \quad (12)$$

3.1. Yang Transform of Some Functions [17]

Yang transform of some useful functions is given below.

1) $Y\{1\}=s$,
2) $Y\{t\}=s^2$,
3) $Y\{t^n\}=n!\cdot s^{n+1}$,
4) $Y\{\mathrm{e}^{\lambda t}\}=\frac{s}{1-\lambda s}$,
5) $Y\{\sin\lambda t\}=\frac{\lambda s^2}{1+\lambda^2 s^2}$,
6) $Y\{\cos\lambda t\}=\frac{s}{1+\lambda^2 s^2}$,
7) $Y\{\sin\lambda t\}=\frac{\lambda s^2}{1-\lambda^2 s^2}$,
8) $Y\{\cos\lambda t\}=\frac{s}{1-\lambda^2 s^2}$.

3.2. Yang Transform of Derivatives [26]

If $Y\left[u(t)\right]=T(s)$ then

1) $Y\left[\dfrac{\partial u(x,t)}{\partial t}\right]=\dfrac{T(x,s)}{s}-u(x,0)$,

2) $Y\left[\dfrac{\partial^2 u(x,t)}{\partial t^2}\right]=\dfrac{T(x,s)}{s^2}-\dfrac{u(x,0)}{s}-\dfrac{\partial u(x,0)}{\partial t}$,

3) $Y\left[\dfrac{\partial^n u(x,t)}{\partial t^n}\right]=\dfrac{T(x,s)}{s^n}-\sum_{k=0}^{n-1}s^{-n+k+1}\dfrac{\partial^{(k)}u(x,0)}{\partial t^k}\quad \forall n=1,2,3,4,\cdots$

4. The Padé Approximants

Suppose that, we are given a power series $\sum_{r=0}^{\infty}c_r x^r$, representing a function $f(x)$, so that

$$f(x)=\sum_{r=0}^{\infty}c_r x^r \tag{13}$$

The Padé approximant is a rational fraction, and its notation is as follows [22],

$$[L/M]=\frac{A_L(x)}{B_M(x)}, \tag{14}$$

where $A_L(x)$ is a polynomial of degree at most L and $B_M(x)$ is a polynomial of degree at most M. We have

$$f(x)=c_0+c_1x+c_2x^2+c_3x^3+c_4x^4+\cdots, \tag{15}$$

$$A_L(x)=a_0+a_1x+a_2x^2+a_3x^3+\cdots+a_Lx^L,$$

$$B_M(x)=b_0+b_1x+b_2x^2+b_3x^3+\cdots+b_Mx^M,$$

there are $L+1$ numerator coefficients and $M+1$ denominator coefficients in Equation (14). We impose the normalizing condition since we can obviously multiply the numerator and denominator by a constant while leaving $[L,M]$ unaltered

$$B_M(0)=1. \tag{16}$$

As a result, there are $L+1$ independent numerator coefficients and M independent denominator coefficients, for a total of $L+M+1$ unknown coefficients. This number implies that the $[L,M]$ should generally fit the power series (13) through the orders $1,x,x^2,\cdots,x^{(L+M)}$. Using the conclusion from [22]. We know that the $[L,M]$ approximation is determined uniquely.

In formal power series notation,

$$\sum_{r=0}^{\infty}c_r x^r=\frac{a_0+a_1x+a_2x^2+\cdots+a_Lx^L}{b_0+b_1x+b_2x^2+\cdots+b_Mx^M}+O\left(x^{L+M+1}\right). \tag{17}$$

Using cross-multiplying Equation (17), we discover

$$\begin{aligned}&\left(b_0+b_1x+b_2x^2+\cdots+b_Mx^M\right)\left(c_0+c_1x+c_2x^2+c_3x^3+c_4x^4+\cdots\right)\\&=a_0+a_1x+a_2x^2+\cdots+a_Lx^L+O\left(x^{L+M+1}\right)\end{aligned} \tag{18}$$

From Equation (18), the set of equations can be found.

$$\begin{cases} c_0 = a_0, \\ c_1 + c_0 b_1 = a_1, \\ c_2 + c_1 b_1 + c_0 b_2 = a_2, \\ \vdots \\ c_L + a_{L-1} b_1 + \cdots + c_0 b_L = a_L, \end{cases} \tag{19}$$

and

$$\begin{cases} c_{L+1} + c_L b_1 + \cdots + c_{L-M+1} b_M = 0, \\ c_{L+2} + c_{L+1} b_1 + \cdots + c_{L-M+2} b_M = 0, \\ \vdots \\ c_{L+M} + c_{L+M-1} b_1 + \cdots + c_L b_M = 0, \end{cases} \tag{20}$$

where $c_n = 0$ for $n < 0$ and $b_j = 0$ for $j > M$.

If Equations (19) and (20) are not unique, we can immediately solve them.

$$\left[\frac{L}{M}\right] = \frac{\begin{vmatrix} c_{L-m+1} & c_{L-m+2} & \cdots & c_{L+1} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{j=M}^{L} c_{j-M} \chi^j & \sum_{j=M-1}^{L} c_{j-M+1} \chi^j & \cdots & \sum_{j=0}^{L} c_j \chi^j \end{vmatrix}}{\begin{vmatrix} c_{L-m+1} & c_{L-m+2} & \cdots & c_{L+1} \\ \vdots & \vdots & \ddots & \vdots \\ c_{L+1} & c_{L+1} & \cdots & c_{L+M} \\ x^M & x^{M-1} & \cdots & 1 \end{vmatrix}}.$$

If the lower index of an amount is greater than the higher index, the sum is replaced by zero. Alternative forms include

$$\begin{aligned} [L/M] &= \sum_{j=0}^{L-M} c_j x^j + x^{L-M+1} w_{L/M}^{\mathrm{T}} W_{L/M}^{-1} w_{L/M} \\ &= \sum_{j=0}^{L+n} c_j x^j + x^{L+n+1} w_{(L+M)/M}^{\mathrm{T}} W_{L/M}^{-1} w_{(L+n)/M}, \end{aligned}$$

for

$$W_{L/M} = \begin{bmatrix} c_{L-M+1} - \chi c_{L-M+2} & \cdots & c_L - \chi c_{L+1} \\ \vdots & \ddots & \vdots \\ c_L - \chi c_{L+1} & \cdots & c_{L+M+1} - \chi c_{L+M} \end{bmatrix},$$

$$W_{L/M} = \begin{bmatrix} c_{L-M+1} \\ c_{L-M+2} \\ \vdots \\ c_L \end{bmatrix}.$$

The construction $[L/M]$ of approximants can be made only by algebraic operations [22]. Each choice of L degree numerator and M degree denominator results in an approximation. How to steer the choice to produce the best approximation is the technique's main point of difficulty. This necessitates the application of a selection criterion that is based on the solution's shape. The choice of $[L/M]$ approximants has proven to be a useful criterion in this situation.

5. The Hybrid Method Algorithm

Consider a general nonlinear non-homogenous partial differential equation with initial conditions of the form:

$$\mathcal{L}u(x,y,t)+\mathcal{R}u(x,y,t)+\mathcal{N}u(x,y,t)=g(x,y,t), \tag{23}$$

with initial condition

$$u(x,y,0)=h(x,y), \tag{24}$$

where $\mathcal{L}=\frac{\partial}{\partial t}$, $\mathcal{R}$ are a linear differential operators, $\mathcal{N}$ is a nonlinear operator and $g(x,y,t)$ is an inhomogeneous term.

Taking the Yang transform on both sides to Equation (23), to get:

$$Y\left[\mathcal{L}u(x,y,t)\right]+Y\left[\mathcal{R}u(x,y,t)\right]+Y\left[\mathcal{N}u(x,y,t)\right]=Y\left[g(x,y,t)\right]. \tag{25}$$

Using the differentiation property of the Yang transforms (2.3) and above initial conditions, we have:

$$Y\left[u(x,y,t)\right]=sY\left[g(x,y,t)\right]+sh(x,y)-sY\left[\mathcal{R}u(x,y,t)+\mathcal{N}u(x,y,t)\right], \tag{26}$$

applying the inverse Yang transform on both sides to Equation (26), to find:

$$u(x,y,t)=G(x,y,t)-Y^{-1}\left\{sY\left[\mathcal{R}u(x,y,t)+\mathcal{N}u(x,y,t)\right]\right\}.$$

where $G(x,y,t)$ represents the term arising from the source term and the prescribed initial conditions.

Now, we apply the reduced differential transform method:

$$U(x,y,0)=G(x,y,t), \tag{27}$$

$$U(x,y,k+1)=-Y^{-1}\left\{sY\left[\mathcal{R}U(x,y,k)+\mathcal{N}U(x,y,k)\right]\right\}, \tag{28}$$

where $RU(x,y,k), NU(x,y,k)$ are transformation of functions.
$\mathcal{R}u(x,y,t), \mathcal{N}u(x,y,t)$, respectively. This is coupling of Yang transform and reduced differential transform method.

Then by YRDTM we have the solution of Equation (23), with initial condition (24) in the form of infinite series which converge to the exact solution as follows:

$$u(x,y,t)=\sum_{k=0}^{\infty}U(x,y,k). \tag{29}$$

After that, we applied its Pad'e approximant of an order $[L/M]$ on the power series solution. The values L and M are arbitrarily selected. In this stage, the Pad's approximant improves the accuracy and convergence of the truncated series solution by expanding the domain of that solution.

6. Applications

In this section, we solve the problem mentioned in (5) using PYRDTM to show the correctness, efficiency, and convergence of the proposed approach.

In order to apply PYRDTM to KRLNS equation to obtain approximate solutions to the lid-driven square cavity flow problem. The exact solution to the problem's steady state [30] [31] [32] takes the initial conditions into account in this test:

$$\begin{aligned} &u(x,y) = 8r(x)c'(y), \\ &v(x,y) = -8r'(x)c(y), \\ &g(x,y) = \frac{8}{Re}\left(F(x)c'''(x) + r'(x)c'(y)\right) + 64F_1(x)\left(c(y)c''(y) - \left(c'(y)\right)^2\right), \end{aligned} \tag{30}$$

where

$$r(x) = x^4 - 2x^3 + x^2, c(y) = y^4 - y^2,$$

$$F(x) = \int r(x)\mathrm{d}x, F_1(x) = \int r(x)r'(x)\mathrm{d}x,$$

such that the stream function ψ and vorticity ω are defined as

$$\begin{aligned} &\psi = 8r(x)p(y), \text{ such that } \psi_y = u, \text{ and } \psi_x = -v, \\ &\omega = v_x - u_y = -8\left(r''(x)c(y) + r(x)c''(y)\right). \end{aligned} \tag{31}$$

PYRDTM can be used to find the analytical-approximate solution to KRLNS, as shown below; by taking the Yang transform on both sides to Equation (5) subject to the initial condition (30), we have

$$Y\left[u(x,y,t)\right] = Su(0) + SY\left[-2u\frac{\partial u}{\partial x} - v\frac{\partial v}{\partial x} - v\frac{\partial u}{\partial y} - \frac{\partial g}{\partial x} + \frac{1}{Re}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)\right], \tag{32}$$

$$Y\left[v(x,y,t)\right] = Sv(0) + SY\left[-u\frac{\partial v}{\partial x} - u\frac{\partial u}{\partial y} - 2v\frac{\partial v}{\partial y} - \frac{\partial g}{\partial y} + \frac{1}{Re}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)\right], \tag{33}$$

$$Y\left[g(x,y,t)\right] = Sg(0) + SY\left[\frac{-1}{ma}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \frac{1}{Re}\left(\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2}\right)\right], \tag{34}$$

by applying the inverse Yang transform to Equations (32), (33), and (34), we have

$$u(x,y,t) = u(0) + Y^{-1}\left(SY\left[-2u\frac{\partial u}{\partial x} - v\frac{\partial v}{\partial x} - v\frac{\partial u}{\partial y} - \frac{\partial g}{\partial x} + \frac{1}{Re}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)\right]\right), \tag{35}$$

$$v(x,y,t) = v(0) + Y^{-1}\left(SY\left[-u\frac{\partial v}{\partial x} - u\frac{\partial u}{\partial y} - 2v\frac{\partial v}{\partial y} - \frac{\partial g}{\partial y} + \frac{1}{Re}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)\right]\right), \tag{36}$$

$$g(x,y,t) = g(0) + Y^{-1}\left(SY\left[\frac{-1}{ma}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \frac{1}{Re}\left(\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2}\right)\right]\right), \tag{37}$$

applying the reduced differential transform method to Equations (35), (36), and (37), we have

$$\begin{aligned} U(x,y,k+1) = Y^{-1}\Bigg(SY\Bigg[&-2A(x,y,k) - B(x,y,k) - C(x,y,k) - \frac{\partial}{\partial x}g(x,y,k) \\ &+ \frac{1}{Re}\left(\frac{\partial^2}{\partial x^2}V(x,y,k) + \frac{\partial^2}{\partial y^2}V(x,y,k)\right)\Bigg]\Bigg), \end{aligned} \tag{38}$$

$$V(x,y,k+1)=Y^{-1}\left(SY\left[-D(x,y,k)-E(x,y,k)-2F(x,y,k)-\frac{\partial}{\partial y}g(x,y,k)\right.\right.$$
$$\left.\left.+\frac{1}{Re}\left(\frac{\partial^2}{\partial x^2}V(x,y,k)+\frac{\partial^2}{\partial y^2}V(x,y,k)\right)\right]\right), \tag{39}$$

$$G(x,y,k+1)=Y^{-1}\left(SY\left[\frac{-1}{ma}\left(\frac{\partial}{\partial x}U(x,y,k)+\frac{\partial}{\partial y}V(x,y,k)\right)\right.\right.$$
$$\left.\left.+\frac{1}{Re}\left(\frac{\partial^2}{\partial x^2}G(x,y,k)+\frac{\partial^2}{\partial y^2}G(x,y,k)\right)\right]\right), \tag{40}$$

with

$$\begin{aligned}
U(x,y,0)&=16x^2y\left(2y^2-1\right)\lambda^2,\\
V(x,y,0)&=-16\left(2x^3-3x^2+x\right)\left(y^4-y^2\right),\\
G(x,y,0)&=\frac{32xy}{Re}\left(\left(4y^2-3x+\frac{6x^2}{5}\right)x^2-\left(2y^2-1\right)(3x-1)\right)\\
&\quad-64\lambda^2x^2y^2\left[x^3\left(10y^4-9y^2+3\right)(x-2)\right.\\
&\quad\left.-x^2\left(8y^6-6y^4-y^2+3\right)-2y^2\mu^2\xi^2(4x-1)\right].
\end{aligned} \tag{41}$$

where $A(x,y,k), B(x,y,k), C(x.y,k), D(x,y,k), E(x,y,k)$ and $F(x,y,k)$ are the reduced differential transformed of $uu_x, vu_x, vu_y, uv_x, uu_y, vv_y$, and having the value,

$$\begin{aligned}
A(x,y,k)&=\sum_{r=0}^{k}U(x,y,r)\left(U(x,y,k-r)\right)_{x,}\\
B(x,y,k)&=\sum_{r=0}^{k}V(x,y,r)\left(V(x,y,k-r)\right)_{y},\\
C(x,y,k)&=\sum_{r=0}^{k}V(x,y,r)\left(U(x,y,k-r)\right)_{y},\\
D(x,y,k)&=\sum_{r=0}^{k}U(x,y,r)\left(V(x,y,k-r)\right)_{x},\\
E(x,y,k)&=\sum_{r=0}^{k}U(x,y,r)\left(U(x,y,k-r)\right)_{y},\\
F(x,y,k)&=\sum_{r=0}^{k}V(x,y,r)\left(V(x,y,k-r)\right)_{y},
\end{aligned} \tag{42}$$

from relationships (38), (39), (40), and (41), give us the values of $U(x,y,k)$, $V(x,y,k)$, and $G(x,y,k)$ as follows;

$$U(x,y,0)=16x^2y\left(2y^2-1\right)\lambda^2, \tag{43}$$

$$U(x,y,1)=0, \tag{44}$$

$$\begin{aligned}
U(x,y,2)&=\frac{1}{5}\frac{1}{Re^2}\left(\left(-184320yx^2Re^2\lambda^2\left(\left(y^6-\frac{1}{2}y^4+\frac{2}{9}y^2-\frac{1}{36}\right)x^8\right.\right.\right.\\
&\quad+\left(-4y^6+2y^4-\frac{8}{9}y^2+\frac{1}{9}\right)x^7+\left(-5y^8+\frac{118}{9}y^6-\frac{1}{6}-\frac{35}{6}y^4\right.\\
&\quad\left.+\frac{37}{18}y^2\right)x^6+\left(15y^8-\frac{76}{3}y^6+\frac{1}{9}+\frac{21}{2}y^4-\frac{55}{18}y^2\right)x^5\\
&\quad+\left(4y^{10}-\frac{53}{2}y^8-\frac{1}{36}+\frac{293}{9}y^6-\frac{143}{12}y^4+\frac{31}{12}y^2\right)x^4
\end{aligned}$$

$$
\begin{aligned}
&+\left(-8y^{10}+28y^{8}-\frac{248}{9}y^{6}+\frac{26}{3}y^{4}-\frac{10}{9}y^{2}\right)x^{3}\\
&+\left(\frac{58}{9}y^{10}-\frac{317}{18}y^{8}+\frac{136}{9}y^{6}-\frac{149}{36}y^{4}+\frac{7}{36}y^{2}\right)x^{2}\\
&-\frac{22}{9}y^{4}\mu^{2}x\xi^{2}\left(y^{2}-\frac{1}{2}\right)+\frac{1}{3}y^{10}-\frac{5}{6}y^{8}+\frac{2}{3}y^{6}-\frac{1}{6}y^{4}\Big)\\
&+\Big(\left(9216y^{2}-1536\right)x^{9}+\left(-41472y^{2}+6912\right)x^{8}\\
&+\left(543744y^{4}-104448y^{2}+1024\right)x^{7}\\
&+\left(-1903104y^{4}+559104y^{2}-35840\right)x^{6}\\
&+\left(645120y^{6}+1880064y^{4}-574464y^{2}+55040\right)x^{5}\\
&+\left(-1612800y^{6}+57600y^{4}-57600y^{2}-32000\right)x^{4}\\
&+\left(92160y^{8}+1192960y^{6}-844800y^{4}+337920y^{2}+6400\right)x^{3}\\
&+\left(-138240y^{8}-176640y^{6}+257280+142080y^{2}\right)x^{2}\\
&+\left(61440y^{8}-79360y^{6}+24320y^{4}+14080y^{2}\right)x\\
&-7680y^{4}\xi^{2}\mu^{2}\Big)Re,
\end{aligned}
\tag{45}
$$

$$\vdots$$

and

$$V(x,y,0)=-16\left(2x^{3}-3x^{2}+x\right)\left(y^{4}-y^{2}\right), \tag{46}$$

$$
\begin{aligned}
V(x,y,1)=\frac{1}{5}\frac{1}{Re}\Bigg(\Bigg(&3840Re\,yx^{8}\left(y^{4}-\frac{1}{3}y^{2}+\frac{1}{6}\right)-15360Re\,yx^{7}\Big(y^{4}-\frac{1}{3}y^{2}\\
&+\frac{1}{6}\Big)-5120\,\mathrm{Re}\;yx^{6}\left(y^{6}-6y^{4}+2y^{2}-\frac{3}{4}\right)\\
&+x^{8}\left(15360Re\,y^{7}-38400Re\,y^{5}+12800Re\,y^{3}-2560Re\,y-192\right)\\
&+x^{4}\left(-17920Re\,y^{7}+30720Re\,y^{5}-10240Re\,y^{3}+640Re\,y+480\right)\\
&+x^{3}\left(10240Re\,y^{7}-15360Re\,y^{5}+5120Re\,y^{3}-3840y^{2}+320\right)\\
&+x^{2}\left(-2560Re\,y^{7}+3840Re\,y^{5}-1280Re\,y^{3}+5760y^{2}-960\right)\\
&+\left(-960y^{4}-960y^{2}+320\right)x+480y^{4}-480y^{2}\Bigg)\tau\Bigg),
\end{aligned}
\tag{47}
$$

$$
\begin{aligned}
V(x,y,2)=\frac{1}{5}\frac{1}{Re^{2}}\Bigg(\Bigg(&-1310720Re^{2}x^{3}y^{10}\beta\lambda^{3}\left(x^{2}-x+\frac{9}{16}\right)\\
&+614400Re^{2}x^{3}y^{8}\beta\lambda^{3}\left(x^{4}-2x^{3}+\frac{27}{5}x^{2}-\frac{22}{5}x+\frac{37}{15}\right)\\
&+2560Re\,y^{7}-737280Re^{2}x^{3}y^{6}\beta\lambda^{3}\left(x^{2}-x+1\right)\left(x^{2}-x+\frac{23}{18}\right)\\
&+414720Re\,y^{5}\left(x^{6}-3x^{5}+\frac{167}{54}x^{4}-\frac{32}{27}x^{3}+\frac{5}{54}x^{2}-\frac{1}{108}\right)\\
&+143360Re^{2}x^{3}y^{4}\beta\lambda^{3}\left(x^{4}-2x^{3}+3x^{2}-2x+\frac{8}{7}\right)
\end{aligned}
$$

$$
\begin{aligned}
&+221184Re\,y^3\left(x^8-4x^7+\frac{1163}{216}x^6-\frac{155}{72}x^5-\frac{355}{432}x^4\right.\\
&\left.+\frac{5}{9}x^3+\frac{5}{144}x^2+\frac{5}{864}\right)-20480y^2\beta\left(\frac{9}{8}+\lambda^5Re^2\right)\\
&-20480y^2\beta\left(\frac{9}{8}+\lambda^5Re^2\right)\\
&-33792Re\,x^2y\lambda\left(x^5-3x^4+\frac{65}{66}x^3+\frac{100}{33}x^2-\frac{105}{44}x+\frac{15}{44}\right)\\
&-9600x^3+14400x^2-960x-1920\Big)\tau^2\Big),\\
&\qquad\vdots
\end{aligned}
\tag{48}
$$

and

$$
\begin{aligned}
G(x,y,0)=\frac{32xy}{Re}&\left(\left(4y^2-3x+\frac{6x^2}{5}\right)x^2-\left(2y^2-1\right)(3x-1)\right)\\
&-64\lambda^2x^2y^2\left[x^3\left(10y^4-9y^2+3\right)(x-2)\right.\\
&\left.-x^2\left(8y^6-6y^4-y^2+3\right)-2y^2\mu^2\xi^2(4x-1)\right].
\end{aligned}
\tag{49}
$$

$$
\begin{aligned}
G(x,y,1)=\frac{1}{Re^2}&\Big(\Big(\Big(\left(-19200y^4+6912y^2-384\right)x^8\\
&+\left(76800y^4-27648y^2+1536\right)x^7\\
&+\left(-64512y^6-52224y^4+24576y^2-2304\right)x^6\\
&+\left(193536y^6-112128y^4+23040y^2+1536\right)x^5\\
&-\left(15360y^8-177664y^6+168960y^4-47616y^2-384\right)x^4\\
&+\left(30720y^8+32768y^6-61440y^4+24576y^2\right)x^3\\
&+\left(-19968y^8+25088y^6-5376y^4-3840y^2\right)x^2\\
&+4608xy^4\mu^2\xi^2-256y^8+512y^6+256y^4\Big)Re\\
&+1536\beta y\left(x^2+\frac{1}{2}y^2-x-\frac{1}{4}\right)\Big)\tau\Big),
\end{aligned}
\tag{50}
$$

$$
\begin{aligned}
G(x,y,2)=\frac{1}{ma^2Re^3}&\left(\left(-1920\lambda^2x^2\left(\left(y^4-\frac{1}{5}y^2+\frac{1}{30}\right)x^4\right.\right.\right.\\
&+\left(-2y^4+\frac{2}{5}y^2-\frac{1}{15}\right)x^3+\left(\frac{28}{15}y^6-2y^4+\frac{2}{5}y^2\right)x\\
&+\left(-\frac{28}{15}y^6+3y^4-\frac{3}{5}y^2+\frac{1}{30}\right)x^2-\frac{14}{15}y^6+y^4-\frac{1}{5}y^2\Big)Re^3\\
&+768Re^2\beta y\left(x^2+\frac{1}{2}y^2-x-\frac{1}{4}\right)-115200\left(\left(y^2-\frac{3}{50}\right)x^8\right.\\
&+\left(-4y^2+\frac{6}{25}\right)x^7+\left(-\frac{3}{25}+\frac{196}{15}y^4+\frac{26}{25}y^2\right)x^6
\end{aligned}
$$

$$+\left(-\frac{12}{25}-\frac{196}{5}-\frac{272}{25}y^2\right)x^5+\left(\frac{107}{150}+\frac{182}{15}y^6+\frac{449}{15}y^4-12y^2\right)x^4+\left(-\frac{26}{75}-\frac{364}{15}y^6+\frac{82}{15}y^4+\frac{6}{5}y^2\right)x^3$$
$$+\left(\frac{4}{5}y^8+\frac{4}{75}+\frac{1058}{75}y^6-\frac{181}{15}y^4+\frac{69}{25}y^2\right)x^2 \quad (51)$$
$$-\frac{4}{5}y^2x\left(y^6+\frac{37}{15}y^4-\frac{7}{2}y^2+\frac{11}{10}\right)+\frac{13}{75}y^8-\frac{7}{45}-\frac{1}{50}y^4$$
$$+\frac{7}{150}y^2\Big)ma^2Re+6912\beta ma^2y\Big)\tau^2\Big),$$

the solutions series obtained by YRDTM is

$$u_2(x,y,\tau)=\sum_{k=0}^{2}U(x,y,k),$$
$$v_2(x,y,\tau)=\sum_{k=0}^{2}V(x,y,k), \quad (52)$$
$$g_2(x,y,\tau)=\sum_{k=0}^{2}G(x,y,k).$$

All of the $[L/M]$ t-Padé approximant of (52), with $L=0, M=2$, and $L=0, M=1$, respectively for u, v, and g yields:

$$\left[\frac{L}{M}\right]u(x,y,\tau)$$
$$=\left(320Re^2y^2\left(y^2-\frac{1}{2}\right)^2x^4\lambda^4\right)\Big/\Bigg(11520y\Bigg(\left(\left(y^2-\frac{1}{6}\right)\left(y^4-\frac{1}{3}y^2+\frac{1}{6}\right)\tau^2x^8\right.$$
$$-4\left(y^2-\frac{1}{6}\right)\left(y^4-\frac{1}{3}y^2+\frac{1}{6}\right)\tau^2x^7-5\left(y^8-\frac{118}{45}y^6+\frac{7}{6}y^4-\frac{37}{90}y^2+\frac{1}{30}\right)\tau^2x^6$$
$$+15\left(y^8-\frac{76}{45}y^6+\frac{7}{10}y^4-\frac{11}{54}y^2-\frac{1}{135}\right)\tau^2x^5+4\left(y^{10}-\frac{53}{8}y^8+\frac{293}{36}y^6\right.$$
$$\left.-\frac{143}{48}y^4+\frac{31}{48}y^2-\frac{1}{144}\right)\tau^2x^4-8y^2\tau^2x^3\mu\xi\left(y^6-\frac{5}{2}y^4+\frac{17}{18}y^2-\frac{5}{36}\right)$$
$$+\frac{58}{9}y^2x^2\tau^2\mu\xi\left(y^6-\frac{201}{116}y^4+\frac{71}{116}y^2-\frac{7}{232}\right)-\frac{22}{9}y^4\tau^2\mu^2\xi^2\left(y^2-\frac{1}{2}\right)x$$
$$+\frac{1}{3}\left(y^2-\frac{1}{2}\right)\left(y^8\tau^2-2y^6\tau^2+\tau^2y^4+\frac{1}{384}\right)\Bigg)x^2\lambda^2Re^2-576\Bigg(\left(\left(y^2-\frac{1}{6}\right)x^9\right.$$
$$+\left(\frac{3}{4}-\frac{9}{2}y^2\right)x^8+\left(\frac{1}{9}+59y^4-\frac{34}{3}y^2\right)x^7+\left(-\frac{35}{9}+\frac{182}{3}y^2-\frac{413}{2}y^4\right)x^6$$
$$+\left(\frac{215}{36}+70y^6+204y^4\right)x^5+\left(-\frac{125}{36}-\frac{25}{4}y^2-175y^6+\frac{25}{4}y^4\right)x^4$$
$$+\left(\frac{25}{36}+10y^8+\frac{1165}{9}y^6-\frac{275}{3}y^4+\frac{110}{3}y^2\right)x^3-15y^2\left(y^6+\frac{23}{18}y^4\right.$$
$$\left.-\frac{67}{36}y^2+\frac{37}{36}\right)x^2+\left(\frac{55}{36}y^2+\frac{20}{3}y^8-\frac{155}{18}y^6+\frac{95}{36}y^4\right)x-\frac{5}{6}y^4\mu^2\xi^2\Bigg)\tau^2Re \quad (53)$$
$$+720\left(x^2+\frac{1}{6}y^2-x+\frac{1}{12}\right)y\tau^2\Bigg),$$

$$\begin{aligned}
&\left[\frac{N}{M}\right]v(x,y,\tau) \\
&=-\left(320\beta^2\lambda^2\xi^2\mu^2y^4x^2Re\right)\Big/\Bigg(240\tau yRe\left(y^4-\frac{1}{3}y^2+\frac{1}{6}\right)x^8 \\
&\quad-960\tau yRe\left(y^4-\frac{1}{3}y^2+\frac{1}{6}\right)x^7-320\tau yRe\left(y^6-6y^4+2y^2-\frac{3}{4}\right)x^6 \\
&\quad+960\tau\left(Rey^7-\frac{5}{2}Rey^5+\frac{5}{6}Rey^3-\frac{1}{6}yRe-\frac{1}{80}\right)x^5 \\
&\quad-1120\tau\left(Rey^7-\frac{12}{7}Rey^5+\frac{4}{7}Rey^3-\frac{1}{28}Rey-\frac{3}{112}\right)x^4 \\
&\quad+\left(640Re\tau y^7-960Re\tau y^5+10Rey^4+320Re\tau y^3\right. \\
&\quad\left.+\left(-10Re-240\tau\right)y^2+20\tau\right)x^3+\left(-160Re\tau y^7+240Re\tau y^5\right. \\
&\quad\left.-15Rey^4-80Re\tau y^3+\left(15Re+360\tau\right)y^2-60\tau\right)x^2 \\
&\quad+\left(\left(5Re-60\tau\right)y^4+\left(-5Re-60\tau\right)y^2+20\tau\right)x+30\tau y^4-30\tau y^2\Bigg)
\end{aligned} \tag{54}$$

$$\begin{aligned}
&\left[\frac{L}{M}\right]g(x,y,\tau) \\
&=-\Bigg(320000y^2x^2\Bigg(Rey\left(y^4-\frac{9}{10}y^2+\frac{3}{10}\right)x^7-4Rey\left(y^4-\frac{9}{10}y^2+\frac{3}{10}\right)x^6 \\
&\quad+\frac{4}{5}yRe\left(y^6+\frac{11}{2}y^4-\frac{23}{4}y^2+\frac{9}{4}\right)x^5+\left(-\frac{12}{5}Rey^7+\frac{4}{5}Rey^5+\frac{6}{5}Rey^3\right. \\
&\quad\left.-\frac{6}{5}yRe-\frac{3}{50}\right)x^4+\left(\frac{3}{20}+\frac{13}{5}Rey^7-\frac{21}{5}Rey^5+\frac{17}{10}Rey^3+\frac{3}{10}yRe\right)x^3 \\
&\quad-\frac{6}{5}y^2\left(Rey^5-2Rey^3+yRe+\frac{1}{6}\right)x^2+\left(\frac{1}{5}Rey^7-\frac{2}{5}Rey^5+\frac{1}{5}Rey^3\right. \\
&\quad\left.-\frac{3}{20}+\frac{3}{10}y^2\right)x-\frac{1}{10}y^2+\frac{1}{20}\Bigg)^2\Bigg)\Bigg/\Bigg(500Re\left(Rey^6+\left(-\frac{9}{10}Re-30\tau\right)y^4\right. \\
&\quad\left.+\left(\frac{3}{10}Re+\frac{54}{5}\tau\right)y^2-\frac{3}{5}\tau\right)x^8-2000Re\left(Rey^6+\left(-\frac{9}{10}Re-30\tau\right)y^4\right. \\
&\quad\left.+\left(\frac{3}{10}Re+\frac{54}{5}\tau\right)y^2-\frac{3}{5}\tau\right)x^7+400\left(Rey^8\left(\frac{11}{2}Re-126\tau\right)y^6\right. \\
&\quad\left.+\left(-\frac{23}{4}Re102\tau\right)y^4+\left(\frac{9}{4}Re+48\tau\right)y^2-\frac{9}{2}\tau\right)Rex^6 \\
&\quad-1200\left(Rey^8+\left(-\frac{1}{3}Re-126\tau\right)y^6+\left(-\frac{1}{2}Re+73\tau\right)y^4+\left(\frac{1}{2}Re-15\tau\right)y^2\right. \\
&\quad\left.+\frac{1}{40}y-\tau\right)Rex^5+1300\left(\left(Re-\frac{120}{13}\tau\right)y^8+\left(-\frac{21}{13}Re-\frac{1388}{13}\tau\right)y^6\right. \\
&\quad\left.+\left(\frac{17}{26}Re+\frac{1320}{13}\tau\right)y^4+\left(\frac{3}{26}Re-\frac{372}{13}\tau\right)y^2+\frac{3}{52}y-\frac{3}{13}\tau\right)Rex^4 \\
&\quad-600y\left(Re\left(Re-40\tau\right)y^7-2Re\left(Re+\frac{64}{3}\tau\right)y^5+Re\left(Re+80\tau\right)y^3\right.
\end{aligned}$$

$$+\frac{1}{6}Re\,y^2 32\tau-2\tau\Big)x^3+100y\big(Re(Re-156\tau)y^7-2R(Re-98\tau)y^5$$
$$+Re(Re-42\tau)y^3+\frac{3}{2}Re\,y^2-30Re\tau y-\frac{3}{4}Re18\tau\Big)x^2$$
$$+3600y\Big(Re\tau y^7-2Re\tau y^5+Re\tau y^3+\Big(-\frac{1}{72}Re+\frac{1}{6}\tau\Big)y^2+\frac{1}{144}Re \tag{55}$$
$$+\frac{1}{12}\tau\Big)x-200\tau\Big(Re\,y^7 2Re\,y^5+Re\,y^3-\frac{3}{4}+\frac{3}{2}y^2\Big)y\Big),$$

where $\xi=y-1, \mu=y+1, \beta=x-\frac{1}{2}$ and $\lambda=x-1$.

7. Convergence Analysis

In this work, we study the convergence analysis of the approximate analytical solutions calculated by the PYRDTM application. The following theorems state the necessary conditions for the series solution to converge.

Theorem 1. Let F be an operator from a Hilbert space H into H Then, the series solution $\{S_n\}_{n=0}^{\infty}$ converges when there is γ such that $0<\gamma<1$, $\gamma=\gamma_1+\gamma_2+\gamma_3$, and $\|\mathcal{B}_{j(k+1)}\|\le\gamma_j\|\mathcal{B}_{jk}\|$. (see for proof Ref. [26]).

Definition 1 [26]. In the case of $j=1,2,3$ and $k\in\mathbb{N}\cup\{0\}$, we define

$$\gamma_{ik}=\begin{cases}\dfrac{\|\mathcal{B}_{j(k+1)}\|}{\|\mathcal{B}_{jk}\|}, & \|\mathcal{B}_{jk}\|\ne 0,\\ 0, & \|\mathcal{B}_{jk}\|=0.\end{cases}$$

After that, we state that the sequence of approximations $\{\mathbf{S}_n\}_0^{\infty}$ converges to the exact solution $u(x,y,t)$ when $\gamma_k=\gamma_{1k}+\gamma_{2k}+\gamma_{3k}$ and $0<\gamma_k<1$ for all $k\in\mathbb{N}\cup\{0\}$.

8. Discussion

In this part, we provide the numerical calculations of the velocity components u, v, the vorticity function ω, and the stream function ψ that were obtained by using PYRDTM. All calculations are performed using the Maple 2016 program with varying Reynolds in the domain $[0,1]^2$. **Figure 1** and **Figure 2** shows that the approximate solution of u and v obtained by applying the suggested method at $t=2$ and $Re=10$ for different values of Mach numbers ($Ma=0.01, 0.05$ and 0.1), and **Figure 2** at $R=1$ and $Ma=0.01$ for different values of t (t = 0.1, 1, and 2), and at [0/2], on two iterations, the drawing was found. In **Table 2**, we examined the estimated values of u velocity along the vertical line and v velocity along the horizontal line at the geometric center of the square cavity that were obtained by using PYRDTM. The results show these values are identical to those given in KRDRM, and KPIM with at least three-digit at $t=0.1$ and $Ma=0.01$ for different values of Renold numbers, and these results represent PYRDTM solutions $L=2$ and $M=1$ for three iterations also the found results were compared by the RDTM method and we got the same results as the KRDTM

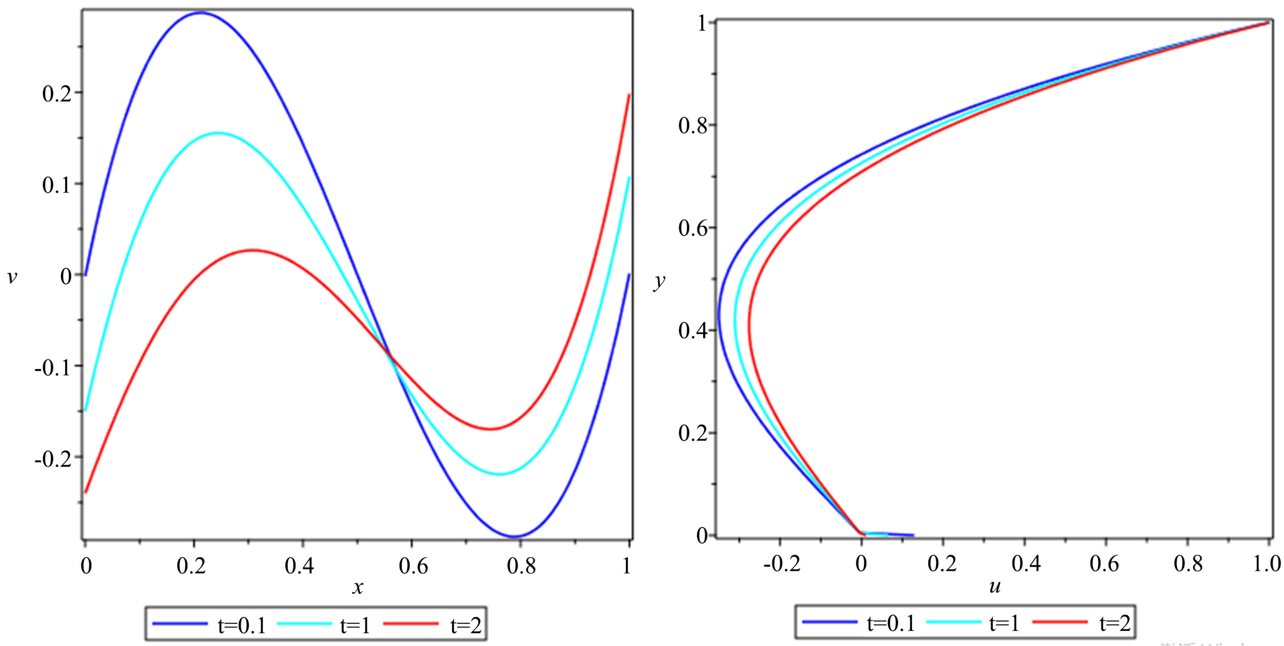

Figure 1. PYRDTM solution [0/2] of $u(0.5,y,t)$ and $v(x,0.5,t)$ at $Re=1$.

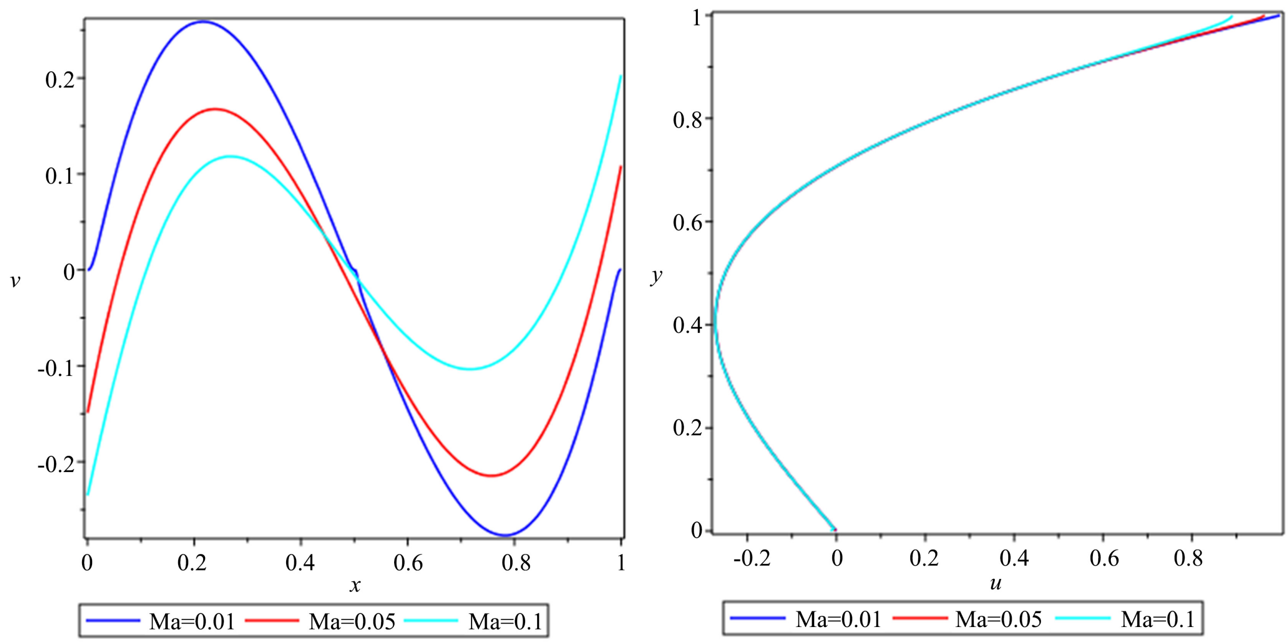

Figure 2. PYRDTM solutions [0/2] of $u(0.5,y,2)$ and $v(x,0.5,2)$ at $Re=10$.

Table 2. Comparison the value of u and v between PYRDTM [0/2] and others method, at $t=0.1$.

	PYRDTM		KRDTM [26]		KPIA [26]	
$u(0.5,y,0.1)$	$Re=10$	$Re=100$	$Re=10$	$Re=100$	$Re=10$	$Re=100$
0.0625	−0.06201162624	−0.06201171782	−0.062011623	−0.0620117178	−0.0620116267	−0.0620117178
0.125	−0.1210934186	−0.1210937382	−0.121093416	−0.1210937382	−0.1210934190	−0.1210937382
0.1875	−0.1743158597	−0.1743163946	−0.174315857	−0.1743163948	−0.1743158600	−0.1743163948

Continued

0.25	−0.2187492696	−0.2187500031	−0.218749267	−0.2514648747	−0.2187492699	−0.2187500032
0.3125	−0.2514639637	−0.2514648744	−0.251463968	−0.2514648747	−0.2514639639	0.25146487477
0.375	−0.2695302502	−0.2695313136	−0.269530253	−0.2695313138	−0.2695302503	−0.2695313137
0.4375	−0.2700184293	−0.2700196175	−0.269530253	−0.2700196175	−0.2700184292	−0.2700196175
0.5	−0.2499987981	−0.2500000797	−0.249998791	−0.2500000797	−0.2499987975	−0.2500000797
0.5625	−0.2065416606	−0.2065430027	−0.206541667	−0.2065430027	−0.2065416599	−0.2065430027
0.625	−0.1367173508	−0.1367187179	−0.136717358	−0.1367187180	−0.1367173502	−0.1367187180
0.6875	−0.03759625741	−0.037597612863	−0.037596254	−0.0375976129	−0.0375962576	−0.0375976130
0.75	0.09375115245	0.09374984662	0.0937511524	0.0937498465	0.0937511505	0.0937498462
0.8125	0.2602543373	0.2602531197	0.2602543373	0.2602531196	0.2602543328	0.2602531191
0.875	0.4648427530	0.4648416624	0.4648427531	0.4648416627	0.4648427455	0.4648416618
0.9375	0.7104460576	0.7104451334	0.7104460577	0.7104451334	0.7104460475	0.7104451322
$v(x,0.5,0.1)$						
0.0625	0.1520352201	0.1535990206	0.1520352246	0.1535990206	0.1520352173	0.1535990205
0.125	0.2443698722	0.2458276284	0.2443698776	0.2458276287	0.2443698698	0.2458276285
0.1875	0.2840233720	0.2853353438	0.2840233785	0.2853353452	0.2840233713	0.2853353451
0.25	0.2797924832	0.2809293342	0.2797924904	0.2809293365	0.2797924846	0.2809293364
0.3125	0.2404575708	0.2413982092	0.2404575825	0.2413982264	0.2404575786	0.2413982263
0.375	0.1747890526	0.1755190030	0.1747890513	0.1755190079	0.1747890492	0.1755190079
0.4375	0.09155465640	0.09206486648	0.0915546641	0.0920648498	0.0915546634	0.0920648498
0.5	−0.0004740105330	−0.0001881204	−0.000474000	−0.0001880250	−0.0004739999	−0.0001880249
0.5625	−0.09251736322	−0.0924557555	−0.092517322	−0.0924556955	−0.0925173258	−0.0924556954
0.625	−0.1757701738	−0.1759451442	−0.175786450	−0.1759449258	−0.1757864500	−0.1759449258
0.6875	−0.2414858424	−0.2418545148	−0.241485327	−0.2418548488	−0.2414853294	−0.2418548488
0.75	−0.2808140296	−0.2813869254	−0.280815842	−0.2813816962	−0.2808158450	−0.2813816962
0.8125	−0.2849842230	−0.2857143754	−0.284985144	−0.2857257912	−0.2849851463	−0.2857257912
0.875	−0.2452147886	−0.2459434762	−0.245213908	−0.2460994800	−0.2452139127	−0.2460994800
0.9375	−0.1527425132	−0.1537296438	−0.152743654	−0.1537341432	−0.1527436637	−0.1537341433

method as in the source [26]. **Table 3** shows values of u and v that be obtained by using PYRDTM at $Ma = 0.01$, $R = 1$, $t = 0.1$, and [0/2], at two iterations are remarkably good with those given by applying the finite volume method and defined by [31]. **Table 4** shows the L_∞ errors for the stream function ψ and the vorticity ω, for three different values of Reynolds numbers $Re = 10,100$ and 1000 at $Ma = 0.001$. The computed errors by the rational fourth-order compact finite difference approach in [31] and RDTM are compared. When compared to the previous method, we can observe that the errors calculated using the new methodology is significantly lower for all Reynolds numbers. According to the calculations presented in the tables and figures, the analytical approximation

Table 3. Comparison the approximate solutions between PYRDTM [0/2] and other methods at $t = 0.1$, and $Re = 1$.

	Ref. [9]	KRDTM [26]	KPIA [26]	PYRDTM
$\psi_{\min}$	−0.125	−0.1263030704	−0.1262878352	−0.127127418
$x(\psi_{\min})$	0.5	0.5	0.5	0.5
$y(\psi_{\min})$	0.70703	0.70703125	0.703125	0.703125
$u_{\min}$	−0.2721659	−0.2720274424	−0.2720273443	−0.271193052
$y(u_{\min})$	0.40869	0.41015625	0.40625	0.49995412
$v_{\min}$	−0.2886756	−0.365196882	−0.3649970593	−0.2814026232
$x(v_{\min})$	0.78857	0.78515625	0.78125	0.78977638
$v_{\max}$	0.2886756	0.3590485063	0.3588814265	0.2746767493
$x(v_{\max})$	0.21143	0.21484375	0.21875	0.1332080151
$u(0.5;0.0625)$	−0.0620117187	−0.0619894102	−0.0619894551	−0.0619894182
$u(0.5;0.125)$	−0.1210937499	−0.1210481233	−0.1210481633	−0.1210481405
$u(0.5;0.1875)$	−0.1743164062	−0.1742486127	−0.1742486451	−0.1742486389
$u(0.5;0.25)$	−0.2187499999	−0.2186617447	−0.2186617667	−0.2186617802
$u(0.5;0.3125)$	−0.2514648436	−0.2513583768	−0.2513583858	−0.2513584216
$u(0.5;0.375)$	0.2695312499	−0.2694093550	−0.2694093550	−0.2694094100
$u(0.5;0.4375)$	−0.2700195312	−0.2698855135	−0.2698855135	−0.2698855797
$u(0.5;0.5)$	−0.2500000000	−0.2498576781	−0.2498576781	−0.2498577590
$u(0.5;0.5625)$	−0.2065429687	−0.2063966790	−0.2063966790	−0.2063967824
$u(0.5;0.625)$	−0.1367187500	−0.1365733705	−0.1365733705	−0.1365735248
$u(0.5;0.6875)$	−0.0375976562	−0.0374586584	−0.0374586584	−0.0374591704
$u(0.5;0.75)$	0.0937499999	0.0938764774	0.0938764774	0.09387664829
$u(0.5;0.8125)$	0.2602539062	0.2603609875	0.2603609875	0 .2603610316
$u(0.5;0.875)$	0.4648437499	0.4649238234	0.4649238234	0.4649238371
$u(0.5;0.9375)$	0.7104492187	0.7104941432	0.7104941432	0.7104941459
$v(x,0.5,0.1)$				
$v(0.0625;0.5)$	0.15380859374	0.1365030994	0.1365023819	0.1365020622
$v(0.125;0.5)$	0.24609374999	0.2298505049	0.2298497694	0.2298500264
$v(0.1875;0.5)$	0.28564453123	0.2709291888	0.2709285329	0.2709289686
$v(0.25;0.5)$	0.28124999999	0.2684293744	0.2684288605	0.2684292722
$v(0.3125;0.5)$	0.2416992187	0.2310463824	0.2310460353	0.2310463296
$v(0.375;0.5)$	0.17578125000	0.1674821344	0.1674819447	0.1674781124
$v(0.4375;0.5)$	0.09228515625	0.0864478792	0.0864478112	0.0864478251
$v(0.5;0.5)$	2.3e−14	−0.00333375	−0.0033337499	−0.0003333899
$v(0.5625;0.5)$	−0.09228515625	−0.0931287051	−0.0931286946	−0.0931295302
$v(0.625;0.5)$	−0.17578125000	−0.1741943413	−0.1741943774	−0.1741938245

Continued

$v(0.6875;0.5)$	−0.24169921874	−0.2377853663	−0.2377854934	−0.2377852184
$v(0.75;0.5)$	−0.28124999998	−0.2751626592	−0.2751629021	−0.2751622548
$v(0.8125;0.5)$	−0.28564453123	−0.2776041425	−0.2776044993	−0.2776041702
$v(0.875;0.5)$	−0.24609374999	−0.2364163347	−0.2364167719	−0.2364156354
$v(0.9375;0.5)$	−0.15380859374	−0.1429446548	−0.1429451057	−0.1429438657

Table 4. Comparisons of L_∞-error between PYRDTM [0/2] and other methods at $t = 0.1$, and $Ma = 0.001$.

Grid size	Ref. [31]		KRDTM [26]		KPIA [26]		PYRDTM		RDTM	
	ψ	ω	ψ	ω	ψ	ω	ψ	ω	ψ	ω
$Re = 10$										
21 × 21	3.23e−7	1.01e−5	2.01e−8	8.13e−7	5.84e−8	1.59e−6	2.00E−09	7.72E−09	5.82E−8	1.48E−6
41 × 41	2.35e−8	7.74e−7	3.41e−8	9.77e−7	7.23e−8	1.85e−6	7.23E−09	5.05E−08	5.82E−8	1.51E−6
81 × 81	1.56e−9	5.17e−8	4.46e−8	1.08e−7	9.29e−8	2.00e−6	2.47E−08	3.00E−07	5.82E−8	1.51E−6
$Re = 100$										
21 × 21	8.09e−5	4.08e−3	4.25e−9	3.62e−7	4.25e−9	3.62e−7	4.69E−11	1.13E−08	6.45E−09	1.59E−07
41 × 41	7.12e−6	2.51e−4	5.11e−9	3.62e−7	5.11e−9	3.62e−7	3.86E−10	1.67E−08	6.45E−09	1.62E−07
81 × 81	4.93e−7	1.72e−5	5.11e−9	3.64e−7	5.66e−9	3.64e−7	1.13E−09	1.61E−07	6.45E−09	1.62E−07
$Re = 1000$										
41 × 41	3.32e−4	1.45e−2	5.07e−9	3.65e−7	5.07e−9	3.64e−7	1.35E−09	3.60E−09	6.50E−09	1.61E−07
81 × 81	3.92e−5	1.66e−3	5.74e−9	3.64e−7	5.74e−9	3.64e−7	2.31E−10	2.53E−07	6.50E−09	1.62E−07
161 × 161	2.79e−6	1.46e−4	6.11e−9	3.64e−7	6.11e−9	3.64e−7	1.96E−09	1.57E−07	6.50E−09	1.62E−07

Table 5. Comparisons of Convergence solutions between PYRDTM[2/1], RDTM, KPLM, and KRDTM.

	PYRDTM	RDTM	KRDTM [26]	KPIA[26]
$Re = 1, Ma = 0.1$ and $t = 0.1$				
γ_0	0.9966346559	0.908851949	0.9991283888	0.9991283888
γ_1	0.7179153236	0.814301296	0.7958329986	0.7958329986
$Re = 10, Ma = 0.1$ and $t = 0.01$				
γ_0	0.0122832227	0.0131350980	0.0129736262	0.0129736262
γ_1	0.0040514387	0.0118098473	0.2936820858	0.2936820858

solution obtained with the new approach is remarkably accurate. In addition, we discovered that PYRDTM is an efficient and effective approach to solve the nonlinear two-dimensional unsteady incompressible Navier-Stokes equation for low Mach numbers and for various Reynolds numbers. We successfully employed definition (1) to discuss the convergence of the suggested method; this is explained in Table 5.

9. Conclusion

In this work, we applied a new approach that combines a reduced differential transform method (RDTM), a resummation method based on the Yang transform, and a Padé approximant to KRLNS equations to find approximate analytical solutions to the lid-driven square cavity flow problem. Using YRDTM, the solution of PDE is first obtained in convergent series forms. This technique then employs a posttreatment Padé approximant to broaden the field of convergence of truncated power series, which we name PYRDTM, considerably improves the convergence rate of the RDTM truncated series solution. One of the advantages of the suggested method is that it reduces the number of iterations required to find approximate analytical solutions. According to the calculations shown in the tables and figures, The PYRDTM procedures are very effective in resolving unstable "viscous incompressible flow problems at low Mach numbers" and for various Reynolds numbers. Finally, further research should be performed to solve fractional differential equations, which are currently widely used.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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