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Oscillation Theorems for Two Classes of Fractional Neutral Differential Equations

Xulong Wang, Siying Long, Anping Liu

School of Mathematics and Physics, China University of Geosciences, Wuhan, China
Email: wang_xulong@qq.com

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Abstract

In this paper, we study the oscillatory theory for two classes of fractional neutral differential equations. By using fractional calculus and the Laplace transform, we obtain several new sufficient conditions for the oscillation of all solutions of this equation. Our results improve and extend some known results in the literature. Furthermore, some examples are provided to show the effectiveness and feasibility of the main results.

Keywords

Oscillation, Fractional Derivative, Laplace, Fractional Differential Equation

1. Introduction

The fractional differential equation (FDE) has gained considerable importance due to its various application in fluid mechanics, viscoelasticity, electrochemistry of corrosion, classical mechanics and particle physics, control theory, diffusive systems, and so on [1]-[7]. Therefore, the theory of fractional calculus has received extensive attention from scholars at home and abroad. In the past few decades, many researchers have done a lot of research on the properties of fractional differential, for example, the existence, uniqueness, stability, asymptoticity of solutions of fractional differential equations and numerical solutions of fractional differential equations, etc. [8]-[13].

However, to the best of our knowledge, there are few results on oscillation for the fractional differential equation. We refer to [14]-[19] and the references therein. In [20], Meng *et al.* studied the linear fractional order delay differential equation

$$ {}^{C}D_{-}^{\alpha}x(t)-px(t-\tau)=0, $$

where $0<\alpha<1$, $p,\tau>0$, ${}^{C}D_{-}^{\alpha}x(t)=-\Gamma^{-1}(1-\alpha)\int_{t}^{\infty}(s-t)^{-\alpha}x'(s)\mathrm{d}s$.

In [21], A. George Maria Selvam and R. Janagaraj establish oscillation theorems for damped fractional order differential equation of the form:

$$\Delta\left[\delta(k)\Delta^{\gamma}y(k)\right]+\rho(k)\Delta^{\gamma}y(k)+\xi(k)Z\left[Y(k)\right]=0,k\geq k_0>0,$$

where $Y(k)=\sum_{u=k_0}^{k-1+\gamma}(k-u-1)^{-\gamma}y(u)$ and Δ^{γ} defined as the difference operator of the Riemann-Liouville derivative of order $\gamma\in(0,1]$.

In [22], Zhu *et al.* studied forced oscillatory properties of solutions to nonlinear fractional differential equations with damping term and time delay:

$$\left(D_{0+}^{1+\alpha}y\right)(t-\tau)+p(t-\tau)\left(D_{0+}^{\alpha}y\right)(t-\tau)+q(t)f\left(y(t)\right)=g(t),$$

where $y(t)=\xi(t)$ when $t\in[-\tau,0)$ and $\xi(t)$ is a given continuous function, where τ and $\alpha\in(0,1)$ are constants, $\lim_{t\to 0^-}\xi(t)=0$, *b* is a real number, $D_{0+}^{\alpha}y$ is the Riemann-Liouville fractional derivative of order α of *y*.

In [23], Zhou *et al.* study the following fractional functional partial differential equation involving Riemann-Liouville fractional derivative:

$$\frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}}=C(t)\Delta u+\sum_{i=1}^{n}P_i(x)u(x,t-\sigma_i)+R(x,t),$$

supplemented with the initial condition

$$\left.\frac{\partial^{\alpha-1}u(x,t)}{\partial t^{\alpha-1}}\right|_{t\in[-\sigma,0]}=\varphi(x,t),x\in\Omega,\ \text{where } \sigma=\max\{\sigma_i,i=1,2,\cdots,n\},$$

and boundary conditions

$$\frac{\partial u(x,t)}{\partial N}=0\ \text{ on } (x,t)\in\partial\Omega\times[0,\infty),$$

$$u(x,t)=0\ \text{ on } (x,t)\in\partial\Omega\times[0,\infty),$$

$$\frac{\partial u(x,t)}{\partial N}+vu=0\ \text{ on } (x,t)\in\partial\Omega\times[0,\infty).$$

Motivated by the analysis above, in this paper, we are concerned with the oscillation of two classes of fractional differential equations as follows:

$$_{0}D_t^{\alpha}\left(x(t)+rx(t-\sigma)\right)+f\left(x(t-\tau)\right)=0, \tag{1}$$

$$\begin{aligned}&_{0}D_t^{\alpha}\left(u(x,t)+\sum_{i=1}^{n}r_iu(x,t-\sigma_i)\right)\\&=a(t)\Delta u(x,t)+\sum_{k=1}^{L}b_k(t)\Delta u(x,t-p_k)-\sum_{i=1}^{m}q_i(t)f\left(u(x,t-\tau_i)\right),\end{aligned} \tag{2}$$

where $0<\alpha=\frac{\text{oddinteger}}{\text{oddinteger}}<1$, $_{0}D_t^{\alpha}x(t)$ is Riemann-Liouville fractional derivative of order α.

This paper is organized as follows. In the next section, we introduce some useful preliminaries. In Section 3, we present various sufficient conditions for the oscillation of all solutions to Equations (1) and (2) by using fractional calculus, Laplace transforms and Green's function. Finally, we provide some examples to show the applications of our criteria.

2. Preliminaries

In this section, we introduce preliminary facts which are used throughout this paper.

Definition 1 ([24]). Let $[a,b](-\infty < a < b < \infty)$ be a finite interval and let $AC[a,b]$ be the space of functions *f* which are absolutely continuous on $[a,b]$. It is known (see [25], p.338) that $AC[a,b]$ coincides with the space of primitives of Lebesgue summable function:

$$f(x) \in AC[a,b] \Rightarrow f(x) = c + \int_a^x \phi(t)\,\mathrm{d}t, \left(\phi(t) \in L(a,b)\right).$$

Definition 2 ([24]). The Riemann-Liouville left-sided fractional integral of order $\alpha > 0$ of a function $f : \mathbb{R}_+ \to \mathbb{R}$ on the half-axis $\mathbb{R}_+$ is given by:

$$\left({}_0D_t^{-\alpha} f\right)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s)}{(t-s)^{1-\alpha}}\,\mathrm{d}s, \ t > 0,$$

provided the left hand side is pointwise defined on $\mathbb{R}_+$, where Γ is the gamma function.

Definition 3 ([24]). The Riemann-Liouville left-sided fractional derivative of order $\alpha > 0$ of a function $f : \mathbb{R}_+ \to \mathbb{R}$ on the half-axis $\mathbb{R}_+$ is given by:

$$\left({}_0D_t^{\alpha} f\right)(t) = \frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t \frac{f(s)}{(t-s)^{\alpha}}\,\mathrm{d}s, \ t > 0,$$

provided the left hand side is pointwise defined on $\mathbb{R}_+$.

We recall some facts about Laplace transforms. If $X(s)$ is the Laplace transform of $x(t)$,

$$X(s) = L\left[x(t)\right](s) = \int_0^\infty \mathrm{e}^{-st} x(t)\,\mathrm{d}t,$$

then the abscissa of convergence of $X(s)$ is defined by:

$$b = \inf\left\{\gamma \in R : X(\gamma) \text{ exists}\right\},$$

Therefore, $X(s)$ exists for $\mathrm{Re}(s) > b$.

Definition 4. A function $x(t)$ is eventually positive if there is a $c \ge 0$ such that $x_c(t) > 0$ for all $t > 0$, where $x_c(t) = x(t+c)$.

Definition 5. By a solution of (3) in $[0,\infty)$ with the initial function $\varphi \in AC[-\xi,0]$, we mean a function $x \in AC[-\xi,\infty]$ such that $x(t) = \varphi(t)$, $t \in [-\xi,0]$, $\left({}_0D_t^{\alpha} x\right)(t)$ exists and $x(t)$ satisfies (3) in $v, w : [0,\infty) \to [0,\infty)$. A nontrivial solution $x(t)$ of equation (3) is said to oscillate if it has an arbitrarily large number of zeros. Otherwise, the solution is called non-oscillatory.

Lemma 1 ([24]). Let $\left(L\,{}_0D_t^{\alpha} x\right)(s)$ be the Laplace transform of the Riemann-Liouville fractional derivative of order α with the lower limit zero for a function *x*, and $X(s)$ is the Laplace transform of $x(t)$. Further, for $x \in AC[0,b]$ and for any $b > 0$,

$$\left|x(t)\right| \le A\mathrm{e}^{m_0 t} \quad (t > b > 0)$$

holds for constant $A > 0$ and $m_0 > 0$. Then the relation

$$\left(L\,{}_0D_t^{\alpha}x\right)(s)=s^{\alpha}X(s)-\left({}_0D_t^{\alpha-1}x\right)(t)\Big|_{t=0},0<\alpha<1$$

is valid for $\mathrm{Re}(s)>m_0$.

Lemma 2 ([23]). For any $c\in R$, the Laplace transform $X_c(s)$ of $x_c(t)$ exists and has the same abscissa of convergence as $X(s)$.

Lemma 3 ([26]). Let $v,w:[0,\infty)\to[0,\infty)$ be continuous function. If w is non-decreasing and there are constants $a>0$ and $0<\beta<1$ such that

$$v(t)\le w(t)+a\int_0^t\frac{v(s)}{(t-s)^{\beta}}\mathrm{d}s,$$

then there exists a constant $k=k(\beta)$ with

$$v(t)\le w(t)+ka\int_0^t\frac{w(s)}{(t-s)^{\beta}}\mathrm{d}s,$$

for every $t\in[0,\infty)$.

3. Main Results

In this section, we present our main results.

Lemma 4. ${}_0D_t^{\alpha}x$ is the Riemann-liouville derivative of order α with the lower limit zero for a function $x(t)$, $X(s)$ is the Laplace transform of $x(t)$, $\sigma>0$, $0<\alpha<1$, then the following relation holds.

$$L\left[{}_0D_t^{\alpha}x(t-\sigma)\right](s)=s^{\alpha}\mathrm{e}^{-s\sigma}X(s)+s^{\alpha}\mathrm{e}^{-s\sigma}\int_{-\sigma}^{0}\mathrm{e}^{-st}x(t)\mathrm{d}t-\left({}_0D_t^{\alpha-1}x(t-\sigma)\right)\Big|_{t=0}$$

Proof.

$$L\left[{}_0D_t^{\alpha}x(t-\sigma)\right](s)=L\left[\frac{\mathrm{d}\left({}_0D_t^{\alpha-1}x(t-\sigma)\right)}{\mathrm{d}t}\right](s)$$

$$=sL\left[{}_0D_t^{\alpha-1}x(t-\sigma)\right](s)-\left({}_0D_t^{\alpha-1}x(t-\sigma)\right)\Big|_{t=0}$$

$$=sL\left[\frac{1}{\Gamma(1-\alpha)}\int_0^t(t-s)^{-\alpha}x(s-\sigma)\mathrm{d}s\right](s)-\left({}_0D_t^{\alpha-1}x(t-\sigma)\right)\Big|_{t=0}$$

$$=sL\left[\frac{t^{-\alpha}}{\Gamma(1-\alpha)}*x(t-\sigma)\right](s)-\left({}_0D_t^{\alpha-1}x(t-\sigma)\right)\Big|_{t=0}$$

$$=sL\left[\frac{t^{-\alpha}}{\Gamma(1-\alpha)}\right]L\left[x(t-\sigma)\right]-\left({}_0D_t^{\alpha-1}x(t-\sigma)\right)\Big|_{t=0}.$$

By Lemma 2, $L\left[x(t-\sigma)\right]$ exists such that

$$L\left[x(t-\sigma)\right]=\mathrm{e}^{-s\sigma}X(s)+\mathrm{e}^{-s\sigma}\int_{-\sigma}^{0}\mathrm{e}^{-st}x(t)\mathrm{d}t,$$

then

$$L\left[{}_0D_t^{\alpha}x(t-\sigma)\right](s)=s^{\alpha}\mathrm{e}^{-s\sigma}X(s)+s^{\alpha}\mathrm{e}^{-s\sigma}\int_{-\sigma}^{0}\mathrm{e}^{-st}x(t)\mathrm{d}t-\left({}_0D_t^{\alpha-1}x(t-\sigma)\right)\Big|_{t=0}$$

The proof is complete. □

We consider the following fractional-order delay differential equation:

$$ {}_0D_t^{\alpha}\left(x(t)+rx(t-\sigma)\right)+qx(t-\tau)=0 \tag{3}$$

where $r\in R$, $q,\sigma,\tau\in R^{+}$. The standard initial condition associated with (3) is

$$x(t)=\varphi(t), t\in[-\xi,0], \tag{4}$$

where $\varphi(t)\in C([-\xi,0],R)$, $\xi=\max\{\sigma,\tau\}$.

Lemma 5. If $|r|<1$, the solution of Equation (3) has an exponent estimate

$$x(t)=o\left(e^{q_0 t}\right) \quad (t>b>0)$$

for constants $q_0>0$.

Proof. Taking the Riemann-Liouville integral of Equation (3), we have

$$x(t)=\frac{t^{\alpha-1}}{\Gamma(\alpha)}x_0-rx(t-\sigma)-\frac{q}{\Gamma(\alpha)}F(t) \tag{5}$$

where $F(t)=\int_0^t (t-s)^{\alpha-1}x(s-\tau)\mathrm{d}s$,

$x_0=\left({}_0D_t^{\alpha-1}x(t)\right)\Big|_{t=0}+r\left({}_0D_t^{\alpha-1}x(t-\sigma)\right)\Big|_{t=0}$.

As $x(t)\in AC[0,b]$, there exists a constant $M>0$ such that $|x(t)|\le M$. For $t>b$, we have

$$\begin{aligned}\|F(t)\| &\le \int_0^t (t-s)^{\alpha-1}\|x(s-\tau)\|\mathrm{d}s\\ &\le \int_0^b (t-s)^{\alpha-1}\max_{s-\tau\le\eta\le s}\|x(\eta)\|\mathrm{d}s+\int_b^t (t-s)^{\alpha-1}\max_{s-\tau\le\eta\le s}\|x(\eta)\|\mathrm{d}s\\ &\le \int_0^b (t-s)^{\alpha-1}\left(M+\|\varphi\|\right)\mathrm{d}s+\int_b^t (t-s)^{\alpha-1}\left(\max_{b\le\eta\le s}\|x(\eta)\|+M+\|\varphi\|\right)\mathrm{d}s\\ &\le \frac{t^{\alpha}}{\alpha}\left(M+\|\varphi\|\right)+\int_b^t (t-s)^{\alpha-1}\max_{b\le\eta\le s}\|x(\eta)\|\mathrm{d}s,\end{aligned} \tag{6}$$

which, together with (5), yields

$$\begin{aligned}\|x(t)\| &\le |x_0|\frac{b^{\alpha-1}}{\Gamma(\alpha)}+|r|\|x(t-\sigma)\|+\frac{q}{\Gamma(\alpha)}\|F(t)\|\\ &\le |x_0|\frac{b^{\alpha-1}}{\Gamma(\alpha)}+|r|\max_{t-\sigma\le\eta\le t}\|x(\eta)\|+\frac{q}{\Gamma(\alpha)}\frac{t^{\alpha}}{\alpha}\left(M+\|\varphi\|\right)\\ &\quad+\frac{q}{\Gamma(\alpha)}\int_b^t (t-s)^{\alpha-1}\max_{b\le\eta\le s}\|x(\eta)\|\mathrm{d}s\\ &\le |x_0|\frac{b^{\alpha-1}}{\Gamma(\alpha)}+|r|\left(M+\|\varphi\|+\max_{b\le\eta\le t}\|x(\eta)\|\right)+\frac{q}{\Gamma(\alpha)}\frac{t^{\alpha}}{\alpha}\left(M+\|\varphi\|\right)\\ &\quad+\frac{q}{\Gamma(\alpha)}\int_b^t (t-s)^{\alpha-1}\max_{b\le\eta\le s}\|x(\eta)\|\mathrm{d}s\\ &\le |x_0|\frac{b^{\alpha-1}}{\Gamma(\alpha)}+\left(|r|+\frac{q}{\Gamma(\alpha)}\frac{t^{\alpha}}{\alpha}\right)\left(M+\|\varphi\|\right)+|r|\max_{b\le\eta\le t}\|x(\eta)\|\\ &\quad+\frac{q}{\Gamma(\alpha)}\int_b^t (t-s)^{\alpha-1}\max_{b\le\eta\le s}\|x(\eta)\|\mathrm{d}s.\end{aligned} \tag{7}$$

In the interval $[b,t]$, taking the maximum value on both sides of inequality (7), then

$$\max_{b\le\eta\le t}\|x(\eta)\| \le |x_0|\frac{b^{\alpha-1}}{\Gamma(\alpha)}+\left(|r|+\frac{q}{\Gamma(\alpha)}\frac{t^{\alpha}}{\alpha}\right)(M+\|\varphi\|)+|r|\max_{b\le\eta\le t}\|x(\eta)\|$$
$$+\frac{q}{\Gamma(\alpha)}\int_b^t (t-s)^{\alpha-1}\max_{b\le\eta\le s}\|x(\eta)\|\mathrm{d}s.$$

One can introduce nondecreasing function $m(t)$ as

$$m(t)=\frac{1}{1-|r|}\left[|x_0|\frac{b^{\alpha-1}}{\Gamma(\alpha)}+\left(|r|+\frac{q}{\Gamma(\alpha)}\frac{t^{\alpha}}{\alpha}\right)(M+\|\varphi\|)\right],$$

then we have

$$\max_{b\le\eta\le t}\|x(\eta)\| \le m(t)+\frac{1}{1-|r|}\frac{q}{\Gamma(\alpha)}\int_b^t (t-s)^{\alpha-1}\max_{b\le\eta\le s}\|x(\eta)\|\mathrm{d}s.$$

By lemma 3, there exists a constant $k=k(1-\alpha)$ such that

$$\|x(t)\|\le\max_{b\le\eta\le t}\|x(\eta)\|\le m(t)+k\frac{1}{1-|r|}\frac{q}{\Gamma(\alpha)}\int_b^t (t-s)^{\alpha-1}m(s)\mathrm{d}s$$
$$\le m(t)\left(1+\frac{kq}{(1-|r|)\Gamma(\alpha)}\frac{t^{\alpha}}{\alpha}\right).$$

Obviously, from the above formula, we infer that $x(t)$ has an exponent estimate. The proof is complete. □

Theorem 1. Assume that $\tau>\sigma$, $r\in R^{+}$. If the equation

$$p(\lambda)=\lambda^{\alpha}+r\lambda^{\alpha}\mathrm{e}^{-\lambda\sigma}+q\mathrm{e}^{-\lambda\tau}=0 \tag{8}$$

has no real roots, then every solution of (3) is oscillatory.

Proof. Suppose that $x(t)$ is a nonoscillatory solution of Equation (3). Without loss of generality, we assume that $x(t)$ is an eventually positive solution of Equation (3) which means that there exists a constant T such that $x(t)>0$ for $t>T$. Since Equation (3) is autonomous, we may assume that $x(t)>0$ for $t\in[-\xi,\infty)$. Taking Laplace transform of both sides of (3), we obtain

$$s^{\alpha}X(s)-A+rs^{\alpha}\mathrm{e}^{-s\sigma}X(s)+rs^{\alpha}\mathrm{e}^{-s\sigma}\int_{-\sigma}^{0}\mathrm{e}^{-st}x(t)\mathrm{d}t-Br$$
$$+q\mathrm{e}^{-s\tau}X(s)+q\mathrm{e}^{-s\tau}\int_{-\tau}^{0}\mathrm{e}^{-st}x(t)\mathrm{d}t=0,$$

where $A=\left({}_0D_t^{\alpha-1}x(t)\right)\big|_{t=0}$, $B=\left({}_0D_t^{\alpha-1}x(t-\sigma)\right)\big|_{t=0}$.

Hence

$$\begin{aligned}&X(s)\left(s^{\alpha}+rs^{\alpha}\mathrm{e}^{-s\sigma}+q\mathrm{e}^{-s\tau}\right)\\&=A+Br-rs^{\alpha}\mathrm{e}^{-s\sigma}\int_{-\sigma}^{0}\mathrm{e}^{-st}x(t)\mathrm{d}t-q\mathrm{e}^{-s\tau}\int_{-\tau}^{0}\mathrm{e}^{-st}x(t)\mathrm{d}t.\end{aligned} \tag{9}$$

Let

$$p(s)=s^{\alpha}+rs^{\alpha}\mathrm{e}^{-s\sigma}+q\mathrm{e}^{-s\tau},$$

$$\Phi(s)=A+Br-rs^{\alpha}\mathrm{e}^{-s\sigma}\int_{-\sigma}^{0}\mathrm{e}^{-st}x(t)\mathrm{d}t-q\mathrm{e}^{-s\tau}\int_{-\tau}^{0}\mathrm{e}^{-st}x(t)\mathrm{d}t,$$

then, from (9), we get

$$X(s)p(s)=\Phi(s). \tag{10}$$

Since $p(s)=0$ has no real roots and $p(0)>0$, $p(s)>0$. By positivity of $x(t)$ in $AC[-\xi,0]$, then exists constant $m,M>0$ such that $m<x(t)<M$. Since $r>0$, $\tau>\sigma$ then

$$\frac{q\mathrm{e}^{-s\tau}\int_{-\tau}^{0}\mathrm{e}^{-st}x(t)\mathrm{d}t}{-rs^{\alpha}\mathrm{e}^{-s\sigma}\int_{-\sigma}^{0}\mathrm{e}^{-st}x(t)\mathrm{d}t}\geq\frac{q\mathrm{e}^{-s\tau}}{-rs^{\alpha}\mathrm{e}^{-s\sigma}}\geq\frac{q\mathrm{e}^{-s(\tau-\sigma)}}{r(-s)^{\alpha}}\to\infty\quad(s\to-\infty).$$

Thus there exists a constant $k<0$ such that for $s<k$,

$$\frac{q\mathrm{e}^{-s\tau}\int_{-\tau}^{0}\mathrm{e}^{-st}x(t)\mathrm{d}t}{-rs^{\alpha}\mathrm{e}^{-s\sigma}\int_{-\sigma}^{0}\mathrm{e}^{-st}x(t)\mathrm{d}t}\geq 2.$$

Then,

$$\begin{aligned}&-rs^{\alpha}\mathrm{e}^{-s\sigma}\int_{-\sigma}^{0}\mathrm{e}^{-st}x(t)\mathrm{d}t-q\mathrm{e}^{-s\tau}\int_{-\tau}^{0}\mathrm{e}^{-st}x(t)\mathrm{d}t\\&=-rs^{\alpha}\mathrm{e}^{-s\sigma}\int_{-\sigma}^{0}\mathrm{e}^{-st}x(t)\mathrm{d}t\left(1-\frac{q\mathrm{e}^{-s\tau}\int_{-\tau}^{0}\mathrm{e}^{-st}x(t)\mathrm{d}t}{-rs^{\alpha}\mathrm{e}^{-s\sigma}\int_{-\sigma}^{0}\mathrm{e}^{-st}x(t)\mathrm{d}t}\right)\\&\leq rs^{\alpha}\mathrm{e}^{-s\sigma}\int_{-\sigma}^{0}\mathrm{e}^{-st}x(t)\mathrm{d}t\leq rs^{\alpha}\mathrm{e}^{-s\sigma}m\int_{-\sigma}^{0}\mathrm{e}^{-st}\mathrm{d}t\\&=rms^{\alpha}\frac{1}{s}\frac{\mathrm{e}^{s\sigma}-1}{\mathrm{e}^{s\sigma}}=rm\frac{1-\mathrm{e}^{-s\sigma}}{(-s)^{1-\alpha}}\to-\infty\quad(s\to-\infty).\end{aligned}$$

Thus we conclude that $\Phi(s)\to-\infty\,(s\to-\infty)$, but $p(s)$ and $X(s)$ are positive. Hence, (10) leads to a contradiction. The proof is complete. □

Corollary 1. Assume that $r=0$, $q,\tau\in R^{+}$. If the equation

$$p(\lambda)=\lambda^{\alpha}+q\mathrm{e}^{-\lambda\tau}=0$$

has no real roots, then every solution of (3) is oscillatory.

Proof. If we modify the function $p(s)$ and $\Phi(s)$, defined in Theorem 1, as

$$p(s)=s^{\alpha}+q\mathrm{e}^{-s\tau}$$

$$\Phi(s)=A-q\mathrm{e}^{-s\tau}\int_{-\tau}^{0}\mathrm{e}^{-st}x(t)\mathrm{d}t.$$

Then, following the method of proof for Theorem 1, one can complete the proof. □

Corollary 2. Assume that $r<0$, $q,\tau,\sigma\in R^{+}$. If the equation

$$p(\lambda)=\lambda^{\alpha}+r\lambda^{\alpha}\mathrm{e}^{-\lambda\sigma}+q\mathrm{e}^{-\lambda\tau}=0$$

has no real roots, then every solution of (3) is oscillatory.

Proof. Proceeding as in the proof of Theorem 1, according to (9) and (10), for $s<0$, we have

$$qm\frac{1-\mathrm{e}^{-s\tau}}{s}\leq q\mathrm{e}^{-s\tau}\int_{-\tau}^{0}\mathrm{e}^{-st}x(t)\mathrm{d}t\leq qM\frac{1-\mathrm{e}^{-s\tau}}{s},$$

$$rm\frac{1-\mathrm{e}^{-s\sigma}}{(-s)^{1-\alpha}}\leq rs^{\alpha}\mathrm{e}^{-s\sigma}\int_{-\sigma}^{0}\mathrm{e}^{-st}x(t)\mathrm{d}t\leq rM\frac{1-\mathrm{e}^{-s\sigma}}{(-s)^{1-\alpha}}.$$

Thus, we obtain

$$qe^{-s\tau}\int_{-\tau}^{0}e^{-st}x(t)\,dt\to\infty\,(s\to-\infty),$$

$$rs^{\alpha}e^{-s\sigma}\int_{-\sigma}^{0}e^{-st}x(t)\,dt\to\infty\,(s\to-\infty).$$

Thus we conclude that $\Phi(s)\to-\infty\,(s\to-\infty)$, but $p(s)$ and $X(s)$ are positive. Hence (10) leads to a contradiction. The proof is complete. □

Theorem 2. Assume that $\tau>\sigma$, $r\in R^{+}$, and if

$$(\tau-\sigma)\left(\frac{q}{1+r}\right)^{\frac{1}{\alpha}}>\frac{1}{e}, \tag{11}$$

then every solution of Equation (3) is oscillatory.

Proof. Assume that Equation (8) has a real roots λ_1, if $\lambda_1\ge 0$, then $p(\lambda_1)>0$, it is impossible. Thus we conclude that $\lambda_1<0$. Since α is the ratio of two odd integers, it follows from (8) that

$$\lambda_1^{\alpha}=-\frac{qe^{-\lambda_1\tau}}{1+re^{-\lambda_1\sigma}}=-\frac{q}{e^{\lambda_1\tau}+re^{\lambda_1(\tau-\sigma)}}\le-\frac{q}{1+r},$$

then

$$(-\lambda_1)^{\alpha}\ge\frac{q}{1+r},$$

$$(-\lambda_1)^{1-\alpha}\ge\left(\frac{q}{1+r}\right)^{\frac{1-\alpha}{\alpha}}. \tag{12}$$

By (12) and the inequality $e^{x}\ge ex$ for $x\ge 0$, we get

$$\begin{aligned}(-\lambda_1)^{\alpha}&=\frac{qe^{-\lambda_1\tau}}{1+re^{-\lambda_1\sigma}}=\frac{qe^{\lambda_1(\sigma-\tau)}}{e^{\lambda_1\sigma}+r}\ge\frac{qe\lambda_1(\sigma-\tau)}{1+r}=\frac{qe(\tau-\sigma)}{1+r}(-\lambda_1)\\&=\frac{qe(\tau-\sigma)}{1+r}(-\lambda_1)^{\alpha}(-\lambda_1)^{1-\alpha}\ge\frac{qe(\tau-\sigma)}{1+r}(-\lambda_1)^{\alpha}\left(\frac{q}{1+r}\right)^{\frac{1-\alpha}{\alpha}},\end{aligned}$$

which implies that

$$1\ge e(\tau-\sigma)\left(\frac{q}{1+r}\right)^{\frac{1}{\alpha}},$$

this is a contradiction. The proof is complete. □

Theorem 3. Assume that $r=0$, $q,\tau\in R^{+}$, and if

$$\tau q^{\frac{1}{\alpha}}>\alpha, \tag{13}$$

then every solution of Equation (3) is oscillatory.

Proof. In regard to $p(\lambda)=\lambda^{\alpha}+qe^{-\lambda\tau}=0$, since $p(\lambda)>0$ for $\lambda\ge 0$, thus we conclude that if $p(\lambda)=0$ has real roots, the roots are definitely less than zero.

Let

$$p'(\lambda)=\alpha\lambda^{\alpha-1}-\tau qe^{-\lambda\tau}=0, \tag{14}$$

from (14), we get

$$\left(\frac{\tau q}{\alpha}\right)^{\frac{1}{1-\alpha}}(-\lambda)\mathrm{e}^{-\lambda\frac{\tau}{1-\alpha}}-1=0,(\lambda<0). \tag{15}$$

Equation (15) has only one real root for $\lambda<0$. Thus (14) has only one real root for $\lambda<0$, and Assume that the root is $l(l<0)$. Meanwhile, as for $\lambda<0$, $p(\lambda)\to\infty(\lambda\to-\infty)$, thus $p(\lambda)$ has only one extreme point, which is also the minimum point for $\lambda<0$.

Let

$$g(\lambda)=\left(\frac{\tau q}{\alpha}\right)^{\frac{1}{1-\alpha}}(-\lambda)\mathrm{e}^{-\lambda\frac{\tau}{1-\alpha}}-1, \tag{16}$$

since $g(0)=-1<0$ and $g\left(-\frac{1}{\left(\frac{\tau q}{\alpha}\right)^{\frac{1}{1-\alpha}}}\right)=\mathrm{e}^{\frac{\tau}{1-\alpha}\frac{1}{\left(\frac{\tau q}{\alpha}\right)^{\frac{1}{1-\alpha}}}}-1>0$, we can obtain

$$-\frac{1}{\left(\frac{\tau q}{\alpha}\right)^{\frac{1}{1-\alpha}}}<l<0. \tag{17}$$

By $p(\lambda)=\lambda^{\alpha}+q\mathrm{e}^{-\lambda\tau}$ and (14), we get

$$p(l)=l^{\alpha}+q\mathrm{e}^{-l\tau}=\frac{q\tau l}{\alpha}\mathrm{e}^{-\tau l}+q\mathrm{e}^{-l\tau}=q\mathrm{e}^{-l\tau}\left(\frac{\tau l}{\alpha}+1\right),$$

and by (17) and (13), we obtain

$$\frac{\tau l}{\alpha}+1>-\frac{1}{\left(\frac{\tau q}{\alpha}\right)^{\frac{1}{1-\alpha}}}\frac{\tau}{\alpha}+1>0.$$

Thus we conclude $p(l)>0$, and then $p(\lambda)>0$ $(\lambda\in R)$. By Corollary 1, every solution of (3) is oscillatory. The proof is complete. □

Theorem 4. Assume that $-1<r<0$, $\tau,\sigma,q>0$, if

$$(\tau+\sigma)(-4qr)^{\frac{1}{\alpha}}>\frac{\alpha}{\mathrm{e}}, \tag{18}$$

then every solution of Equation (3) is oscillatory.

Proof. To prove the result, it suffices to prove that (8) has no real roots under the conditions (18). One can note that any real root of (8) cannot be positive and since $p(0)=q>0$, $\lambda=0$ is not a root. Thus any real root of (8) can only be negative if it is possible. Let us set $\lambda=-\mu$ for convenience and show that

$$p(-\mu)=-\mu^{\alpha}-r\mu^{\alpha}\mathrm{e}^{\mu\sigma}+q\mathrm{e}^{\mu\tau}=0,\mu>0. \tag{19}$$

From (19), we have

$$-1-r\mathrm{e}^{\mu\sigma}+\frac{q\mathrm{e}^{\mu\tau}}{\mu^{\alpha}}=0$$

$$1=-r\mathrm{e}^{\mu\sigma}+\frac{q\mathrm{e}^{\mu\tau}}{\mu^{\alpha}}\geq 2\sqrt{\frac{-rq\mathrm{e}^{\mu(\tau+\sigma)}}{\mu^{\alpha}}}$$

$$1 \geq -4rq\frac{e^{\mu(\tau+\sigma)}}{\mu^{\alpha}}. \tag{20}$$

Let

$$h(\mu) = \frac{e^{\mu(\tau+\sigma)}}{\mu^{\alpha}}, \mu > 0.$$

Then

$$h(\mu) = \frac{e^{\mu(\tau+\sigma)}}{\mu^{\alpha}} \geq \min_{\mu>0} h(\mu) = h\left(\frac{\alpha}{\tau+\sigma}\right) = e^{\alpha}\left(\frac{\tau+\sigma}{\alpha}\right)^{\alpha}.$$

So, we get

$$1 \geq -4rqe^{\alpha}\left(\frac{\tau+\sigma}{\alpha}\right)^{\alpha} \Rightarrow \frac{\alpha}{e} \geq (-4rq)^{\frac{1}{\alpha}}(\tau+\sigma).$$

This contradicts (18) which implies that (8) has no real roots. By Corollary 2, then every solution of (3) is oscillatory. The proof is complete. □

Now, we consider fractional differential equations with multiple delays

$${}_0D_t^{\alpha}\left(x(t) + \sum_{i=1}^{n} r_i x(t-\sigma_i)\right) + \sum_{i=1}^{m} q_i x(t-\tau_i) = 0 \tag{21}$$

where $r_i, q_i, \sigma_i, \tau_i \in R^+$, $0 < \alpha = \frac{\text{oddinteger}}{\text{oddinteger}} < 1$.

Lemma 6. If $nr < 1$, $r = \max\{r_i, i = 1, 2, \cdots, n\}$, then the solution of Equation (21) has an exponent estimate

$$x(t) = o\left(e^{At}\right) \quad (t > b > 0),$$

for constants $A > 0, b > 0$.

Proof. The proof of this conclusion is similar to that of Lemma 5, and thus we omit it. □

Theorem 5. Assume that $\tau > \sigma$, $\tau = \min\{\tau_i, i = 1, 2, \cdots, m\}$, $\sigma = \max\{\sigma_i, i = 1, 2, \cdots, n\}$, if

$$p(\lambda) = \lambda^{\alpha} + \lambda^{\alpha}\sum_{i=1}^{n} r_i e^{-\lambda\sigma_i} + \sum_{i=1}^{m} q_i e^{-\lambda\tau_i} = 0 \tag{22}$$

has no real roots, then every solution of (21) is oscillatory.

Proof. Taking Laplace transform of both sides of (21), we obtain

$$\begin{aligned} &s^{\alpha}X(s) - A + X(s)\sum_{i=1}^{n} r_i s^{\alpha} e^{-s\sigma_i} + \sum_{i=1}^{n} r_i s^{\alpha} e^{-s\sigma_i}\int_{-\sigma_i}^{0} e^{-st}x(t)dt \\ &-\sum_{i=1}^{n} B_i r_i + X(s)\sum_{i=1}^{m} q_i e^{-s\tau_i} + \sum_{i=1}^{m} q_i e^{-s\tau_i}\int_{-\tau_i}^{0} e^{-st}x(t)dt = 0, \end{aligned}$$

where $A = \left({}_0D_t^{\alpha-1}x(t)\right)\Big|_{t=0}$, $B_i = \left({}_0D_t^{\alpha-1}x(t-\sigma_i)\right)\Big|_{t=0}$.

Hence,

$$\begin{aligned} &X(s)\left(s^{\alpha} + \sum_{i=1}^{n} r_i s^{\alpha} e^{-s\sigma_i} + \sum_{i=1}^{m} q_i e^{-s\tau_i}\right) \\ &= A + \sum_{i=1}^{n} B_i r_i - \sum_{i=1}^{n} r_i s^{\alpha} e^{-s\sigma_i}\int_{-\sigma_i}^{0} e^{-st}x(t)dt - \sum_{i=1}^{m} q_i e^{-s\tau_i}\int_{-\tau_i}^{0} e^{-st}x(t)dt. \end{aligned} \tag{23}$$

Let

$$p(s)=s^{\alpha}+\sum_{i=1}^{n}r_i s^{\alpha}\mathrm{e}^{-s\sigma_i}+\sum_{i=1}^{m}q_i\mathrm{e}^{-s\tau_i},$$

$$\Phi(s)=A+\sum_{i=1}^{n}B_i r_i-\sum_{i=1}^{n}r_i s^{\alpha}\mathrm{e}^{-s\sigma_i}\int_{-\sigma_i}^{0}\mathrm{e}^{-st}x(t)\mathrm{d}t-\sum_{i=1}^{m}q_i\mathrm{e}^{-s\tau_i}\int_{-\tau_i}^{0}\mathrm{e}^{-st}x(t)\mathrm{d}t,$$

then, from (23), we get

$$X(s)p(s)=\Phi(s). \tag{24}$$

Then, the following proof process is similar to Theorem 1. one can complete the proof. □

Corollary 3. Assume that $\tau>\sigma$, $\tau=\min\{\tau_i,i=1,2,\cdots,m\}$, $\sigma=\max\{\sigma_i,i=1,2,\cdots,n\}$, if

$$\left(\frac{1}{m}\sum_{i=1}^{m}\tau_i-\frac{1}{n}\sum_{i=1}^{n}\sigma_i\right)\left(\frac{m\left(\prod_{i=1}^{m}q_i\right)^{\frac{1}{m}}}{1+n\left(\prod_{i=1}^{n}r_i\right)^{\frac{1}{n}}}\right)^{\frac{1}{\alpha}}>\frac{1}{\mathrm{e}}, \tag{25}$$

then every solution of (21) is oscillatory.

Proof. By using the arithmetic-geometric mean inequality

$$\sum_{i=1}^{m}a_i\ge m\left(\prod_{i=1}^{m}a_i\right)^{\frac{1}{m}},$$

for $s<0$, we find

$$\begin{aligned}
&\lambda^{\alpha}+\lambda^{\alpha}\sum_{i=1}^{n}r_i\mathrm{e}^{-\lambda\sigma_i}+\sum_{i=1}^{m}q_i\mathrm{e}^{-\lambda\tau_i}\\
&\ge\lambda^{\alpha}+\lambda^{\alpha}n\left(\prod_{i=1}^{n}r_i\mathrm{e}^{-\lambda\sigma_i}\right)^{\frac{1}{n}}+m\left(\prod_{i=1}^{m}q_i\mathrm{e}^{-\lambda\tau_i}\right)^{\frac{1}{m}}\\
&=\lambda^{\alpha}+\lambda^{\alpha}n\left(\prod_{i=1}^{n}r_i\right)^{\frac{1}{n}}\mathrm{e}^{-\lambda\frac{\sum_{i=1}^{n}\sigma_i}{n}}+m\left(\prod_{i=1}^{m}q_i\right)^{\frac{1}{m}}\mathrm{e}^{-\lambda\frac{\sum_{i=1}^{m}\tau_i}{m}}\\
&=\lambda^{\alpha}+\lambda^{\alpha}A\mathrm{e}^{-\lambda B}+C\mathrm{e}^{-\lambda D}.
\end{aligned} \tag{26}$$

Let

$$f(\lambda)=\lambda^{\alpha}+\lambda^{\alpha}A\mathrm{e}^{-\lambda B}+C\mathrm{e}^{-\lambda D} \tag{27}$$

where $A=n\left(\prod_{i=1}^{n}r_i\right)^{\frac{1}{n}}$, $B=\frac{1}{n}\sum_{i=1}^{n}\sigma_i$, $C=m\left(\prod_{i=1}^{m}q_i\right)^{\frac{1}{m}}$, $D=\frac{1}{m}\sum_{i=1}^{m}\tau_i$ and $D>B$.

By (25) and the proof process of Theorem 2, we can obtain $f(\lambda)>0$ for $\lambda\in R$. This is, (22) has no real roots, by Theorem 4, every solution of (21) is oscillatory. The proof is complete. □

We consider the following fractional order delay differential equation

$$ {}_0D_t^{\alpha}\left(x(t)+rx(t-\sigma)\right)+f\left(x(t-\tau)\right)=0 \tag{28}$$

(H): $f(u)\in C(R,R)$, $f(u)/u\ge C=const>0$, for $u\ne0$, $r\in R$,

$q, \sigma, \tau \in R^+$.

The standard initial condition associated with (28) is

$$x(t) = \varphi(t), t \in [-\xi, 0], \tag{29}$$

where $\varphi(t) \in C([-\xi, 0], R)$, $\xi = \max\{\sigma, \tau\}$.

Theorem 6. Assume that (H) holds, and if

$$(\tau - \sigma)\left(\frac{C}{1+r}\right)^{\frac{1}{\alpha}} > \frac{1}{\mathrm{e}}, \tag{30}$$

then every solution of Equations (28) is oscillatory.

Proof. Assume that (28) has an eventually positive solution $x(t)$, that is, there exists a sufficiently large positive constant T such that $x(t) > 0$, $x(t-\sigma) > 0$, $x(t-\tau) > 0$ for $t > T$, from (28), we obtain

$$0 = {}_0D_t^{\alpha}\left(x(t) + rx(t-\sigma)\right) + f\left(x(t-\tau)\right) \geq D_t^{\alpha}\left(x(t) + rx(t-\sigma)\right) + Cx(t-\tau).$$

Then we can see that the eventually positive solution $x(t)$ satisfies the inequality

$$D_t^{\alpha}\left(x(t) + rx(t-\sigma)\right) + Cx(t-\tau) \leq 0, t > T. \tag{31}$$

According to (30) and Theorem 2, it follows that Equation $p(\lambda) = \lambda^{\alpha} + r\lambda^{\alpha}\mathrm{e}^{-\lambda\sigma} + C\mathrm{e}^{-\lambda\tau} = 0$ has no real roots. Therefore, similarly to the proof of Theorem 1, inequality (31) has no eventually positive solution, which implies that every solution of (28) oscillates. □

We consider the delay fractional order partial differential equation

$$\begin{aligned} &{}_0D_t^{\alpha}\left(u(x,t) + \sum_{i=1}^{n} r_i u(x, t-\sigma_i)\right) \\ &= a(t)\Delta u(x,t) + \sum_{i=1}^{l} b_i(t)\Delta u(x, t-p_i) - \sum_{i=1}^{m} q_i(t) f_i\left(u(x, t-\tau_i)\right) \end{aligned} \tag{32}$$

where $(x,t) \in \Omega \times (0, \infty) = G$. Here $\Omega \subset R^N$ is a bounded domain with boundary $\partial\Omega$ smooth enough. The hypotheses are always true as follows:

(H1): $r_i, \sigma_i, p_i, \tau_i \in R^+$; $a(t), b_i(t), q_i(t) \in C(R^+, R^+)$; $q_i = \inf q_i(t) > 0$.

(H2): $f_i(u) \in C(R, R)$, $f_i(u)/u \geq C_i = const > 0$, for $u \neq 0$.

The initial condition $u(x,t) = \phi(x,t)$, $(x,t) \in \Omega \times [-\zeta, 0]$, $\zeta = \max\{\sigma_i, p_j, \tau_k, i = 1,2,\cdots,n; j = 1,2,\cdots,l; k = 1,2,\cdots,m\}$. Consider the boundary conditions as follows:

$$\frac{\partial u(x,t)}{\partial N} = 0, \text{ on } (x,t) \in \partial\Omega \times R^+, \tag{33}$$

where N is the unit exterior normal vector in $\partial\Omega$.

Theorem 7. Assume that $\tau > \sigma$, $\tau = \min\{\tau_i, i = 1,2,\cdots,m\}$, $\sigma = \max\{\sigma_i, i = 1,2,\cdots,n\}$, if

$$\left(\frac{1}{m}\sum_{i=1}^{m}\tau_i - \frac{1}{n}\sum_{i=1}^{n}\sigma_i\right)\left(\frac{m\left(\prod_{i=1}^{m} q_i C_i\right)^{\frac{1}{m}}}{1 + n\left(\prod_{i=1}^{n} r_i\right)^{\frac{1}{n}}}\right)^{\frac{1}{\alpha}} > \frac{1}{\mathrm{e}},$$

then every solution of (32) with the boundary condition (33) is oscillatory.

Proof. Assume that (32) with the boundary condition (33) has no oscillation solution, without loss of generality, we assume that $u(x,t)$ is an eventually positive solution of (32) which implies that there exists $T>0$ such that $u(x,t)>0$, $u(x,t-\sigma_i)>0$, $u(x,t-\tau_i)>0$ in $\Omega\times[T,\infty)$. Since (H1) and (H2), from (32), we can obtain

$$\begin{aligned}&{}_0D_t^{\alpha}\left(u(x,t)+\sum_{i=1}^{n}r_iu(x,t-\sigma_i)\right)\\&\le a(t)\Delta u(x,t)+\sum_{i=1}^{l}b_i(t)\Delta u(x,t-p_i)-\sum_{i=1}^{m}q_iC_iu(x,t-\tau_i).\end{aligned}\tag{34}$$

Integrating (34) with respect to x over Ω yields

$$\begin{aligned}&{}_0D_t^{\alpha}\left(\int_\Omega u(x,t)\mathrm{d}x+\sum_{i=1}^{n}r_i\int_\Omega u(x,t-\sigma_i)\mathrm{d}x\right)\\&\le a(t)\int_\Omega\Delta u(x,t)\mathrm{d}x+\sum_{i=1}^{l}b_i(t)\int_\Omega\Delta u(x,t-p_i)\mathrm{d}x-\sum_{i=1}^{m}q_iC_i\int_\Omega u(x,t-\tau_i)\mathrm{d}x.\end{aligned}\tag{35}$$

By Green's formula and the boundary condition (33), we have

$$\begin{aligned}&\int_\Omega\Delta u(x,t)\mathrm{d}x=\int_{\partial\Omega}\frac{\partial u(x,t)}{\partial N}\mathrm{d}S=0\\&\int_\Omega\Delta u(x,t-p_i)\mathrm{d}x=\int_{\partial\Omega}\frac{\partial u(x,t-p_i)}{\partial N}\mathrm{d}S=0.\end{aligned}\tag{36}$$

Let $v(t)=\int_\Omega u(x,t)\mathrm{d}x$, from (35) and (36), we get

$${}_0D_t^{\alpha}\left(v(t)+\sum_{i=1}^{n}r_iv(t-\sigma_i)\right)+\sum_{i=1}^{m}q_iC_iv(t-\tau_i)\le 0.\tag{37}$$

That is, there exists eventually positive solution for inequality (37). According to the conditions of Theorem 7 and Corollary 3, it follows that Equation $p(\lambda)=\lambda^{\alpha}+\lambda^{\alpha}\sum_{i=1}^{n}r_i\mathrm{e}^{-\lambda\sigma_i}+\sum_{i=1}^{m}q_iC_i\mathrm{e}^{-\lambda\tau_i}=0$ has no real roots. Therefore, similarly to the proof of Theorem 5, inequality (37) has no eventually positive solution which implies that every solution of (32) oscillates. □

4. Example

Example 1. Consider the following fractional differential equation

$${}_0D_t^{1/3}\left(x(t)+0.5x(t-0.5)\right)+\left(x(t-1)\right)^3+2x(t-1)=0,t>0.\tag{38}$$

Notice $\alpha=1/3$, $r=0.5$, $\sigma=0.5$, $\tau=1$, $f(u)=u^3+2u$, then it is easy to find $C=2$ and $(\tau-\sigma)\left(\frac{C}{1+r}\right)^{\frac{1}{\alpha}}=\frac{32}{27}>\frac{1}{\mathrm{e}}$, then (38) is oscillatory by Theorem 6.

Example 2. Consider the following fractional differential equation

$${}_0D_t^{1/5}\left(u(x,t)+\frac{1}{4}u\left(x,t-\frac{1}{3}\right)+u\left(x,t-\frac{1}{2}\right)\right)$$

$$= \mathrm{e}^t \Delta u(x,t) + \Delta u\left(x, t-\frac{1}{2}\right) + t^2 \Delta u\left(x, t-\frac{4}{5}\right) \\ -\left[\left(t+\frac{1}{t}\right) u\left(x, t-\frac{3}{2}\right) + \sin\left(u(x,t-1)\right) + 2u(x,t-1)\right] \tag{39}$$

with the boundary conditions

$$\frac{\partial u(0,t)}{\partial x} = \frac{\partial u(5,t)}{\partial x} = 0, (x,t) \in (0,5) \times (0,\infty). \tag{40}$$

Notice $\alpha = \frac{1}{5}$, $r_1 = \frac{1}{4}$, $r_2 = 1$, $\sigma_1 = \frac{1}{3}$, $\sigma_2 = \frac{1}{2}$, $a(t) = \mathrm{e}^t$, $b_1(t) = 1$, $b_2(t) = t^2$, $p_1 = \frac{1}{2}$, $p_2 = \frac{4}{5}$, $q_1(t) = t + \frac{1}{t}$, $q_2(t) = 1$, $f_1(u) = u$, $f_2(u) = \sin(u) + 2u$, $\tau_1 = \frac{3}{2}$, $\tau_2 = 1$, then it is easy to find $q_1 = 2$, $q_2 = 1$, $C_1 = 1$, $C_2 = 1$.

Therefore, $\left(\frac{1}{m}\sum_{i=1}^{m}\tau_i - \frac{1}{n}\sum_{i=1}^{n}\sigma_i\right)\left(\frac{m\left(\prod_{i=1}^{m} q_i C_i\right)^{\frac{1}{m}}}{1+n\left(\prod_{i=1}^{n} r_i\right)^{\frac{1}{n}}}\right)^{\frac{1}{\alpha}} = \frac{10\sqrt{2}}{3} > \frac{1}{\mathrm{e}}$ then (39) is oscillatory by Theorem 7.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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