

Poisson Process Modeling of Pure Jump Equities on the Ghana Stock Exchange

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Abstract

Although Geometric Brownian Motion and Jump Diffusion Models have largely dominated the literature on asset price modeling, studies of the empirical stock price data on the Ghana Stock Exchange have led to the conclusion that there are some stocks in which the return processes consistently depart from these models in theory as well as in its statistical properties. This paper gives a fundamental review of the development of a stock price model based on pure jump processes to capture the unique behavior exhibited by some stocks on the Exchange. Although pure jump processes have been examined thoroughly by other authors, there is a lack of mathematical clarity in terms of deriving the underlying stock price process. This paper provides a link between stock prices existing on a measure space to its development as a pure jump Levy process. We test the suitability of the model to the empirical evidence using numerical procedures. The simulation results show that the trajectories of the model are a better fit for the empirical data than those produced by the diffusion and jump diffusion models.

Keywords

Poisson Process, Pure Jump Process, Compound Poisson Process, Jump Diffusion

1. Introduction

The study of asset price behavior is a fundamental link in the development of derivative pricing models. Consequently, having a model that accurately captures the behavior of the underlying asset is crucial in stock price modeling. A time series observation of the empirical distribution of some stock returns on the Ghana Stock Exchange (GSE) exhibits behavior that significantly deviates from

existing normality assumption models. In this study, we develop a model to capture the observed behavior and proceed to test its suitability using numerical procedures. The basic premise in Merton's Jump Diffusion Model (JDM) is that the total change in an asset price is driven by a continuous and pure jump process. However, empirical observation of the stock price on the Ghana stock exchange shows that changes in some stock prices are driven strictly by jumps. It is therefore, imperative that we examine pure jump models to analyse stock behavior on the Exchange. One problem with the pure jump model is that researchers hardly give the mathematical background leading to the derivation of the model. This poses a problem for practitioners with limited mathematical backgrounds. In this study, we focus on deriving the pure jump model mainly from an abstract framework and give a step-by-step approach using basic mathematical language. The aim is to provide an accessible overview of the fundamentals of the pure jump model to the non-specialist reader and bridge the link between the abstract conception of the stock price process and the corresponding Levy process. To achieve this, emphasis is placed on certain aspects in the development of the pure jump model, specifically from the abstract perspective, that has not been well addressed in previous studies. In addition, although models with jumps in the underlying asset have been used extensively in asset price modeling, especially when market returns have only discontinuities, practical developments in this area are rare as most stocks have sufficiently high frequency data to depart from purely discontinuous processes. This arises because European and American markets are mostly characterized by high level trading activity resulting in high frequency data such that although the underlying assets evolve structurally as discrete, they can without any loss of generality be approximated by diffusion and jump diffusion models. This is, however, not the case in some African markets such as Nigeria, Kenya and more specifically, as observed on the GSE. Recent empirical data and time series analysis from the exchange has shown that stocks assume specific prices for long periods and do not change for a considerable length of time. Price changes are irregular and lead to either small or large price jumps. To demonstrate these empirical observations, we present in **Figures 1-3** plots of a five-year time series of the actual movement of six trading stocks: Ecobank Transnational Inc, HFC Bank, SIC Insurance Limited, SG-SSB, Cocoa Processing Company and Starwin Products Limited totaling 1174 of trading days on GSE, from July 2013 to July 2018.

Figure 1(a) and **Figure 1(b)** show the plots of stock prices of Ecobank Transnational and HFC Bank within the trading period. Clearly, the trajectories are purely continuous and are suitably modeled by Geometric Brownian Motion. Pricing follows diffusion processes and is modeled under Black-Scholes option pricing model. **Figure 2(a)** and **Figure 2(b)** also show the paths of SIC Insurance Ltd and SG-SSB Limited during the same trading period. The trajectories are continuous but punctuated by occasional jumps at arbitrary time intervals. The pricing trajectories are thus suitably modeled by Merton's Jump Diffusion processes. On the other hand, the plots in **Figure 3(a)** and **Figure 3(b)** show the

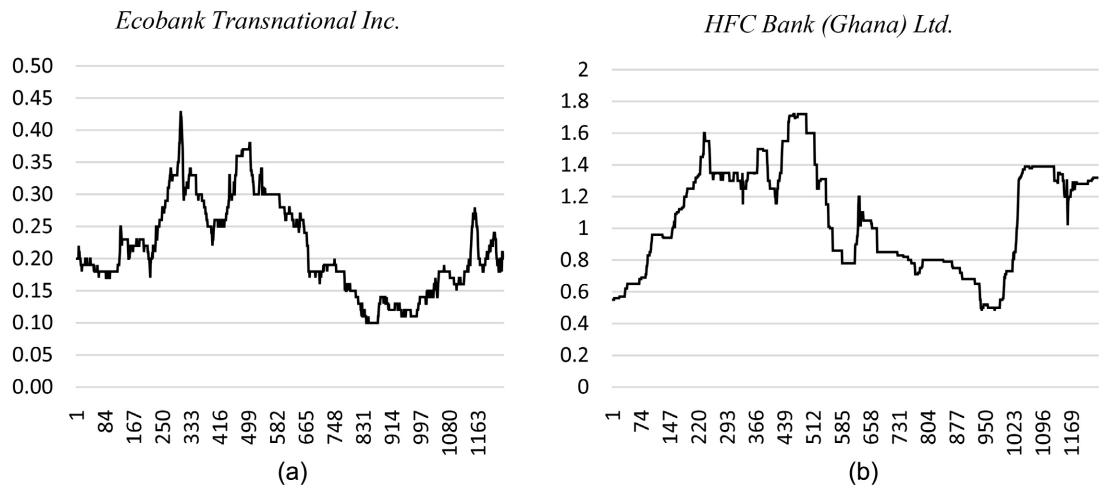


Figure 1. (a & b) Trajectories of Ecobank transnational and HFC—from July 2013 to July 2018. Source: Annual reports Ghana.

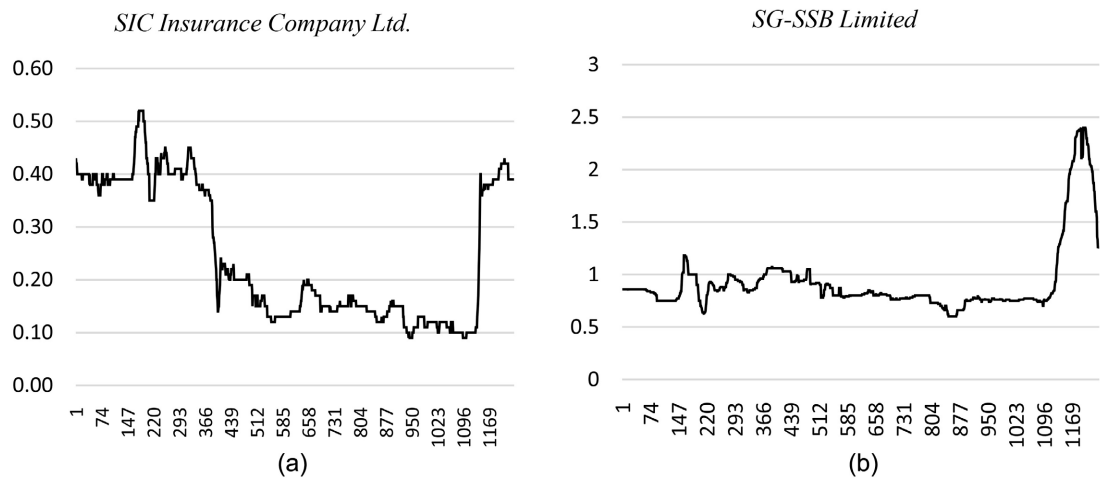


Figure 2. (a & b) Trajectories of SIC insurance Ltd and SG-SSB limited: July 2013-July 2018. Source: Annual reports Ghana.

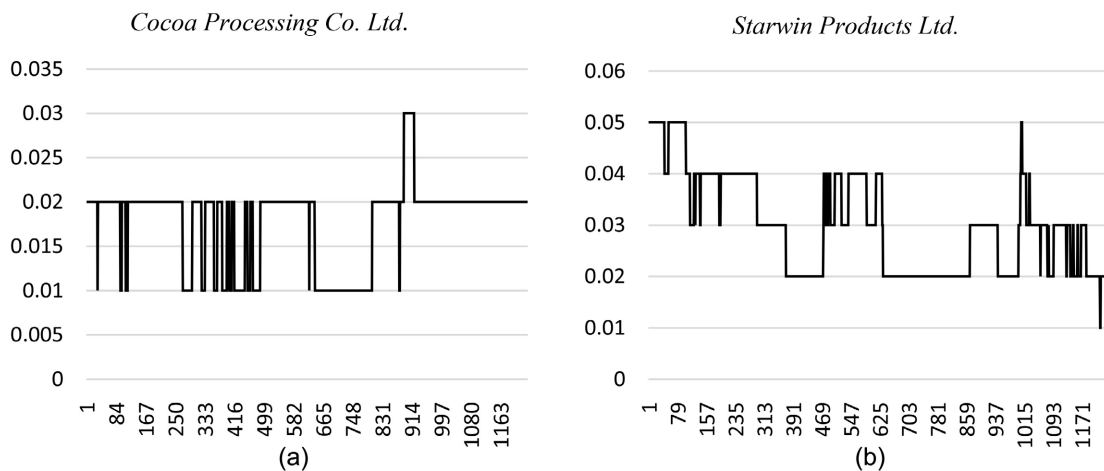


Figure 3. (a & b) Trajectories of cocoa processing company and Starwin products: July 2013-July 2018. Source: Annual reports Ghana.

trajectories of Cocoa Processing Company and Starwin Products Limited for the same period. It is observed that the price paths have only jumps occurring at different times and the stock price either goes up or come down. In between these occasional jumps the price of the stock remains constant. Clearly, the trajectories are dimensionally different and cannot be modeled using diffusion or jump diffusion models. To this end, we develop the pure jump processes which captures all the empirical facts associated with such stocks. Strictly speaking, the model is characterized purely by discontinuities such that at any given point in time, the sample paths are almost surely discontinuous with points of continuity forming a set of zero measures. The paper is organized as follows: In Section 2, we develop the pure jump process from the framework of an abstract probability space. This is done under the physical probability measure which describes the evolution of an arrival process into a Poisson jump process and consequently a compound Poisson process. Section 2.2 considers the compound Poisson process under the levy measure and derives the characteristic functions and the Levy-Ito decomposition. Section 3 discusses the results of the simulation of pure jumps for stocks listed on GSE.

2. Literature Review

Models of stock price processes with the Geometric Brownian Motion (GBM) were proposed by [1] and popularized as a pricing model in the [2] seminal paper. However, to provide a realistic description of empirical stock price dynamics and accommodate several practical situations many extensions have been made to the GBM. The most popular are Jump Diffusion Models (JDM), stochastic volatility, and regime switching models. Models with jumps have their roots from [3], in which a compound Poisson process modeling pure jumps is added to the continuous GBM. Whilst Merton assumed a normal distribution for the jump sizes, [4] demonstrates that the dynamics of the asset price can be modeled as a combination of GBM and a Poisson process such that jump sizes have double exponential distribution. The Merton model underlying jump diffusion models has since been thoroughly examined by many authors. [5] developed a discrete time version of Merton's jump diffusion model so that the values of the discrete model converge weakly to the desired corresponding continuous time model values. [6] further showed that the sequence of American option values obtained from discrete-time models also converges to the corresponding value obtained from the continuous-time model. Option pricing on Jump Diffusion Models has been examined extensively leading to closed form analytical solutions or numerical simulations of European and American call-and-put options. [7] [8] presented closed form solutions of options written on JDMs. [9] presented a finite difference method for solving parabolic partial integro-differential equations of underlying assets driven by jump-diffusions. The Variance-Gamma (VG) model of [10] and the CGMY model of [11] have all proposed options prices on pure jump processes.

Stochastic volatility models that generalize the traditional GBM by allowing the constant volatility to be stochastic have been presented by [12] [13] [14]. [15] in particular, allows for a volatility risk premium that is proportional to the square root of the stochastic variance. Options pricing models under stochastic volatility have been further examined by [16] [17]. Monte–Carlo simulations that numerically analyse the pricing of options in this regime have been examined extensively by [18] [19] [20]. Regime switching models that account for markets that may switch from time to time between a stable low-volatility state and an unstable high-volatility regime have been proposed by [21] and further examined by [22] [23] [24] [25]. The class of hyperbolic distributions that capture empirical data following pure jump distributions and characterized by infinitely divisible fat-tails have been introduced in [26] [27]. Levy models that can rigorously analyse pure jumps and produce explicit computations of finite results have been analyzed [28] [29] [30].

3. Methodology

3.1. The Model

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with filtration $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$, \mathcal{F}_t a sigma algebra generated by $\mathcal{F}_t = \sigma(X(s), 0 \leq s \leq t)$. All processes that we shall consider in this section will be defined in this space. Consider the price of a typical stock on the Ghana Stock Exchange specifically Starwin Limited, from June 1 2018 to June 30 2018 as shown in **Table 1**.

Table 1 shows that there were four price changes within a given month. Let's consider these prices changes as jumps and assign $\omega_1, \omega_2, \dots, \omega_n$ as the event that there is a jump in the stock price. Since stock price changes occur randomly it can be considered as a stochastic process. Let X_1, X_2, \dots, X_n be the real valued function of the random variables describing jumps in the stock price and let T_1, T_2, T_3, \dots represent the arrival times of the jumps as shown in **Figure 4**.

Suppose the process starts at time 0 then when the first jump event occurs at T_1 we wait for a time period S_1 until the next jump occurs at T_2 . The waiting time between the first and the second jump is S_2 and the waiting time between the second and third jump is S_3 and so on. Suppose S_i 's are independent and the T_i 's are such that T_0 is the time we start to observe the process and $T_0 = 0$. Then we can write

$$\begin{aligned} T_1 &= S_1 \\ T_2 &= S_1 + S_2 \\ T_3 &= S_1 + S_2 + S_3 \\ &\vdots \\ T_n &= S_1 + S_2 + S_3 + \dots + S_n \end{aligned}$$

So that

$$T_n = \sum_{i=1}^n S_i$$

Table 1. Typical trading day stock prices.

Trading Day	1	2	3	4	5	6	7	8	9	10
Stock Price	2.06	2.06	2.06	2.06	2.06	2.08	2.08	2.08	2.08	2.13
Trading Day	11	12	13	14	15	16	17	18	19	20
Stock Price	2.13	2.13	2.13	2.19	2.19	2.19	2.19	2.57	2.57	2.78

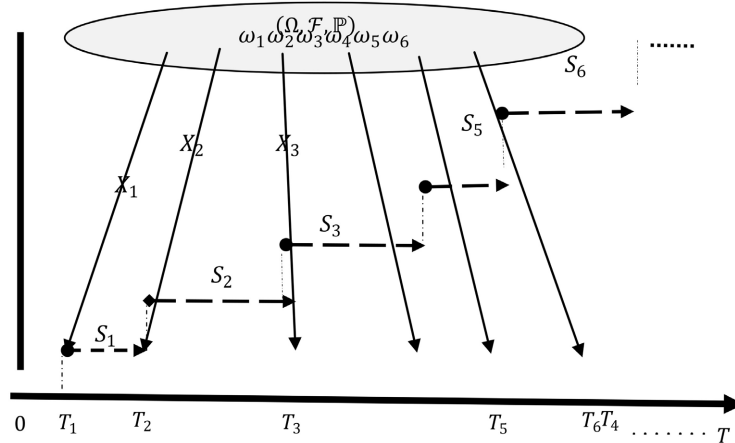


Figure 4. Abstract representation of the stock price process.

Let N_t represent a process that counts the arrival of a jump event and let the value of the process at time t be equal to j . Then we can write $N_t = j$ and the probability of the event as $\mathbb{P}[N_t = j]$. In the time step Δt , there are three possibilities before the next observation:

- 1) One jump occurs and the process value increase by 1 to $j + 1$.

$$N_{(t+\Delta t)} = j + 1$$

- 2) More than one jump occurs and the process value increases to more than $j + 1$.

$$N_{(t+\Delta t)} > j + 1$$

- 3) No jump occurs and the value stays the same.

$$N_{(t+\Delta t)} = j$$

If one jump occurs then in an infinitesimal time interval the probability of j increasing by 1, conditional on the number of jumps at t is given by

$$\mathbb{P}[N_{(t+\Delta t)} = j + 1 | N_t = j]$$

Let's assume that the intensity of the jump process is λ , then we can write that

$$\mathbb{P}[N_{(t+\Delta t)} = j + 1 | N_t = j] = \lambda \Delta t + o(\Delta t)$$

As Δt is infinitesimal we expect the probability of the next jump (two or more jumps) to be negligible.

$$\mathbb{P}[N_{(t+\Delta t)} \geq j + 1 | N_t = j] = o(\Delta t)$$

$$\text{and } \mathbb{P}\left[N_{(t+\Delta t)} = j \mid N_t = j\right] = 1 - (\lambda\Delta t + o(\Delta t))$$

Now in the reverse order suppose the process value was $j - 1$ at time t and 1 jump occurred in Δt then

$$\begin{aligned} \mathbb{P}\left[N_t = j - 1\right] &= \lambda\Delta t + o(\Delta t) && \text{one jump occurred} \\ \mathbb{P}\left[N_t > j - 1\right] &= o(\Delta t) && \text{more than one jump occurred} \\ \mathbb{P}\left[N_t = j\right] &= 1 - \lambda\Delta t + o(\Delta t) && \text{no jump occurred} \\ \mathbb{P}\left[N_{(t+\Delta t)} = j \mid N_t = j - 1\right] &= \lambda\Delta t + o(\Delta t). \end{aligned}$$

Thus, the from time t to time $t + \Delta t$ we have

$$\begin{aligned} \mathbb{P}\left[N_{(t+\Delta t)} = j \mid N_t = j\right] &= 1 - (\lambda\Delta t + o(\Delta t)) \\ \mathbb{P}\left[N_{(t+\Delta t)} = j\right] &= (\lambda\Delta t + o(\Delta t))\mathbb{P}\left[N(t) = j - 1\right] \\ &\quad + (1 - (\lambda\Delta t + o(\Delta t)))\mathbb{P}\left[N_t = j\right] \end{aligned}$$

Let's denote $\mathbb{P}\left[N_t = j\right] = P_j(t)$ then

$$\begin{aligned} P_j(t + \Delta t) &= (\lambda\Delta t + o(\Delta t))P_{j-1}(t) + (1 - (\lambda\Delta t + o(\Delta t)))P_j(t) \\ P_j(t + \Delta t) &= \lambda\Delta t P_{j-1}(t) + (1 - \lambda\Delta t)P_j(t) + o(\Delta t)P_{j-1}(t) + o(\Delta t) \\ P_j(t + \Delta t) &= \lambda\Delta t P_{j-1}(t) + P_j(t) - \lambda\Delta t P_j(t) + o(\Delta t) \\ P_j(t + \Delta t) - P_j(t) &= \lambda\Delta t P_{j-1}(t) - \lambda\Delta t P_j(t) + o(\Delta t) \\ \frac{P_j(t + \Delta t) - P_j(t)}{\Delta t} &= \lambda P_{j-1}(t) - \lambda P_j(t) + \frac{o(\Delta t)}{\Delta t} \\ \lim_{\Delta t \rightarrow 0} \frac{P_j(t + \Delta t) - P_j(t)}{\Delta t} &= \lambda P_{j-1}(t) - \lambda P_j(t) \\ \frac{dP_j(t)}{dt} &= \lambda P_{j-1}(t) - \lambda P_j(t) \end{aligned} \tag{1}$$

Since j is an integer, we can have the following sequence

$$\begin{aligned} &\left\{ \frac{dP_0(t)}{dt} = -\lambda P_0(t) \right\}, \left\{ \frac{dP_1(t)}{dt} = \lambda P_0(t) - \lambda P_1(t) \right\}, \dots, \\ &\left\{ \frac{dP_n(t)}{dt} = \lambda P_{n-1}(t) - \lambda P_n(t) \right\} \end{aligned}$$

This system of equations can be written as a matrix

$$\begin{pmatrix} \frac{dP_0(t)}{dt} \\ \frac{dP_1(t)}{dt} \\ \vdots \\ \frac{dP_n(t)}{dt} \end{pmatrix} = \begin{pmatrix} -\lambda & 0 & 0 \\ \lambda & -\lambda & 0 \\ 0 & \lambda & -\lambda \end{pmatrix} \begin{pmatrix} P_0(t) \\ P_1(t) \\ P_2(t) \end{pmatrix} \tag{2}$$

Hence, we can write the system of differential equations as

$$\frac{d\mathbb{P}}{dt} = \mathbb{Q}\mathbb{P}(t) \tag{3}$$

where \mathbb{Q} is a matrix.

The process starts at zero with probability 1, that is $\mathbb{P}_0(0) = 1$,

$$\frac{d\mathbb{P}_0(t)}{dt} = -\lambda\mathbb{P}_0(t), \quad \mathbb{P}_0(0) = 1.$$

For $j > 0$ the condition is $\mathbb{P}_j(0) = 0$,

$$\frac{d\mathbb{P}_j(t)}{dt} = \lambda\mathbb{P}_{j-1}(t) - \lambda\mathbb{P}_j(t) \tag{4}$$

Essentially, the process starts at zero, so the probability of the process taking a value zero at time zero is 1 and the probability of the process taking any other value at time zero is 0 by the same logic.

Now from (4) we have

$$\begin{aligned} \frac{d\mathbb{P}_0(t)}{dt} &= -\lambda\mathbb{P}_0(t) \\ \mathbb{P}_0(t) &= \mathbb{P}_0(0)e^{-\lambda t} \end{aligned}$$

From the initial conditions $\mathbb{P}_0(0) = 1$ hence,

$$\mathbb{P}_0(t) = e^{-\lambda t} \tag{5}$$

Now from 4 $\frac{d\mathbb{P}_j(t)}{dt} = \lambda\mathbb{P}_{j-1}(t) - \lambda\mathbb{P}_j(t)$,

When

$$j = 1, \quad \frac{d\mathbb{P}_1(t)}{dt} + \lambda\mathbb{P}_1(t) = \lambda\mathbb{P}_0(t), \text{ but } \mathbb{P}_0(t) = e^{-\lambda t}.$$

$$j = 2, \quad \mathbb{P}_1(t) = \frac{1}{2}\lambda^2 t^2 e^{-\lambda t} = \frac{(\lambda t)^2}{2} e^{-\lambda t}.$$

$$j = 3, \quad \frac{d\mathbb{P}_3(t)}{dt} = \lambda\mathbb{P}_3(t) - \lambda \cdot \frac{1}{2}\lambda^2 t^2 e^{-\lambda t}.$$

Iteratively for

$$j = n, \quad \mathbb{P}_n(t) = \frac{(\lambda t)^n}{n!} e^{-\lambda t}.$$

We recognise this as the Poisson random variable. The random variables X_i 's representing the arrival of jumps are independently distributed such that if N_t counts the number of jumps in the interval $[0, T]$ then,

$$N_t = \max\{n : T_n \leq t\}.$$

If \mathcal{F}_t is the natural filtration of the process then,

- 1) $N_0 = 0$, almost surely,
- 2) N_t has independent increments, that is if $s < t$ then $N_t - N_s$ is inde-

pendent of \mathcal{F}_s ,

3) The increment $N_t - N_s$ is a Poisson distributed with parameter $\lambda(t - s)$ for all $s < t$.

N_t has density

$$\mathbb{P}(N_t = x) = \frac{(\lambda t)^x}{x!} e^{-\lambda t} \tag{6}$$

We have established that the stock process having distribution as in **Table 1**, follows the Poisson distribution. The distribution of waiting times between jumps is obtained by setting $x = 0$ in (6) and considering waiting at least t time units to see the first jump. This yield,

$$\mathbb{P}(X = 0) = \frac{(\lambda t)^0}{0!} e^{-\lambda t} = e^{-\lambda t}$$

$$\text{Hence } 1 - [\mathbb{P}(X = 0), T > t] = 1 - e^{-\lambda t}$$

Differentiating equation above we have

$$\mathbb{P}(T = t) = \lambda e^{-\lambda t}$$

3.2. The Jump Model

The Poisson process developed for the jump events appears too limited to obtain a realistic stock price model as its jump size is always equal to 1. In reality, jump sizes in stock prices vary hence there is some interest in considering jumps with random sizes. Consider the stock price S_t , understood to be right-continuous function with left limits so that S_{t-} represents the value of S_t just before a possible jump at t . We write S_t as the limit from the left, *i.e.*

$$S_{t-} = \lim_{s \uparrow t} S_t \tag{7}$$

Suppose that in the small-time interval Δt the stock price jumps by Y_t so that it jumps from S_t to $Y_t S_t$. The percentage change in the stock price is thus given by

$$\frac{\Delta S_t}{S_{t-}} = \frac{Y_t S_t - S_{t-}}{S_{t-}}$$

$$\frac{\Delta S_t}{S_{t-}} = \frac{Y_t S_t}{S_{t-}} - \frac{S_t}{S_{t-}}$$

$$\frac{\Delta S_t}{S_{t-}} = Y_t - 1$$

In the infinitesimal limit

$$\frac{dS_t}{S_{t-}} = Y_t - 1 \tag{8}$$

Y_t 's are non-negative random variables modeling the distribution of the jump sizes such that they are independently identically distributed and the trajectories of jump sizes are right-continuous with left limits. Equation (8) gives the relative

jump amplitude or percentage change in stock price as $(Y_i - 1)$. In addition to the jump sizes, the inter-arrival times of the jumps needed to be modeled. As shown earlier, the arrival times of jumps T_1, T_2, \dots, T_m is generated by a Poisson process N_t independent of the jump sizes Y_i with average arrival intensity λ . By combining the jump times and the jump sizes, the trajectory of the stock price process is now completely characterised by the compound Poisson process.

$$S_t = \sum_{j=1}^{N_t} Y_j - 1$$

The compound Poisson process gives us all the information about the jumps in the stock price process and tells us when the jumps occur and how big they are.

3.3. Jump Measures of the Compound Poisson Process

Now let $X_t = \ln(S_t)$ then the stochastic process $\{X_t\}_{t \geq 0}$ is a compound Poisson process and the following proposition suffice.

Proposition 1.1.

The stochastic process $\{X_t\}_{t \geq 0}$ is a compound Poisson process if and only if it is a Levy process and its sample paths are piecewise constant.

Proposition 1.1 allows the stock price process to be characterized as a Levy process. There are ample reasons for introducing Levy processes at this stage. First of all, stock prices, when they jump carries associated risk and such risks can only be rigorously analysed by Levy models which allows for the explicit computations of finite results. In essence, the distributional properties of the empirical data admit the Levy-Khintchine formula which describes the structure of the sample paths via the Levy-Ito decomposition.

We now calibrate the trajectories of X_t as a process having infinite number of jumps in a finite time grid. The increments $(X_t - X_s), s < t$ depends only on $(t - s)$ and X_t has the infinite divisibility property such that for a given path X_t there exist a sequence of *i.i.d* random variables $Y_1^n, Y_2^n, \dots, Y_n^n$ such that X_t can be decomposed as:

$$X_t = Y_1^n + Y_2^n + \dots + Y_n^n \tag{9}$$

$$Y_i^n = X_{\frac{t}{n}} + \left(X_{\frac{2t}{n}} - X_{\frac{t}{n}} \right) + \dots + \left(X_t - X_{\frac{(n-1)t}{n}} \right)$$

and the increment $\left(X_{\frac{kt}{n}} - X_{\frac{(k-1)t}{n}} \right)$ yields a sequence of stationary *i.i.d* random variables.

$\{X_t\}_{t \geq 0}$ is a compound process on $\mathbb{R} \times [0, T]$, having intensity λ and jump size distribution f . We define a random measure \mathbb{J}_X on $\mathbb{R} \times [0, \infty]$ describing the jumps of the stock price such that for any measurable Borel set $\mathcal{B} \in \mathbb{R} \times [0, \infty]$, \mathbb{J}_X counts the number of jumps between t_1 and t_2 . That is,

$$\mathbb{J}_X = \#\{t, X_t - X_{t-} \in \mathcal{B}\}$$

Then \mathbb{J}_X is the Poisson random measure on $\mathbb{R} \times [0, \infty]$ with intensity measure

$$\mu(dx \times dt) = \nu(dx)dt = \lambda f(dx)dt$$

where ν is the Levy measure defined as the expected number of jumps per unit time and given by

$$\nu = \mathbb{E}[\#\{t \in [0, 1]: \Delta X_t \neq 0, \Delta X_t \in A\} \mid A \in \mathcal{B}(\mathbb{R})]$$

X_t is thus given by

$$X_t = bt + \int_{\mathbb{R} \times [0, \infty)} x \mathbb{J}_x(ds \times dx) \tag{10}$$

For a one-dimensional distribution

$$X_t = bt + \int_{-\infty}^{\infty} x \mathbb{J}_x(ds \times dx) \tag{11}$$

where b is the linear drift term.

X_t has the characteristic triplet $[\lambda \int_{0 < |x| < 1} xf(dx), 0, \lambda f(dx)]$ and its characteristic exponent can be expressed as

$$\mathbb{E}[\exp\{iuX_t\}] = \exp\left\{t \int_{-\infty}^{\infty} (e^{iux} - 1) f(dx)\right\} \tag{12}$$

Given the Levy measure ν we can rewrite (12) as

$$\mathbb{E}[\exp\{iuX_t\}] = \exp\left\{t \int_{-\infty}^{\infty} (e^{iux} - 1) \nu(dx)\right\} \tag{13}$$

All paths of X_t has a finite variation if and only if $\int_{|x| \leq 1} |x| \nu(dx) < \infty$.

By the levy-Ito decomposition, if $X(t)$ is a pure jump then it can be decomposed as

$$X_t = bt + M_t^1 + M_t^2$$

where b is the drift defined as $b = \begin{cases} \int_{|x| \leq 1} |x| \nu(dx) & \text{if } \int_{|x| \leq 1} |x| \nu(dx) < \infty \\ 0 & \text{otherwise} \end{cases}$

M_t^1 represents the behavior of large jumps and M_t^2 represents the behavior of small jumps. We assume that $\int \nu(dx) < \infty$, so the process has a finite expected number of small ($|x| < 1$) and large ($|x| \geq 1$) jumps per time interval time interval of $[0, T]$.

4. Results and Discussion

To test the extent to which the model predicts the actual stock price, we simulate the path of each stock using Monte Carl simulations and the results are compared to the actual stock price for the coming year. The previous year was from January 2018 to December 2018 and the coming year was January 2019 to December 2019. The estimates for the coming year are obtained from the simulation-based estimator in \mathcal{R} using log maximum likelihood as the sampling toll. The data for the input parameters were obtained for 36 common stocks on Ghana Stock Exchange and consisted of daily share prices quoted on GSE for the period. The data source is a daily Ghana Annual Reports stock price file wherein we assume no dividend payments within the period. We define a jump as when

the return on the stock is greater or less than 10% from the previous day. So a jump occurs if

$$\frac{X_{k+1}}{X_k} > \pm 0.1$$

The Monte Carlo simulation results allow us to pin down the parameters of stocks that follow pure jumps, jump diffusion or strictly diffusion processes. The input parameters for the simulation comprising the mean and variance of the diffusion process, the mean and variance of the jump processes, the intensity of the Poisson process and the number of jumps per period. These parameters are summarised in **Table 2**. The simulation procedure for pure jumps and jump diffusion processes are well documented in Cont and Tankov (2003) and we adopt the procedure here. The codes for the simulation are given in the Appendix and a cross-section of the simulated and actual results is also given in **Table A1**.

The simulated results and the actual stock price processes that follow pure jump and jump diffusion processes are presented graphically in **Figures 5-13**. The figures show the log returns for 9 stocks whose price processes are characterized by pure jump and jump diffusion processes. They include Total Petroleum, Fanmilk, Calbank Ltd., Ecobank, Ecobank transnational, Unilever, Produce

Table 2. Input parameters for simulation process.

	<i>m</i>	<i>s</i>	<i>l</i>	<i>a</i>	<i>D</i>	<i>ts</i> ,	<i>J</i>
Total Petroleum	(0.0023096	0.01343	1	0.12	0.0	0.00405	122)
Ecobank	(0.0006356	0.0133	2	0.02	0.00405	0.00405	122)
Fanmilk	(0.00186	0.01192	1	0.10	0.01	0.00405	122)
CALBANK	(0.001428	0.03198	9	0.14	0.02	0.00405	122)
Ecobanktrans	(0.00191	0.03410	2	0.57	0.61	0.00405	122)
Unilever	(0.00191	0.03410	2	0.57	0.61	0.00405	122)
PBC	(0.0	0.04573	12	0.20	0.03	0.00405	122)
SIC	(-0.0007411	0.037589	12	0.14	0.03	0.00405	122)
Starwin	(0.0	0.0844	14	0.35	0.57	0.00405	122)

m = mean of the data, *s* = standard deviation of the data, *l* = number of jumps, *a* = mean of the jumps, *D* = standard deviation of the jumps, *ts* = Δ*x*, *J* = number of trading days.

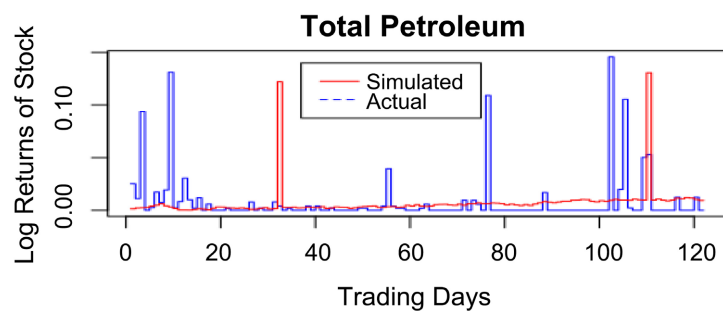


Figure 5. Graphs of simulated and actual stock price—Total Petroleum.

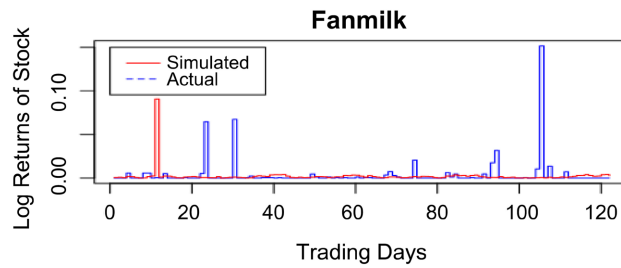


Figure 6. Graphs of simulated and. actual stock price—Fanmilk Ghana.

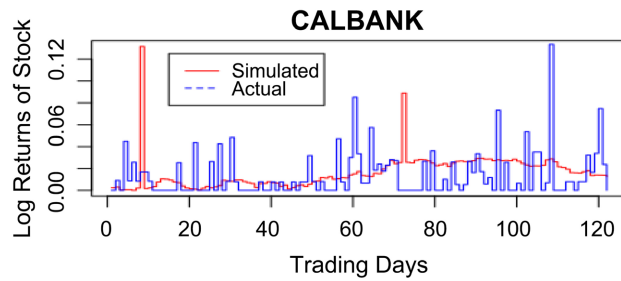


Figure 7. Graphs of simulated and. actual stock price—CalBank.

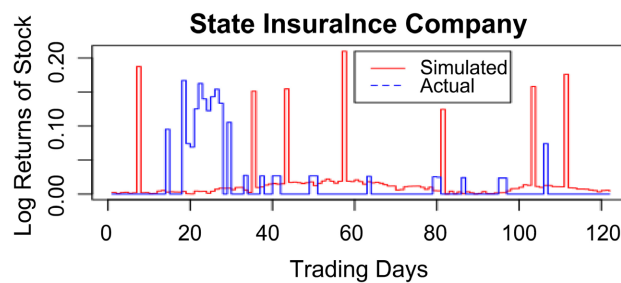


Figure 8. Graphs of simulated and. actual stock price—SIC.

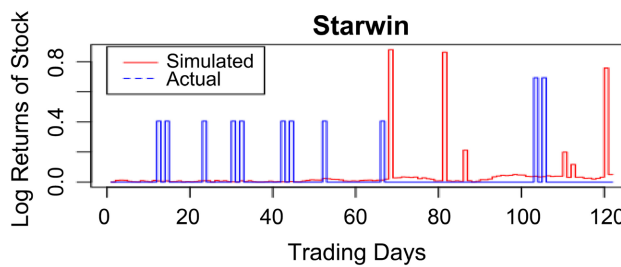


Figure 9. Graphs of simulated and. actual stock price—Starwin.

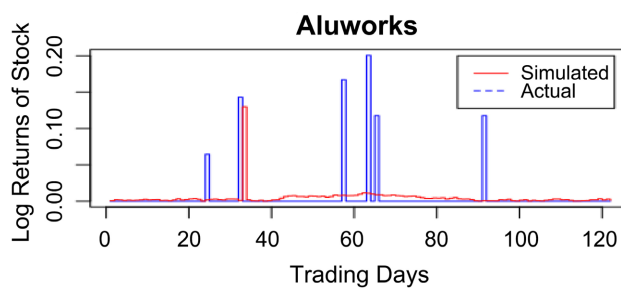


Figure 10. Graphs of simulated and. actual stock price—Aluworks.

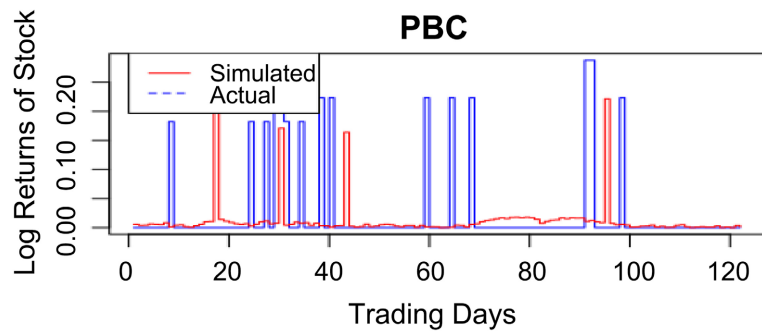


Figure 11. Graphs of simulated and. actual stock price—PBC.

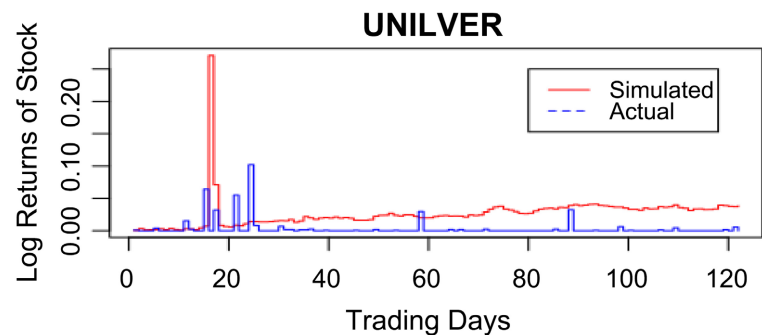


Figure 12. Graphs of simulated and. actual stock price—Unilever.

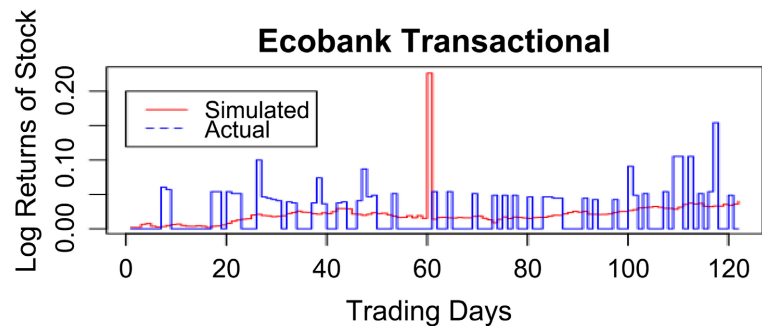


Figure 13. Graphs of simulated and. actual stock price—Ecobank Transnational.

Buying Company, State Insurance Company and Starwin Ltd. Other stocks that follow diffusion processes are not shown here. The actual and simulated trajectories are shown in the graphs and the summary statistics and t -test analysis are also presented in **Table 3**. Using the t -test, the null hypothesis is that the population means of the simulated and actual stock prices are equal assuming that the variances are equal. In this case if the t -stat is greater than the p -value at a given significance level and degrees of freedom, then for a two-tail test the population means will be statistically different. It is realised that this is the case for PBC, Starwin, SIC, Total Petroleum and Fanmilk. Thus it was realised that the trajectories of Starwin, PBC, Total Petroleum, Fanmilk, PBC, Starwin and SIC follow pure jumps processes. However, the paths of Calbank, Unilever and Ecobank Transnational cannot be confirmed from the results as pure jump processes.

Table 3. T-test table of simulated and actual stock prices.

	TOTAL		FANMILK		Calbank		Ecobank Transnational	
	<i>t-Test: Two-Sample Assuming Equal Variances</i>							
	Variable 1	Variable 2	Variable 1	Variable 2	Variable 1	Variable 2	Variable 1	Variable 2
Mean	0.006109952	0.001126385	0.00080081	-0.0021171	-0.0112057	0.001392615	-0.01644359	0.001829046
Variance	0.000262714	0.000670899	7.10164E-05	0.000279493	0.000381012	0.000613197	0.000616941	0.001501239
Observations	122	122	121	121	122	122	122	122
Pooled Variance	0.000466807		0.000175255		0.000497105		0.00105909	
Hypothesized Mean Difference	0		0		0		0	
df	242		240		242		242	
t Stat	1.801509858		1.714408738		-4.41319644		-4.3853038	
P(T ≤ t) one-tail	0.036433605		0.043872318		7.67395E-06		8.64339E-06	
t Critical one-tail	1.651174514		1.651227393		1.651174514		1.651174514	
P(T ≤ t) two-tail	0.072867209		0.087744636		1.53479E-05		1.72868E-05	
t Critical two-tail	1.969815134		1.969897635		1.969815134		1.969815134	
	Unilever		PBC		Starwin Ltd.		SIC	
Mean	-0.02034714	0.002677321	0.007608434	-0.00149444	0.01658298	-0.00332348	0.016461597	0.011155545
Variance	0.000927635	0.000166699	0.00124384	0.006031404	0.018271564	0.02015848	0.001458876	0.001613947
Observations	122	122	122	122	122	122	122	122
Pooled Variance	0.000547167		0.003637622		0.019215022		0.001536411	
Hypothesized Mean Difference	0		0		0		0	
df	242		242		242		242	
t Stat	-7.68766269		1.178785096		1.121601552		1.05726195	
P(T ≤ t) one-tail	1.87386E-13		0.119820731		0.131571785		0.145723148	
t Critical one-tail	1.651174514		1.651174514		1.651174514		1.651174514	
P(T ≤ t) two-tail	3.74773E-13		0.239641461		0.26314357		0.291446296	
t Critical two-tail	1.969815134		1.969815134		1.969815134		1.969815134	

5. Conclusion

The main results of the paper are: developing an underlying price model of a pure jump model from an abstract framework of empirical stock price data. The model is extended as Levy pure jump model that captures the features associated with pure jumps. The numerical results were able to isolate the stocks that follow pure jump processes from those that follow diffusion and jump diffusion processes. The summary statistics of the trajectories show that stock price processes on the Ghana stock exchange are characterized by diffusion, jump diffusion and pure jump processes. In conclusion, it is realised that the paths of Starwin, PBC, Total Petroleum, Fanmilk, PBC, Starwin and SIC follow pure jumps. The paths of Calbank, Unilever and Ecobank Transnational cannot be confirmed as pure jumps.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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Appendix

Codes for simulation

```

SimulatePureJump = function(m, s, l, a, D, ts, J)
{L = length(ts);
  T = ts[L];
  # simulate number of jumps;
  N = rpois(J, l * T);
  Jumps = matrix(0, J, L);
  for(j in 1 : J);
  {# simulate jump arrival time;
    t = T * runif(N[j]);
    t = sort(t);
    # simulate jump size;
    S = a + D * rnorm(N[j]);
    # put things together;
    CumS = cumsum(S);
    Jumps_ts = matrix(0, 1, L);
    for(n in 1 : L);
    {Events = sum(t ≤ ts[n]);
      if(Events);
        {Jumps_ts[n] = CumS[Events]; }
      } Jumps[j,] = Jumps_ts;}
  D_Diff = matrix(NA, J, L);
  for(l in 1 : L);
  {Dt = ts[l];
    if(l > 1);
    {Dt = ts[l] - ts[l - 1];}
  D_Diff[l, 1] = m * Dt + s * sqrt(Dt) * rnorm(J);}
X = cbind(matrix(0, J, 1), apply(D_Diff, 2, cumsum) + Jumps);
return(X);}

tot<-SimulatePureJump(.0023096,0.01343,1,0.12,0,0.00405,122)
Total1<-tot[,-1]
plot(abs(Total1),type = "s",col="red", xlab="Trading Days",
      ylab="Log Returns of Stock", main = "Total Petroleum", xlim=c(0, 40))
lines(abs(doc$Total),type = "s",col="blue")
legend(15, 0.125, legend=c("Simulated", "Actual"),col=c("red", "blue"), lty=1:2,
       cex=0.8)
eco<-SimulatePureJump(.0006356,0.0133,2,0.02,0.00405,0.00405,122)
Ecobank<-eco[,-1]
plot(abs(doc$ecobank),type = "s",col="blue", xlab="Trading Days", ylab="Log
Returns of f(Y)", main = "Ecobank")
lines(abs(Ecobank),type = "s",col="red")

```

```
legend(0, 0.11, legend=c("Simulated", "Actual"),col=c("red", "blue"), lty=1:2,
cex=0.8)
```

```
fun<- SimulatePureJump(.00186,0.01192,1,0.10,0.01,0.00405,122)
Funmilk1<-fun[,-1]
sim_2<-cbind(Funmilk1,doc$Funmilk)
plot(abs(doc$Funmilk),type = "s",col="blue", xlab="Trading Days", ylab="Log
Returns of Stock", main = "Fanmilk")
lines(abs(Funmilk1),type = "s",col="red")
legend(0, 0.15, legend=c("Simulated", "Actual"),col=c("red", "blue"), lty=1:2,
cex=0.8)
```

```
cal<- SimulatePureJump(.001428,0.03198,9,0.14,0.02,0.00405,122)
ca<-cal[,-1]
ca1<-cbind(ca,doc$calbank)
View(ca)
plot(abs(ca),type = "s",col="red", xlab="Trading Days", ylab="Log Returns of
Stock", main = "CALBANK")
lines(abs(doc$calbank),type = "s",col="blue")
legend(15, 0.125, legend=c("Simulated", "Actual"),col=c("red", "blue"), lty=1:2,
cex=0.8)
```

```
ecot<- SimulatePureJump(.00191,0.03410,2,0.57,0.61,0.00405,122)
Ecotrans<-ecot[,-1]
e1<-cbind(Ecotrans,doc$eco_trans)
plot(abs(Ecotrans),type = "s",col="red", xlab="Trading Days", ylab="Log Re-
turns of Stock",main = "Ecobank Transactional")
lines(abs(doc$eco_trans),type = "s",col="blue")
legend(0, 0.2, legend=c("Simulated", "Actual"),col=c("red", "blue"), lty=1:2,
cex=0.8)
```

```
uni<- SimulatePureJump(.00191,0.03410,2,0.57,0.61,0.00405,122)
Uniliver<-uni[,-1]
u1<-cbind(Uniliver,doc$uniliver)
plot(abs(Uniliver),type = "s",col="red", xlab="Trading Days", ylab="Log Re-
turns of Stock",main = "UNILIVER")
lines(abs(doc$uniliver),type = "s",col="blue")
legend(80, 0.25, legend=c("Simulated", "Actual"),col=c("red", "blue"), lty=1:2,
cex=0.8)
```

```
pb<- SimulatePureJump(0,0.04573,12,0.20,0.03,0.00405,122)
PBC<-pb[,-1]
p1<-cbind(PBC,doc$pb)
plot(abs(doc$pb),type = "s",col="blue", xlab="Trading Days", ylab="Log Re-
```

```

turns of Stock",main = "PBC")
lines(abs(PBC),type = "s",col="red")
legend(0, 0.3, legend=c("Simulated", "Actual"),col=c("red", "blue"), lty=1:2,
cex=0.8)

sic<- SimulatePureJump(-.0007411,0.037589,12,0.14,0.03,0.00405,122)
SIC<-sic[,-1]
s1<-cbind(SIC,doc$sic)
plot(abs(SIC),type = "s",col="red", xlab="Trading Days", ylab="Log Returns of
Stock",main = "State Insurance Company")
lines(abs(doc$sic),type = "s",col="blue")
legend(60, 0.21, legend=c("Simulated", "Actual"),col=c("red", "blue"), lty=1:2,
cex=0.8)

star<- SimulatePureJump(0,0.0844,14,0.35,0.57,0.00405,122)
Starwin<-star[,-1]
st1<-cbind(Starwin,doc$starwin)
plot(abs(Starwin),type = "s",col="red", xlab="Trading Days", ylab="Log Returns
of Stock",main = "Starwin")
lines(abs(doc$starwin),type = "s",col="blue")
legend(0, 0.9, legend=c("Simulated", "Actual"),col=c("red", "blue"), lty=1:2,
cex=0.8)

```

Table A1. Cross section of simulated and actual results.

TOTAL GH		FANMILK GH		CAL BANK		ECOBANK TRANSNATIONAL		UNILEVER	
Simulated	Actual	Simulated	Actual	Simulated	Actual	Simulated	Actual	Simulated	Actual
-0.0013772	0.0251762	6.94E-04	0	2.39E-03	0	0.0021152	0	-0.0005893	0.0007785
-0.0021608	0.0109891	9.47E-04	0	3.08E-03	0.0092167	0.0019802	0	-0.0032496	0
-0.0023799	0.0938188	-4.21E-04	0	3.84E-04	0	0.0064807	0	-0.001178	0
-0.0026601	0	-1.86E-03	0.0056338	-6.16E-04	0.0448506	0.0076638	0	0.0015265	0
-0.0038927	0.0024845	-1.59E-03	0	6.60E-04	0.0087337	0.0037751	0	-0.001663	0.0038835
-0.004748	0.0172206	-7.03E-04	0	-2.79E-05	0.0257525	0.0026251	0	-0.0029136	0
-0.0061715	0.0072904	-3.14E-05	0	2.14E-03	0.0084389	0.0033053	0.0606246	-0.0015818	0
-0.0041518	0.0191852	5.57E-05	-0.0056338	1.32E-01	-0.0169496	0.0047569	0.0571584	-0.0033456	0
-0.0030211	0.1311533	1.74E-03	0.0056338	1.73E-03	0.0169496	0.0062335	0	0.0018894	0
-0.0016838	0.0020812	2.34E-03	0	2.34E-03	0.0083683	0.0068486	0	0.0007667	0
0.0002284	0.0082816	9.07E-02	-0.000562	3.54E-03	0	0.0047891	0	0.0029069	0.0153849
0.000204	0.0304592	1.05E-03	0	7.67E-03	0	0.0038499	0	0.0025675	0.0022875
0.0001307	0.0099503	2.24E-03	-0.0050719	1.05E-02	0	0.0037602	0	-0.0014071	0
-0.0007464	0.0019782	2.21E-03	0	1.03E-02	0	0.0045441	0	-0.0044147	0
-0.0013594	0.011788	1.28E-03	0	9.51E-03	0	0.0038458	0	-0.0070985	0.0641576