Journal of Applied Mathematics and Physics, 2022, 10, 2951-2964
https://www.scirp.org/journal/jamp
ISSN Online: 2327-4379
ISSN Print: 2327-4352

On the Far Field Pattern for Acoustic Scattering by a Piecewise Homogeneous Obstacle in Two Dimensions

Evagelia S. Athanasiadou[1], Paraskevi Roupa[2*]

[1]Department of Mathematics, University of Athens, Athens, Greece
[2]Department of Informatics and Telecommunications, University of Athens, Athens, Greece
Email: eathan@math.uoa.gr, *parask_roupa@di.uoa.gr

How to cite this paper: Athanasiadou, E.S. and Roupa, P. (2022) On the Far Field Pattern for Acoustic Scattering by a Piecewise Homogeneous Obstacle in Two Dimensions. *Journal of Applied Mathematics and Physics*, **10**, 2951-2964.
https://doi.org/10.4236/jamp.2022.1010198

Received: July 30, 2022
Accepted: October 15, 2022
Published: October 18, 2022

Abstract

A time-harmonic plane acoustic wave is scattered by a piecewise homogeneous obstacle with a penetrable or impenetrable core. We construct in the close form an integral representation for the far field pattern in which we have incorporated the physical and geometrical characteristics of the scatterer. Through this representation, we obtain the far field pattern for this scatterer. We prove scattering relations between the far field patterns of two scattering problems due to two distinct incident waves on the same scatterer. In particular, we prove reciprocity and general scattering theorems. The optical theorem, connecting the total power that the scatterer extracts from the incident plane wave either by radiation or by absorption with the corresponding far field pattern of an incident plane wave, is recovered as a corollary of the general scattering theorem. Moreover, if we consider incident waves to be both a plane and a spherical, we derive a mixed reciprocity theorem. We define the corresponding far field operators and using these relations, we prove some properties that can be used for solving inverse scattering problems.

Keywords

Two-Dimensional Electromagnetic Scattering, Piecewise Obstacle, Reciprocity Principle, Optical Theorem

1. Introduction

We consider a scattering problem of time-harmonic acoustic waves by a piecewise homogeneous scatterer in two dimensions. This type of scatterer arises when a layered infinitely long cylinder, which is oriented parallel to the x_3-axis, is intersected by the x_1x_2-plane (for more details, see [1]). We define the far field pattern (or scattering amplitude) which is an analytic function defined on the unit disc (or unit sphere for 3D-scattering). This function plays an important role in studying inverse scattering problems. Given the far field pattern for one or several incident plane waves, we can define some geometrical and physical characteristics of the scatterer [2]. Specifically, we study properties of the far field pattern for a scatterer that is a piecewise homogeneous obstacle with a core that may be sound soft, hard, penetrable or impedance.

Scattering theorems in three dimensions have been proved for various scattering models in the case of acoustic [3], electromagnetic [4] and elastic [5] waves. Since 1954 Twersky has proved reciprocity, scattering and optical theorems for acoustic in [6] and electromagnetic in [4] waves. Using these results and low-frequency expansions, he derived the leading-term approximation of the real part of the far field pattern. Moreover, at the same period of time, De Hoop proved a reciprocity theorem for electromagnetic waves in [7] and for liner viscoelastic media in [8]. Reciprocity relations for acoustic and electromagnetic far field patterns have been recorded by Colton and Kress in the books [2] [9]. Reciprocity, general and optical theorems have also been stated and proved by Dassios and Kleinman in their book [10]. In [11], it is proved a reciprocity theorem which corresponds to an impedance boundary value problem in two and three dimensions. In [5], Dassios, Kiriaki and Polyzos proved scattering relations for elastic waves defining far field patterns in a spherical coordinate system. In [12], Athanasiadis has studied far field patterns for electromagnetic scattering by a chiral obstacle in a chiral environment. In [13], it is shown that the correlation-type reciprocity theorem for the scattered field is the progenitor of the generalized optical theorem. Gintides and Kiriaki in [14], using a dyadic representation for the displacement field constructed the longitudinal and transverse parts of the dyadic far-field patterns for the Dirichlet and Neumann problem. A generalization of the optical theorem for the case of excitation of a local body by a multipole can be found in [15].

Moreover, three-dimensional scattering theorems for multi-layered scatterers have been proved in [3] for acoustic waves. Scattering relations for spherical acoustic and electromagnetic waves as well as a mixed reciprocity principle have been proved in [16]. Whereas in [17], corresponding relations for elastic waves are proved. In [18], the authors, using a mixed reciprocity theorem and the factorization method, solved a two-layered background medium inverse scattering problem. In [19], the two-phase acoustic streaming characteristics and droplet properties generated by a dental ultrasonic scaler are investigated.

Concerning the two-dimensional case, reciprocity and other scattering relations have been stated and proved in [1]. In [20], reciprocity, general and optical relations for chiral obstacles are established.

The far field pattern is the most important function in scattering theory. Hence it is worth its study for a complicated scattering model. The novelty of the present work lies in the kind of scatterer. As it is shown from the references, the research has been done for simple scatterers and especially in three dimensions.

To our knowledge, there are no results for multi-layered scatterers in two dimensions. The derived formulae show the dependence of the far field pattern, the extinction cross-section and the far field operator of the scatterer's structure. The reciprocity theorem that is proved for the far field pattern implies that the far field operator is injective, has a dense range and is normal.

In Section 2, we formulate a two-dimensional scattering problem for a piecewise homogeneous obstacle with a core which may be sound soft, hard, penetrable or impedance. In Section 3, we obtain far field patterns for this scatterer. Using plane incidence two-dimensional reciprocity, general and optical scattering theorems are proved in Section 4. Also, assuming that the scatterer is excited by a plane and a point-source wave, we derive mixed reciprocity theorems. In Section 5, we define the far field operator and we prove various properties which are useful in solving inverse scattering problems. Finally, in Section 6, we discuss some special cases of the described scattering problems.

2. Formulation of the Problem

We consider scattering by a piecewise homogeneous obstacle in $\mathbb{R}^2$. Let D be a bounded subset of $\mathbb{R}^2$ with a C^2-boundary $S_0 = \partial D$. The exterior $D_0 = \mathbb{R}^2 \setminus \bar{D}$ of the obstacle is an infinite homogeneous isotropic medium with mass density ρ_0 and mean compressibility γ_0. The interior of D is divided by means of closed and nonintersecting C^2-curves S_j, $j = 1, 2, \cdots, N$, into layers D_j, $j = 1, 2, \cdots, N+1$, with $\partial D_{j-1} \cap \partial D_j = S_{j-1}$. The curve S_{j-1} surrounds S_j and there is one normal unit vector $\hat{\boldsymbol{\nu}}(\boldsymbol{x})$ at each point $\boldsymbol{x}$ of any curve S_j pointing into D_j. The region D_{N+1}, within which lies the origin, is the core of the scatterer which may be sound soft, hard, penetrable or impedance, see Figure 1. The layer D_j is a homogeneous isotropic medium with mass density ρ_j associated with the velocity field and mean compressibility γ_j associated with the pressure field in D_j. All the physical parameters ρ_j and γ_j are positive constants and the real wave numbers in D_j are given by

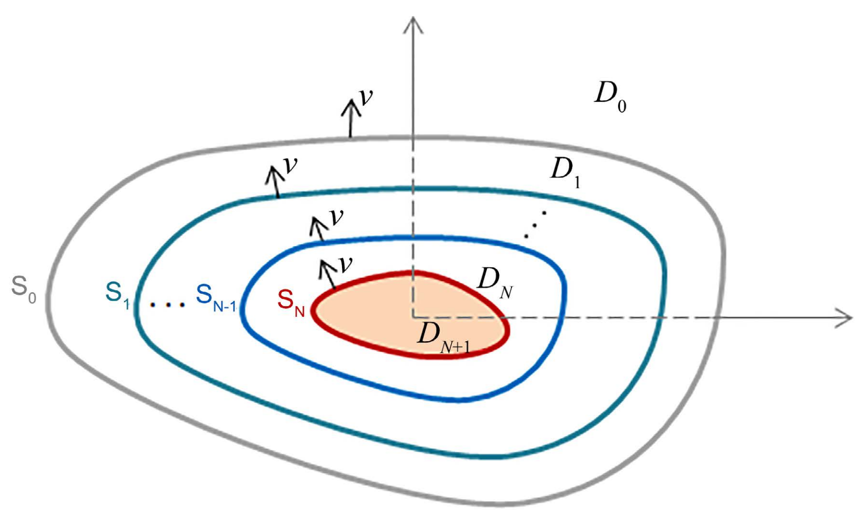

Figure 1. The piecewise homogeneous scatterer.

$$k_j = \omega\sqrt{\gamma_j \rho_j}, \quad j = 0,1,\cdots,N, \tag{1}$$

where ω is the angular frequency.

The total acoustic field in D_j satisfies the Helmholtz equation

$$\Delta u_j + k_j^2 u_j = 0 \text{ in } D_j, \quad j = 0,1,\cdots,N, \tag{2}$$

where $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$ is the two-dimensional Laplace operator. In the core D_{N+1} it holds the same Equation (2) for $j = N+1$.

The total exterior acoustic field u_0 is given by

$$u_0 = u^{\text{inc}} + u^{\text{sc}} \text{ in } \mathbb{R}^2 \setminus D, \tag{3}$$

where u^{sc} is the scattered field which satisfies the Sommerfeld radiation condition:

$$\lim_{|\boldsymbol{x}|\to\infty} \sqrt{|\boldsymbol{x}|}\left(\frac{\partial u^{\text{sc}}}{\partial \nu} - ik_0 u^{\text{sc}}\right) = 0. \tag{4}$$

On the curve S_j the following transmission conditions are valid for $j = 0,1,\cdots,N-1$,

$$u_j = u_{j+1} \text{ on } S_j, \tag{5}$$

$$\frac{\partial u_j}{\partial \nu} = \frac{\rho_j}{\rho_{j+1}} \frac{\partial u_{j+1}}{\partial \nu} \text{ on } S_j. \tag{6}$$

On the curve S_N of the core, we consider either the Dirichlet boundary condition (soft core):

$$u_N = 0 \text{ on } S_N, \tag{7}$$

or the Neumann boundary condition (hard core):

$$\frac{\partial u_N}{\partial \nu} = 0 \text{ on } S_N, \tag{8}$$

or the transmission boundary condition (penetrable core):

$$u_N = u_{N+1} \text{ on } S_N, \tag{9}$$

$$\frac{\partial u_N}{\partial \nu} = \frac{\rho_N}{\rho_{N+1}} \frac{\partial u_{N+1}}{\partial \nu} \text{ on } S_N, \tag{10}$$

or the Robin boundary condition (impedance core):

$$\frac{\partial u_N}{\partial \nu} = -i\lambda u_N \text{ on } S_N, \tag{11}$$

where $\lambda > 0$ is the surface impedance.

Summarizing the above analysis, we formulate the following two-dimensional scattering problems. The first problem is defined by Equations (2)-(7) and is denoted by (P_D); the second one is defined by Equations (2)-(6), (8) and is denoted by (P_N); the third one is defined by Equations (2)-(6), (9), (10) and is denoted by (P_T) and the last one is defined by Equations (2)-(6), (11) and is denoted by (P_I).

3. The Far Field Patterns

The scattered field u^{sc} is a radiating solution of the Helmholtz equation in $\mathbb{R}^2 \setminus \bar{D}$ and it has the integral representation [1],

$$u^{\mathrm{sc}}(\boldsymbol{x}) = \int_{S_0} \left(u^{\mathrm{sc}}(\boldsymbol{y}) \frac{\partial \Phi(\boldsymbol{x},\boldsymbol{y})}{\partial \nu(\boldsymbol{y})} - \Phi(\boldsymbol{x},\boldsymbol{y}) \frac{\partial u^{\mathrm{sc}}(\boldsymbol{y})}{\partial \nu} \right) \mathrm{d}s(\boldsymbol{y}), \tag{12}$$

where $\Phi(\boldsymbol{x},\boldsymbol{y}) = \frac{i}{4} H_0^{(1)}\left(k_0 |\boldsymbol{x}-\boldsymbol{y}|\right)$ is the two-dimensional fundamental solution of the Helmholtz equation in D_0. $H_0^{(1)}$ is the Hankel function of the first kind of order zero and k_0 is the wave number [1].

Using the asymptotic relations for $|\boldsymbol{x}-\boldsymbol{y}| \to 0$, [1]

$$\Phi(\boldsymbol{x},\boldsymbol{y}) = \frac{1}{2\pi} \log \frac{1}{|\boldsymbol{x}-\boldsymbol{y}|} + \mathcal{O}(1), \tag{13}$$

$$\frac{\partial}{\partial \nu(\boldsymbol{y})} \Phi(\boldsymbol{x},\boldsymbol{y}) = \frac{1}{2\pi} \frac{1}{|\boldsymbol{x}-\boldsymbol{y}|} + \mathcal{O}\left(|\boldsymbol{x}-\boldsymbol{y}| \log |\boldsymbol{x}-\boldsymbol{y}|\right), \tag{14}$$

and taking into account the radiation condition (4) we obtain

$$u^{\mathrm{sc}}(\boldsymbol{x}) = \frac{\mathrm{e}^{ik_0|\boldsymbol{x}|}}{\sqrt{|\boldsymbol{x}|}} u^{\infty}(\hat{\boldsymbol{x}}) + \mathcal{O}\left(|\boldsymbol{x}|^{-3/2}\right), \quad |\boldsymbol{x}| \to \infty, \tag{15}$$

where $u^{\infty}(\hat{\boldsymbol{x}})$ is the far field pattern which is defined in the unit disc in $\mathbb{R}^2$ S^1 and it is given by

$$u^{\infty}(\hat{\boldsymbol{x}}) = \frac{\mathrm{e}^{i\pi/4}}{\sqrt{8\pi k_0}} \int_{S_0} \left[u^{\mathrm{sc}}(\boldsymbol{y}) \frac{\partial}{\partial \nu(\boldsymbol{y})} \mathrm{e}^{-ik_0 \hat{\boldsymbol{x}}\cdot\boldsymbol{y}} - \mathrm{e}^{-ik_0 \hat{\boldsymbol{x}}\cdot\boldsymbol{y}} \frac{\partial}{\partial \nu} u^{\mathrm{sc}}(\boldsymbol{y}) \right] \mathrm{d}s(\boldsymbol{y}). \tag{16}$$

In (16) we substitute $u^{\mathrm{sc}} = u_0 - u^{\mathrm{inc}}$ and we get

$$\begin{aligned} u^{\infty}(\hat{\boldsymbol{x}}) &= \frac{\mathrm{e}^{i\pi/4}}{\sqrt{8\pi k_0}} \int_{S_0} \left[u_0(\boldsymbol{y}) \frac{\partial}{\partial \nu(\boldsymbol{y})} \mathrm{e}^{-ik_0 \hat{\boldsymbol{x}}\cdot\boldsymbol{y}} - \mathrm{e}^{-ik_0 \hat{\boldsymbol{x}}\cdot\boldsymbol{y}} \frac{\partial}{\partial \nu} u_0(\boldsymbol{y}) \right] \mathrm{d}s(\boldsymbol{y}) \\ &\quad - \frac{\mathrm{e}^{i\pi/4}}{\sqrt{8\pi k_0}} \int_{S_0} \left[u^{\mathrm{inc}}(\boldsymbol{y}) \frac{\partial}{\partial \nu(\boldsymbol{y})} \mathrm{e}^{-ik_0 \hat{\boldsymbol{x}}\cdot\boldsymbol{y}} - \mathrm{e}^{-ik_0 \hat{\boldsymbol{x}}\cdot\boldsymbol{y}} \frac{\partial}{\partial \nu} u^{\mathrm{inc}}(\boldsymbol{y}) \right] \mathrm{d}s(\boldsymbol{y}). \end{aligned} \tag{17}$$

The last integral is equal to zero, since $\mathrm{e}^{-ik_0 \hat{\boldsymbol{x}}\cdot\boldsymbol{y}}$ and u^{inc} are entire solutions of the Helmholtz Equation (2) for $j = 0$. For the first integral we use the transmission conditions (5), (6), we apply successively Green's first theorem on $u_j(\boldsymbol{x})$ and $\mathrm{e}^{-ik_0 \hat{\boldsymbol{x}}\cdot\boldsymbol{y}}$ in D_j and we get

$$\begin{aligned} &\int_{S_0} \left[u_0(\boldsymbol{y}) \frac{\partial}{\partial \nu(\boldsymbol{y})} \mathrm{e}^{-ik_0 \hat{\boldsymbol{x}}\cdot\boldsymbol{y}} - \mathrm{e}^{-ik_0 \hat{\boldsymbol{x}}\cdot\boldsymbol{y}} \frac{\partial}{\partial \nu} u_0(\boldsymbol{y}) \right] \mathrm{d}s(\boldsymbol{y}) \\ &= I_{D_N}(\hat{\boldsymbol{x}}) + I_{S_N}^{(1)}(\hat{\boldsymbol{x}}) + I_{S_N}^{(2)}(\hat{\boldsymbol{x}}), \end{aligned} \tag{18}$$

where

$$\begin{aligned} I_{D_N}(\hat{\boldsymbol{x}}) &= k_0^2 \sum_{j=1}^{N} \left(\frac{\gamma_j}{\gamma_0} - 1 \right) \int_{D_j} u_j(\boldsymbol{y}) \mathrm{e}^{-ik_0 \hat{\boldsymbol{x}}\cdot\boldsymbol{y}} \mathrm{d}V(\boldsymbol{y}) \\ &\quad + \sum_{j=1}^{N} \left(1 - \frac{\rho_0}{\rho_j} \right) \int_{D_j} \mathit{grad}\, u_j(\boldsymbol{y}) \cdot \mathit{grad}\, \mathrm{e}^{-ik_0 \hat{\boldsymbol{x}}\cdot\boldsymbol{y}} \mathrm{d}V(\boldsymbol{y}) \end{aligned} \tag{19}$$

$$I_{S_N}^{(1)}(\hat{\boldsymbol{x}}) = \int_{S_N} u_N(\boldsymbol{y}) \frac{\partial \mathrm{e}^{-ik_0\hat{\boldsymbol{x}}\cdot\boldsymbol{y}}}{\partial \nu(\boldsymbol{y})} \mathrm{d}s(\boldsymbol{y}), \tag{20}$$

$$I_{S_N}^{(2)}(\hat{\boldsymbol{x}}) = -\frac{\rho_0}{\rho_N} \int_{S_N} \mathrm{e}^{-ik_0\hat{\boldsymbol{x}}\cdot\boldsymbol{y}} \frac{\partial u_N(\boldsymbol{y})}{\partial \nu} \mathrm{d}s(\boldsymbol{y}). \tag{21}$$

Now we apply the boundary and transmission conditions on the core, we take the following far field patterns:

$$u_D^\infty(\hat{\boldsymbol{x}}) = \frac{\mathrm{e}^{i\pi/4}}{\sqrt{8\pi k_0}} \left(I_{D_N}(\hat{\boldsymbol{x}}) + I_{S_N}^{(2)}(\hat{\boldsymbol{x}}) \right), \tag{22}$$

for Dirichlet condition on core,

$$u_N^\infty(\hat{\boldsymbol{x}}) = \frac{\mathrm{e}^{i\pi/4}}{\sqrt{8\pi k_0}} \left(I_{D_N}(\hat{\boldsymbol{x}}) + I_{S_N}^{(1)}(\hat{\boldsymbol{x}}) \right), \tag{23}$$

for Neumann condition on core,

$$u_T^\infty(\hat{\boldsymbol{x}}) = \frac{\mathrm{e}^{i\pi/4}}{\sqrt{8\pi k_0}} I_{D_{N+1}}(\hat{\boldsymbol{x}}), \tag{24}$$

for the transmission conditions on core,

$$u_I^\infty(\hat{\boldsymbol{x}}) = \frac{\mathrm{e}^{i\pi/4}}{\sqrt{8\pi k_0}} \left(I_{D_N}(\hat{\boldsymbol{x}}) + I_{S_N}^{(1)}(\hat{\boldsymbol{x}}) + I_{S_N}^{(3)}(\hat{\boldsymbol{x}}) \right), \tag{25}$$

for the impedance boundary condition on core, where

$$I_{S_N}^{(3)}(\hat{\boldsymbol{x}}) = \frac{i\lambda\rho_0}{\rho_N} \int_{S_N} u_N(\boldsymbol{y}) \mathrm{e}^{-ik_0\hat{\boldsymbol{x}}\cdot\boldsymbol{y}} \mathrm{d}s(\boldsymbol{y}). \tag{26}$$

4. Scattering Theorems

In the sequel, as we have already mentioned before, we consider an incident plane electric wave $u^{\mathrm{inc}}(\boldsymbol{x}, \hat{\boldsymbol{d}}) = \mathrm{e}^{ik_0\boldsymbol{x}\cdot\hat{\boldsymbol{d}}}$, $\hat{\boldsymbol{d}} \in S^1$. We write $u_{a,j}(\boldsymbol{x}, \hat{\boldsymbol{d}})$, $u_a^{\mathrm{sc}}(\boldsymbol{x}, \hat{\boldsymbol{d}})$, $u_a^\infty(\hat{\boldsymbol{x}}, \hat{\boldsymbol{d}})$ for representing the dependence of the total field in D_j, the scattered field and the far field pattern on the incident direction $\hat{\boldsymbol{d}}$ for the scattering problem (P_a), $a = D, N, T, I$.

The vectors $\boldsymbol{x}$ and $\hat{\boldsymbol{d}}$ are expressed in terms of polar coordinates as $\boldsymbol{x} = (r\cos\theta, r\sin\theta)$, $r = |\boldsymbol{x}|$, $\hat{\boldsymbol{d}} = (\cos\phi, \sin\phi)$, where $\theta, \phi \in [0, 2\pi]$ are the polar angles of $\boldsymbol{x}$ and $\hat{\boldsymbol{d}}$, respectively. For convenient reasons we write for the far field pattern $u_a^\infty(\hat{\boldsymbol{x}}, \hat{\boldsymbol{d}}) = u_a^\infty(\theta, \phi)$. The far field pattern $u_a^\infty(\theta, \phi)$ (16) is given by [1],

$$u_a^\infty(\theta, \phi) = \frac{\mathrm{e}^{i\pi/4}}{\sqrt{8\pi k_0}} \int_{S_0} \left[u_a^{\mathrm{sc}} \frac{\partial}{\partial \nu(\boldsymbol{y})} \mathrm{e}^{-ik_0 r_y \cos(\theta - \theta_y)} - \frac{\partial u_a^{\mathrm{sc}}}{\partial \nu(\boldsymbol{y})} \mathrm{e}^{-ik_0 r_y \cos(\theta - \theta_y)} \right] \mathrm{d}s(\boldsymbol{y}), \tag{27}$$

where (r_y, θ_y) are the polar coordinates of $\boldsymbol{y}$.

In the rest of the paper, we will use Twersky's notation [6],

$$\{u, v\}_S = \int_S \left(u \frac{\partial v}{\partial \nu} - v \frac{\partial u}{\partial \nu} \right) \mathrm{d}s. \tag{28}$$

As we can see the far field pattern is expressed through the Twersky's notation as

$$u_a^\infty(\theta,\phi) = u_a^\infty(\hat{\boldsymbol{x}},\hat{\boldsymbol{d}}) = \frac{\mathrm{e}^{i\pi/4}}{\sqrt{8\pi k_0}}\left\{u_a^{\mathrm{sc}}(\cdot,\hat{\boldsymbol{d}}), u^{\mathrm{inc}}(\cdot,-\hat{\boldsymbol{x}})\right\}_{S_0}. \tag{29}$$

Next, we formulate and prove the classical reciprocity theorem in two dimensions for a multi-layered scatterer.

Theorem 1 (Reciprocity). Let $u^{\mathrm{inc}}(\cdot,\hat{\boldsymbol{d}})$ and $u^{\mathrm{inc}}(\cdot,-\hat{\boldsymbol{x}})$, with $\hat{\boldsymbol{d}} = (\cos\phi,\sin\phi)$, $\hat{\boldsymbol{x}} = (\cos\theta,\sin\theta)$, be two incident plane waves. Then the far field pattern $u_a^\infty(\theta,\phi)$ corresponding to the scattering problem (P_a), $a = D,N,T,I$, satisfies the reciprocity principle

$$u_a^\infty(\theta,\phi) = u_a^\infty(\phi+\pi,\theta+\pi), \tag{30}$$

for all $\theta,\phi\in[0,2\pi]$.

Proof. In view of (3) and the bilinearity of (28), we get

$$\begin{aligned}\left\{u_{a,0}(\cdot,\hat{\boldsymbol{d}}), u_{a,0}(\cdot,-\hat{\boldsymbol{x}})\right\}_{S_0} &= \left\{u^{\mathrm{inc}}(\cdot,\hat{\boldsymbol{d}}), u^{\mathrm{inc}}(\cdot,-\hat{\boldsymbol{x}})\right\}_{S_0} + \left\{u^{\mathrm{inc}}(\cdot,\hat{\boldsymbol{d}}), u_a^{\mathrm{sc}}(\cdot,-\hat{\boldsymbol{x}})\right\}_{S_0} \\ &\quad + \left\{u_a^{\mathrm{sc}}(\cdot,\hat{\boldsymbol{d}}), u^{\mathrm{inc}}(\cdot,-\hat{\boldsymbol{x}})\right\}_{S_0} + \left\{u_a^{\mathrm{sc}}(\cdot,\hat{\boldsymbol{d}}), u_a^{\mathrm{sc}}(\cdot,-\hat{\boldsymbol{x}})\right\}_{S_0}.\end{aligned} \tag{31}$$

We observe that

$$\int_{D_j}\left[u_{a,j}(\cdot,\hat{\boldsymbol{d}})\Delta u_{a,j}(\cdot,-\hat{\boldsymbol{x}}) - u_{a,j}(\cdot,-\hat{\boldsymbol{x}})\Delta u_{a,j}(\cdot,\hat{\boldsymbol{d}})\right]\mathrm{d}V = 0, \tag{32}$$

since both $u_{a,j}(\cdot,-\hat{\boldsymbol{x}})$ and $u_{a,j}(\cdot,\hat{\boldsymbol{d}})$ are solutions of (2) in D_j. We apply successively the scalar Green's second theorem on $u_{a,j}(\cdot,-\hat{\boldsymbol{x}})$ and $u_{a,j}(\cdot,\hat{\boldsymbol{d}})$ and by using the transmission conditions (5), (6) we conclude that

$$\begin{aligned}&\left\{u_{a,0}(\cdot,\hat{\boldsymbol{d}}), u_{a,0}(\cdot,-\hat{\boldsymbol{x}})\right\}_{S_0} \\ &= \frac{\rho_0}{\rho_N}\int_{S_N}\left(u_{a,N}(\cdot,\hat{\boldsymbol{d}})\frac{\partial u_{a,N}(\cdot,-\hat{\boldsymbol{x}})}{\partial\nu} - u_{a,N}(\cdot,-\hat{\boldsymbol{x}})\frac{\partial u_{a,N}(\cdot,\hat{\boldsymbol{d}})}{\partial\nu}\right)\mathrm{d}s = 0,\end{aligned} \tag{33}$$

due to the imposed boundary condition (7) for (P_D), (8) for (P_N) and (11) for (P_I) on the core. For the scattering problem (P_T) we apply again the scalar Green's second theorem in D_{N+1} and we obtain

$$\left\{u_{a,0}(\cdot,\hat{\boldsymbol{d}}), u_{a,0}(\cdot,-\hat{\boldsymbol{x}})\right\}_{S_0} = 0. \tag{34}$$

For the first integral of the right-hand side of (31), we apply the scalar Green's second theorem and since the incident waves are entire solutions of the Helmholtz equation (2) for $j=0$, we get

$$\left\{u^{\mathrm{inc}}(\cdot,\hat{\boldsymbol{d}}), u^{\mathrm{inc}}(\cdot,-\hat{\boldsymbol{x}})\right\}_{S_0} = 0. \tag{35}$$

For the evaluation of the last integral of (31), we consider a disc S_R centred at the origin with radius R large enough to include $\overline{D}$ in its interior. We apply once more the scalar Green's second theorem on $u_a^{\mathrm{sc}}(\cdot,\hat{\boldsymbol{d}})$ and $u_a^{\mathrm{sc}}(\cdot,-\hat{\boldsymbol{x}})$ in the region exterior to S_0 and interior to ∂S_R. Hence, we get that the desired integral is equal to the line integral on ∂S_R. By letting $R\to\infty$ and taking into account the asymptotic behaviour (15), we have

$$\left\{u_a^{\mathrm{sc}}\left(\cdot,\hat{\boldsymbol{d}}\right),u_a^{\mathrm{sc}}\left(\cdot,-\hat{\boldsymbol{x}}\right)\right\}_{S_0}=0. \tag{36}$$

From (29), we can write

$$\left\{u_a^{\mathrm{sc}}\left(\cdot,\hat{\boldsymbol{d}}\right),u^{\mathrm{inc}}\left(\cdot,-\hat{\boldsymbol{x}}\right)\right\}_{S_0}=\sqrt{8\pi k_0}\,\mathrm{e}^{-i\pi/4}u_a^{\infty}\left(\hat{\boldsymbol{x}},\hat{\boldsymbol{d}}\right), \tag{37}$$

$$\left\{u^{\mathrm{inc}}\left(\cdot,\hat{\boldsymbol{d}}\right),u_a^{\mathrm{sc}}\left(\cdot,-\hat{\boldsymbol{x}}\right)\right\}_{S_0}=-\sqrt{8\pi k_0}\,\mathrm{e}^{-i\pi/4}u_a^{\infty}\left(-\hat{\boldsymbol{d}},-\hat{\boldsymbol{x}}\right). \tag{38}$$

Therefore, we have

$$u_a^{\infty}\left(\hat{\boldsymbol{x}},\hat{\boldsymbol{d}}\right)=u_a^{\infty}\left(-\hat{\boldsymbol{d}},-\hat{\boldsymbol{x}}\right), \tag{39}$$

and the proof of the theorem is complete. □

Next, we state and prove a general scattering theorem which is useful for the study of the far field operator. In what follows, $\overline{w}$ will denote the complex conjugate of *w*.

Theorem 2 (General). Let $u^{\mathrm{inc}}\left(\cdot,\hat{\boldsymbol{d}}\right)$ and $u^{\mathrm{inc}}\left(\cdot,\hat{\boldsymbol{x}}\right)$, with $\hat{\boldsymbol{d}}=\left(\cos\phi,\sin\phi\right)$ and $\hat{\boldsymbol{x}}=\left(\cos\theta,\sin\theta\right)$, be two incident plane waves. Then the far field pattern $u_a^{\infty}\left(\theta,\phi\right)$ corresponding to the scattering problem (P_a), $a=D,N,T,I$, satisfies

$$\mathrm{e}^{-i\pi/4}u_a^{\infty}\left(\phi,\theta\right)-\mathrm{e}^{i\pi/4}\overline{u_a^{\infty}\left(\theta,\phi\right)}-i\sqrt{\frac{k_0}{2\pi}}\int_0^{2\pi}\overline{u_a^{\infty}\left(\theta_y,\phi\right)}u_a^{\infty}\left(\theta_y,\theta\right)\mathrm{d}s\left(\theta_y\right)=\mathcal{E}_a\left(\phi,\theta\right), \tag{40}$$

for all $\theta,\phi\in\left[0,2\pi\right]$, where $\mathcal{E}_a\left(\phi,\theta\right)$ depends on the scatterer. In particular,

$$\mathcal{E}_a\left(\phi,\theta\right)=0,\ \text{for}\ a=D,N,T, \tag{41}$$

$$\mathcal{E}_a\left(\phi,\theta\right)=-\frac{i\lambda\rho_0}{\sqrt{2\pi k_0}\,\rho_N}\int_{S_N}\overline{u_{I,N}\left(\boldsymbol{y},\hat{\boldsymbol{d}}\right)}u_{I,N}\left(\boldsymbol{y},\hat{\boldsymbol{x}}\right)\mathrm{d}s\left(\boldsymbol{y}\right),\ \text{for}\ a=I. \tag{42}$$

Proof. This theorem is proved in a similar way as Theorem 1. In view of (3), we have

$$\begin{aligned}\left\{\overline{u_{a,0}\left(\cdot,\hat{\boldsymbol{d}}\right)},u_{a,0}\left(\cdot,\hat{\boldsymbol{x}}\right)\right\}_{S_0}&=\left\{\overline{u^{\mathrm{inc}}\left(\cdot,\hat{\boldsymbol{d}}\right)},u^{\mathrm{inc}}\left(\cdot,\hat{\boldsymbol{x}}\right)\right\}_{S_0}+\left\{\overline{u^{\mathrm{inc}}\left(\cdot,\hat{\boldsymbol{d}}\right)},u_a^{\mathrm{sc}}\left(\cdot,\hat{\boldsymbol{x}}\right)\right\}_{S_0}\\&\quad+\left\{\overline{u_a^{\mathrm{sc}}\left(\cdot,\hat{\boldsymbol{d}}\right)},u^{\mathrm{inc}}\left(\cdot,\hat{\boldsymbol{x}}\right)\right\}_{S_0}+\left\{\overline{u_a^{\mathrm{sc}}\left(\cdot,\hat{\boldsymbol{d}}\right)},u_a^{\mathrm{sc}}\left(\cdot,\hat{\boldsymbol{x}}\right)\right\}_{S_0}.\end{aligned} \tag{43}$$

As in Theorem 1, we get

$$\left\{\overline{u^{\mathrm{inc}}\left(\cdot,\hat{\boldsymbol{d}}\right)},u^{\mathrm{inc}}\left(\cdot,\hat{\boldsymbol{x}}\right)\right\}_{S_0}=0, \tag{44}$$

$$\left\{\overline{u^{\mathrm{inc}}\left(\cdot,\hat{\boldsymbol{d}}\right)},u_a^{\mathrm{sc}}\left(\cdot,\hat{\boldsymbol{x}}\right)\right\}_{S_0}=-\sqrt{8\pi k_0}\,\mathrm{e}^{-i\pi/4}u_a^{\infty}\left(\hat{\boldsymbol{d}},\hat{\boldsymbol{x}}\right), \tag{45}$$

$$\left\{\overline{u_a^{\mathrm{sc}}\left(\cdot,\hat{\boldsymbol{d}}\right)},u^{\mathrm{inc}}\left(\cdot,\hat{\boldsymbol{x}}\right)\right\}_{S_0}=\sqrt{8\pi k_0}\,\mathrm{e}^{i\pi/4}\overline{u_a^{\infty}\left(\hat{\boldsymbol{x}},\hat{\boldsymbol{d}}\right)}, \tag{46}$$

$$\begin{aligned}\left\{\overline{u_a^{\mathrm{sc}}\left(\cdot,\hat{\boldsymbol{d}}\right)},u_a^{\mathrm{sc}}\left(\cdot,\hat{\boldsymbol{x}}\right)\right\}_{S_0}&=\left\{\overline{u_a^{\mathrm{sc}}\left(\cdot,\hat{\boldsymbol{d}}\right)},u_a^{\mathrm{sc}}\left(\cdot,\hat{\boldsymbol{x}}\right)\right\}_{S_0}\\&=2ik_0\int_0^{2\pi}\overline{u_a^{\infty}\left(\hat{\boldsymbol{y}},\hat{\boldsymbol{d}}\right)}u_a^{\infty}\left(\hat{\boldsymbol{y}},\hat{\boldsymbol{x}}\right)\mathrm{d}s\left(\hat{\boldsymbol{y}}\right),\end{aligned} \tag{47}$$

$$\left\{\overline{u_{a,0}\left(\cdot,\hat{\boldsymbol{d}}\right)},u_{a,0}\left(\cdot,\hat{\boldsymbol{x}}\right)\right\}_{S_0}$$
$$=\frac{\rho_0}{\rho_N}\int_{S_N}\left(\overline{u_{a,N}\left(\cdot,\hat{\boldsymbol{d}}\right)}\frac{\partial u_{a,N}\left(\cdot,\hat{\boldsymbol{x}}\right)}{\partial\nu}-u_{a,N}\left(\cdot,\hat{\boldsymbol{x}}\right)\frac{\partial\overline{u_{a,N}\left(\cdot,\hat{\boldsymbol{d}}\right)}}{\partial\nu}\right)\mathrm{d}s. \tag{48}$$

This integral is equal to zero for $a=D,N,T$ due to the boundary conditions (7), (8) and (9), (10), respectively. For $a=I$, by applying the boundary condition (11), we get that the integral of (48) is equal to

$$\left\{\overline{u_{I,0}\left(\cdot,\hat{\boldsymbol{d}}\right)},u_{I,0}\left(\cdot,\hat{\boldsymbol{x}}\right)\right\}_{S_0}=-\frac{2i\lambda\rho_0}{\rho_N}\int_{S_N}\overline{u_{I,N}\left(\boldsymbol{y},\hat{\boldsymbol{d}}\right)}u_{I,N}\left(\boldsymbol{y},\hat{\boldsymbol{x}}\right)\mathrm{d}s\left(\boldsymbol{y}\right). \tag{49}$$

From the above relations the theorem is proved. □

The scattering cross-section σ_a^{sc} constitutes a measure of the disturbance caused by the scatterer to the incident wave [10] and it is given by

$$\sigma_a^{\mathrm{sc}}=\int_0^{2\pi}\left|u_a^{\infty}\right|^2\mathrm{d}\theta. \tag{50}$$

We also define the absorption cross-section σ^{ab}, given by

$$\sigma_a^{\mathrm{ab}}=\frac{1}{k_0}\mathrm{Im}\int_{S_0}u_{a,0}\left(\boldsymbol{x}\right)\frac{\partial\overline{u_{a,0}\left(\boldsymbol{x}\right)}}{\partial\nu}\mathrm{d}s, \tag{51}$$

which expresses the total energy absorbed by the scatterer. In particular, the energy which is taken from the incident plane wave is adsorbed by the boundary of the core of the scatterer in the impedance case.

$$\sigma_a^{\mathrm{ab}}=\frac{\lambda\rho_0}{k_0\rho_N}\int_{S_N}\left|u_{a,N}\left(\boldsymbol{x}\right)\right|^2\mathrm{d}s\left(\boldsymbol{x}\right). \tag{52}$$

Moreover, the extinction cross-section σ_a^{ex} is defined by

$$\sigma_a^{\mathrm{ex}}=\sigma_a^{\mathrm{sc}}+\sigma_a^{\mathrm{ab}}, \tag{53}$$

and it describes the total power that the scatterer extracts from the incident plane wave either by radiation or by absorption [10].

In the sequel we formulate a two-dimensional optical theorem for the scattering problem (P_a), $a=D,N,T,I$.

Theorem 3 (Optical). Let $u^{\mathrm{inc}}\left(\cdot,\hat{\boldsymbol{d}}\right)$ with $\hat{\boldsymbol{d}}=\left(\cos\phi,\sin\phi\right)$ be an incident wave and $u_a^{\infty}\left(\cdot,\phi\right)$ be the corresponding far field pattern. Then the extinction cross-section σ_a^{ex} for the problem (P_a), $a=D,N,T,I$, satisfies

$$\sigma_a^{\mathrm{ex}}=2\sqrt{\frac{2\pi}{k_0}}\mathrm{Im}\left[\mathrm{e}^{-i\pi/4}u_a^{\infty}\left(\phi,\phi\right)\right]. \tag{54}$$

Proof. We apply Theorem 2, for $\theta=\phi$, *i.e.* $\hat{\boldsymbol{x}}=\hat{\boldsymbol{d}}$, and we get

$$2i\,\mathrm{Im}\left[\mathrm{e}^{-i\pi/4}u_a^{\infty}\left(\phi,\phi\right)\right]=i\sqrt{\frac{k_0}{2\pi}}\int_0^{2\pi}\left|u_a^{\infty}\left(\theta_y,\phi\right)\right|^2\mathrm{d}s\left(\theta_y\right)+\mathcal{E}_a\left(\phi,\phi\right). \tag{55}$$

Taking into account (42), (50) and (52), the relation (55) is written

$$2i\,\mathrm{Im}\left[\mathrm{e}^{-i\pi/4}u_a^{\infty}\left(\phi,\phi\right)\right]=i\sqrt{\frac{k_0}{2\pi}}\left(\sigma_a^{\mathrm{sc}}+\sigma_a^{\mathrm{ab}}\right), \tag{56}$$

which proves the theorem. □

Next, we prove a mixed reciprocity theorem which connects the far field pattern of a point-source wave and the scattered field of a plane wave. This theorem can be used in studying inverse scattering problems, according to the Potthast point-source method [21] [22]. For this purpose, we have to define point-source waves with the position of the source to be outside the scatterer. A similarly mixed reciprocity theorem can be proved when the source is inside the scatterer. We consider for an incident point-source wave at $\boldsymbol{z}\in\mathbb{R}^2\setminus\bar{D}$ the fundamental solution of the Helmholtz equation, *i.e.*

$$\Phi^{\text{inc}}(\boldsymbol{x},\boldsymbol{z})=\frac{i}{4}H_0^{(1)}\left(k_0\left|\boldsymbol{x}-\boldsymbol{z}\right|\right). \tag{57}$$

We denote by $\Phi_{a,j}(\boldsymbol{x},\boldsymbol{z})$, $\Phi_a^{\text{sc}}(\boldsymbol{x},\boldsymbol{z})$ and $\Phi_a^{\infty}(\hat{\boldsymbol{x}},\boldsymbol{z})$ for representing the dependence of the total field in D_j, $j=0,1,\cdots,N+1$, the scattered field and the far field pattern on the position of the source $\boldsymbol{z}\in\mathbb{R}^2\setminus\bar{D}$.

Theorem 4 (Mixed Reciprocity). Let $\Phi^{\text{inc}}(\boldsymbol{x},\boldsymbol{z})$ be an incident point-source wave at $\boldsymbol{z}\in\mathbb{R}^2\setminus\bar{D}$ and let $u^{\text{inc}}(\boldsymbol{x},-\hat{\boldsymbol{d}})$ be an incident plane wave with propagation direction $-\hat{\boldsymbol{d}}$. Then,

$$\Phi_a^{\infty}(\hat{\boldsymbol{d}},\boldsymbol{z})=\frac{e^{i\pi/4}}{\sqrt{8k_0\pi}}u_a^{\text{sc}}(\boldsymbol{z},-\hat{\boldsymbol{d}}). \tag{58}$$

Proof. Taking into account that $\Phi_{a,0}=\Phi^{\text{inc}}+\Phi_a^{\text{sc}}$ and $u_{a,0}=u^{\text{inc}}+u_a^{\text{sc}}$ we get again the analysis (31) by replacing $u_{a,0}(\cdot,\hat{\boldsymbol{x}})$ by $\Phi_{a,0}(\cdot,\boldsymbol{z})$.

Since $\Phi^{\text{inc}}(\cdot,\boldsymbol{z})$ and $u^{\text{inc}}(\cdot,-\hat{\boldsymbol{d}})$ are regular solutions of the Helmholtz equation in *D*, the scalar Green's second theorem gives

$$\left\{\Phi^{\text{inc}}(\cdot,\boldsymbol{z}),u^{\text{inc}}(\cdot,-\hat{\boldsymbol{d}})\right\}_{S_0}=0. \tag{59}$$

For the integral $\left\{\Phi^{\text{inc}}(\cdot,\boldsymbol{z}),u_a^{\text{sc}}(\cdot,-\hat{\boldsymbol{d}})\right\}_{S_0}$, we consider a small disc $S_{\boldsymbol{z},\varepsilon}$ centered at $\boldsymbol{z}$ with radius ε and a large disc $S_{O,R}$ centered at the origin with radius *R* surrounding the scatterer and the small disc $S_{\boldsymbol{z},\varepsilon}$. Applying the scalar Green's second theorem for $\Phi^{\text{inc}}(\boldsymbol{x},\boldsymbol{z})$, $\boldsymbol{x}\neq\boldsymbol{z}$ and $u_a^{\text{sc}}(\cdot,-\hat{\boldsymbol{d}})$ in the space between the curves $S_{O,R}$, $S_{\boldsymbol{z},\varepsilon}$ and S_0, we get

$$\left\{\Phi^{\text{inc}}(\cdot,\boldsymbol{z}),u_a^{\text{sc}}(\cdot,-\hat{\boldsymbol{d}})\right\}_{S_0}=\left\{\Phi^{\text{inc}}(\cdot,\boldsymbol{z}),u_a^{\text{sc}}(\cdot,-\hat{\boldsymbol{d}})\right\}_{S_{O,R}}-\left\{\Phi^{\text{inc}}(\cdot,\boldsymbol{z}),u_a^{\text{sc}}(\cdot,-\hat{\boldsymbol{d}})\right\}_{S_{\boldsymbol{z},\varepsilon}}. \tag{60}$$

Letting $R\to\infty$ and taking into account that $\Phi^{\text{inc}}(\cdot,\boldsymbol{z})$, $u_a^{\text{sc}}(\cdot,-\hat{\boldsymbol{d}})$ are radiating solutions of the Helmholtz equation we have that $\left\{\Phi^{\text{inc}}(\cdot,\boldsymbol{z}),u_a^{\text{sc}}(\cdot,-\hat{\boldsymbol{d}})\right\}_{S_{O,R}}$.

Letting $\varepsilon\to 0$, using the asymptotic relations (13) and (14) and applying the mean value theorem, we obtain that $\left\{\Phi^{\text{inc}}(\cdot,\boldsymbol{z}),u_a^{\text{sc}}(\cdot,-\hat{\boldsymbol{d}})\right\}_{S_{\boldsymbol{z},\varepsilon}}=u_a^{\text{sc}}(\boldsymbol{z},-\hat{\boldsymbol{d}})$. Hence,

$$\left\{\Phi^{\text{inc}}(\cdot,\boldsymbol{z}),u_a^{\text{sc}}(\cdot,-\hat{\boldsymbol{d}})\right\}_{S_0}=-u_a^{\text{sc}}(\boldsymbol{z},-\hat{\boldsymbol{d}}). \tag{61}$$

From the definition of the far field pattern (29) we have

$$\left\{\Phi_a^{\mathrm{sc}}(\cdot,\boldsymbol{z}),u^{\mathrm{inc}}\left(\cdot,-\hat{\boldsymbol{d}}\right)\right\}_{S_0}=\sqrt{8k_0\pi}\mathrm{e}^{-i\pi/4}\Phi_a^{\infty}\left(\hat{\boldsymbol{d}},\boldsymbol{z}_0\right). \tag{62}$$

As in Theorem 1, we have:

$$\left\{\Phi_a^{\mathrm{sc}}(\cdot,\boldsymbol{z}),u_a^{\mathrm{sc}}\left(\cdot,-\hat{\boldsymbol{d}}\right)\right\}_{S_0}=0, \tag{63}$$

$$\left\{\Phi_{a,0}(\cdot,\boldsymbol{z}),u_{a,0}\left(\cdot,-\hat{\boldsymbol{d}}\right)\right\}_{S_0}=0. \tag{64}$$

Therefore, the above formulae prove the theorem. □

5. The Far Field Operator

In this section, we prove some basic properties of the far field operator for the two-dimensional scattering problem (P_a), $a=D,N,T,I$. The far field operator plays a central role for the study of inverse scattering problems. We consider the Herglotz wave function:

$$u_g(\theta)=\int_0^{2\pi}g(\phi)\mathrm{e}^{ik_0r\cos(\theta-\phi)}\mathrm{d}\phi, \tag{65}$$

with kernel $g\in L^2[0,2\pi]$. The Herglotz wave function u_g is an entire solution of the Helmholtz equation $\Delta u_0+k_0^2u_0=0$.

We now consider as incident field u_g^{inc} a Herglotz wave function of the form (65). We denote the dependence of the total field in D_j, the scattered field and the far field pattern on the kernel g by writing $u_{j,g}$, $u_{a,g}^{\mathrm{sc}}$ and $u_{a,g}^{\infty}$, respectively, and we prove the following results.

Corollary 1. We consider two incident Herglotz waves u_g^{inc} and u_h^{inc}. Let $u_{a,g}^{\mathrm{sc}}$, $u_{a,h}^{\mathrm{sc}}$ and $u_{a,g}^{\infty}$, $u_{a,h}^{\infty}$ be the corresponding scattered fields and far field patterns, $a=D,N,T,I$. Then it holds

$$\left\{u_{a,g}^{\mathrm{sc}},\overline{u_h^{\mathrm{inc}}}\right\}_{S_0}=\sqrt{8\pi k_0}\mathrm{e}^{-i\pi/4}\int_0^{2\pi}\overline{h(\phi)}u_{a,g}^{\infty}(\phi)\mathrm{d}\phi, \tag{66}$$

$$\left\{\overline{u_{a,g}^{\mathrm{sc}}},u_{a,h}^{\mathrm{sc}}\right\}_{S_0}=2ik_0\int_0^{2\pi}\overline{u_{a,g}^{\infty}(\phi)}u_{a,h}^{\infty}(\phi)\mathrm{d}\phi. \tag{67}$$

Proof. For $\hat{\boldsymbol{d}}=(\cos\phi,\sin\phi)$, $\boldsymbol{x}=(r\cos\theta,r\sin\theta)$ and taking into account the relations (65) and (29), we have

$$\begin{aligned}\left\{u_{a,g}^{\mathrm{sc}},\overline{u_h^{\mathrm{inc}}}\right\}_{S_0}&=\int_0^{2\pi}\overline{h(\phi)}\left\{u_{a,g}^{\mathrm{sc}},\mathrm{e}^{-ik_0\boldsymbol{x}\cdot\hat{\boldsymbol{d}}}\right\}_{S_0}\mathrm{d}\phi\\&=\sqrt{8\pi k_0}\mathrm{e}^{-i\pi/4}\int_0^{2\pi}\overline{h(\phi)}u_{a,g}^{\infty}(\phi)\mathrm{d}\phi.\end{aligned} \tag{68}$$

The relation (67) is immediate consequence of (47). □

The far field operator $F_a:L^2[0,2\pi]\to L^2[0,2\pi]$ corresponding to the far field pattern u_a^{∞} is defined by

$$(F_ag)(\theta):=\int_0^{2\pi}u_a^{\infty}(\theta,\phi)g(\phi)\mathrm{d}\phi. \tag{69}$$

Let us now consider the inner product on $L^2[0,2\pi]$ which is defined by $\langle g,h\rangle=\int_0^{2\pi}g\bar{h}\mathrm{d}s$.

Theorem 5. Let u_g^{inc}, u_h^{inc} be two incident Herglotz waves, $j=0,1,\cdots,N+1$. Then the far field operator $F_a:L^2[0,2\pi]\to L^2[0,2\pi]$ corresponding to the scattering problem (P_a), $a = D, N, T$, satisfies the relation

$$e^{-i\pi/4}\langle F_a g, h\rangle - e^{i\pi/4}\langle g, F_a h\rangle = i\sqrt{\frac{k_0}{2\pi}}\langle F_a g, F_a h\rangle. \quad (70)$$

Proof. By using the relation (40) and taking into account that the far field operator is superposition of far field patterns ([2], Lemma 3.20), we get the relation (70). □

If the core of the piecewise homogeneous scatterer is soft, hard or penetrable, then the far field operator is normal, *i.e.*, $F_a^* F_a = F_a F_a^*$, which is an important property and plays a crucial role in solving inverse scattering problems.

In particular, based on Theorem 5, the following corollary can be proved in a similar way as Theorem 7.15 presented in ([1], p. 144).

Corollary 2. The far field operator $F_a : L^2[0, 2\pi] \to L^2[0, 2\pi]$ corresponding to the scattering problem (P_a), $a = D, N, T$, is normal and its eigenvalues lie on the circle of radius $\sqrt{\frac{2\pi}{k_0}}$ with center at $e^{3\pi i/4}\sqrt{\frac{2\pi}{k_0}}$.

When the far field operator is normal, the factorization method ([1], Ch. 7) for solving inverse scattering problems can be applied. The reconstruction of a scatterer is described explicitly in Theorem 7.24 which is given in ([1], p. 150).

6. Concluding Remarks

In this work, we defined far field patterns for multi-layered scatterers with different imposed boundary conditions on the core. When $\rho_0 = \rho_1 = \cdots = \rho_N \neq \rho_{N+1}$ and $\gamma_0 = \gamma_1 = \cdots = \gamma_N \neq \gamma_{N+1}$, then scattering occurs only on the core since the layers disappear. Hence, the generated scattering theorems imply simpler results for sound soft, hard, penetrable or impedance problems. Corresponding scattering theorems can be proved in the case of more general imposed boundary conditions on the core, such as in [23] (resistive and conductive transmission conditions) and in [24] (generalized impedance boundary condition). Applying the derived results, we aim to study inverse scattering problems for multi-layered obstacles in two dimensions. The appropriate adjustment of our results can lead to the following extensions:

- the reciprocity theorem leads to the proof that the far field operator is injective, normal and has a dense range [1];
- the general scattering theorem can be used in low-frequency theory for the rapid computation of the low-frequency coefficients of the far field pattern [22];
- the optical theorem can be used for computing the total power that the scatterer extracts from the incident plane wave either by radiation or by absorption [22];
- the mixed reciprocity theorem can be applied to computing the layers' curves as well as the impedance constant [21].

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

References

[1] Cakoni, F. and Colton, D. (2005) Qualitative Methods in Inverse Electromagnetic Scattering Theory. Springer-Verlag, Berlin.

[2] Colton, D. and Kress, R. (2013) Inverse Acoustic and Electromagnetic Scattering Theory. 3rd Edition, Springer-Verlag, New York. https://doi.org/10.1007/978-1-4614-4942-3

[3] Athanasiadis, C. (1998) On the Acoustic Scattering Amplitude for a Multi-Layered Scatterer. *The Journal of the Australian Mathematical Society Series B: Applied Mathematics*, **39**, 431-448. https://doi.org/10.1017/S0334270000007736

[4] Twersky, V. (1967) Multiple Scattering of Electromagnetic Waves by Arbitrary Configurations. *Journal of Mathematical Physics*, **8**, 589-598. https://doi.org/10.1063/1.1705237

[5] Dassios, G., Kiriaki, K. and Polyzos, D. (1987) On the Scattering Amplitudes for Elastic Waves. *Journal of Applied Mathematics and Physics*, **318**, 856-873. https://doi.org/10.1007/BF00945822

[6] Twersky, V. (1954) Certain Transmissions and Reflection Theorems. *Journal of Applied Physics*, **25**, 859-862. https://doi.org/10.1063/1.1721758

[7] De Hoop, A.T. (1959) On the Plane-Wave Extinction Cross-Section of an Obstacle. *Applied Scientific Research*, *Section B*, **7**, 463-469. https://doi.org/10.1007/BF02921932

[8] De Hoop, A.T. (1966) An Elastodynamic Reciprocity Theorem for Linear, Viscoelastic Media. *Applied Scientific Research*, **16**, 39-45. https://doi.org/10.1007/BF00384053

[9] Colton, D. and Kress, R. (2013) Integral Equation Methods in Scattering Theory. SIAM, Philadelphia. https://doi.org/10.1137/1.9781611973167

[10] Dassios, G. and Kleinman, R. (2000) Low Frequency Scattering. Oxford University Press, Oxford.

[11] Angell, T.S., Colton, D. and Kress, R. (1989) Far Field Patterns and Inverse Scattering Problems for Imperfectly Conducting Obstacles. *Mathematical Proceedings of the Cambridge Philosophical Society*, **106**, 553-569. https://doi.org/10.1017/S0305004100068274

[12] Athanasiadis, C. (2005) On the Far Field Patterns for Electromagnetic Scattering by a Chiral Obstacle in a Chiral Environment. *Journal of Mathematical Analysis and Applications*, **309**, 517-533. https://doi.org/10.1016/j.jmaa.2004.09.058

[13] Douma H., Vasconcelos I. and Snieder R. (2011) The Reciprocity Theorem for the Scattered Field Is the Progenitor of the Generalized Optical Theorem. *The Journal of the Acoustical Society of America*, **129**, 2765-2771. https://doi.org/10.1121/1.3569728

[14] Gintides, D. and Kiriaki, K. (2001) The Far-Field Equations in Linear Elasticity—An Inversion Scheme. *Journal of Applied Mathematics and Mechanics*, **81**, 305-316. https://doi.org/10.1002/1521-4001(200105)81:5<305::AID-ZAMM305>3.0.CO;2-T

[15] Eremin, Y.A. (2017) Generalization of the Optical Theorem to the Case of Excitation of a Local Obstacle by a Multipole. *Computational Mathematics and Modeling*, **28**, 158-163. https://doi.org/10.1007/s10598-017-9354-5

[16] Athanasiadis, C., Martin, P.A., Spyropoulos, A. and Stratis, I.G. (2002) Scattering Relations for Point Sources: Acoustic and Electromagnetic Waves. *Journal of Mathematical Physics*, **43**, 5683-5697. https://doi.org/10.1063/1.1509089

[17] Athanasiadis, C., Sevroglou, V. and Stratis, I.G. (2008) 3D Elastic Scattering Theo-

rems for Point-Generated Dyadic Fields. *Mathematical Methods in the Applied Sciences*, **31**, 987-1003. https://doi.org/10.1002/mma.964

[18] Bazán, F., Francisco, J., Leem, K., Pelekanos, G. and Sevroglou, V. (2021) On an Application of the Improved Maximum Product Criterion to Inverse Acoustic Scattering in a Layered Medium. *Journal of Applied Mathematics and Physics*, **9**, 661-682. https://doi.org/10.4236/jamp.2021.94048

[19] Yamagata, T., Kato, K., Takahashi, N., Mineo, S. and Tabeta, K. (2022) Quantitative Visualization of Two-Phase Acoustic Streaming Emitted from Ultrasonic Scaler. *Journal of Flow Control, Measurement & Visualization*, **10**, 87-97. https://doi.org/10.4236/jfcmv.2022.103006

[20] Athanasiadis, C.E., Athanasiadou, E.S. and Roupa, P. (2022) Scattering Relations for Two-Dimensional Electromagnetic Waves in Chiral Media. *Journal of Applied Mathematics and Physics*, **10**, 1200-1216. https://doi.org/10.4236/jamp.2022.104085

[21] Liu, X., Zhang, B. and Hu, G. (2010) Uniqueness in the Inverse Scattering Problem in a Piecewise Homogeneous Medium. *Inverse Problems*, **26**, Article ID: 015002. https://doi.org/10.1088/0266-5611/26/1/015002

[22] Potthast, R. (2004) A New Non-Iterative Singular Sources Method for the Reconstruction of Piecewise Constant Media. *Numerische Mathematik*, **98**, 703-730. https://doi.org/10.1007/s00211-004-0524-y

[23] Angell, T.S., Kleinman, R.E. and Hettlich, F. (1990) The Resistive and Conductive Problems for the Exterior Helmholtz Equation. *Journal on Applied Mathematics*, **50**, 1607-1623. https://doi.org/10.1137/0150095

[24] Guo, J., Yan, G. and Cai, M. (2015) Multilayered Scattering Problem with Generalized Impedance Boundary Condition on the Core. *Journal of Applied Mathematics*, **2015**, Article ID: 195460. https://doi.org/10.1155/2015/195460