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Exact Solution for Equilibrium Configurations of Two-Component Plasma Confined between Parallel Plates

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Abstract

It is the fifth part of the study published under the common umbrella of "The Gibbs Variational Method in Thermodynamics of Equilibrium Plasma". In Parts 1 - 4, we formulated a novel approach to thermodynamics of one- and two-component heterogeneous systems completely or partially filled with a liquid substance in the plasma state. The approach is based on the use of Gibbs variational principles, and it enables efforts to address a variety of problems relating to the equilibrium and stability of such systems. In this fifth part, the results of Parts 1 - 4 are applied to the analysis of equilibrium configurations of a two-component charged plasma trapped between two parallel plates (the geometry often used in various applications).

Keywords

Plasma, Thermodynamics, Gibbs Variational Principles, Plasma Stability, Equations of State

1. Introduction

In Part 1 of this series of reports, Grinfeld and Grinfeld [1] [2] [3] [4] formulated a novel approach to the thermodynamics of heterogeneous systems completely or partially filled with a liquid or gaseous substance in a plasma state. The approach is based on the use of Gibbs variational principles, and it enables efforts to address a variety of problems relating to the equilibrium and stability of such systems.

The general motivation for this series of reports is discussed in Grinfeld and Grinfeld [1], in which we also demonstrated how the Gibbs approach could be applied to heterogeneous systems with charged gases. The main motivation is to provide a relatively simple model applicable to substances in the plasma state and compatible with the basic methodology suggested by Gibbs. In part Grinfeld and Grinfeld [3], we developed a general thermodynamic methodology applicable to gaseous two-component plasma with arbitrary Equations of State (EOS). The general analysis is the most effective tool to elucidate the universal features of the approach. On the other hand, it puts obvious limitations on the application of mathematically rigorous tools.

The analytical difficulties appear because of two main reasons: 1) the difficulties caused by the geometrical complexities of the problems under study, and 2) the general relationships lead to essentially nonlinear systems of the partial differential equation. Therefore, further simplifying assumptions is unavoidable if one needs to proceed with exact mathematics. The exact solutions are the main tools for a deeper understanding of the gross physical features of the models and for verification of the theory.

In this fifth part of our study, we consider the equilibrium configurations of a gaseous plasma confined between two infinite parallel plates, which are sketched in **Figure 1**.

Fortunately, the equations of electrostatics in a vacuum are linear. The only source of nonlinearity in the static problems is the EOS. To address this difficulty, we choose the EOS, suggested in Grinfeld [5]. This choice results in dealing with the linear ordinary differential equations (see also Gibbs [6], Hidalgo, Acosta, Hinojosa [7]).

2. Formulation of the One-Dimensional Boundary Value Problem for Two-Component Charged Mixture

We follow here the report by Grinfeld and Grinfeld [3] and the publication Grinfeld [5]. Per these reports, the entire system of equilibrium equations includes the following three elements:

1) The condition of thermal equilibrium

$$T = T^{\circ} = const \tag{1.1}$$

through out the whole configuration,

2) The electrostatics system for the electrostatic potential φ

$$\frac{d^2\varphi}{dz^2} = -4\pi\left(\sigma_e\rho_e + \sigma_i\rho_i\right) \tag{1.2}$$

where σ_e, σ_i are the charge densities of the components per unit mass, and ρ_e, ρ_i are the mass densities of the components.

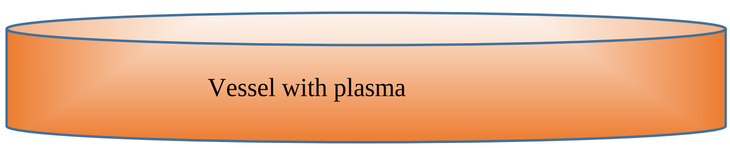

Figure 1. Model of a charged plasma system.

3) The electrochemistry equations

$$\left(\rho e\right)_{\rho_I} + \sigma_I \varphi = \Lambda_I \tag{1.3}$$

where I assumes the values e and i and Λ_I are the indefinite Lagrange multipliers.

To determine the Lagrange multipliers, we have to use the equation dealing with the total charge (or mass) of the system. Let M_I be the total mass of the gas per unit cross section. This leads to the relationships

$$\int_{-H}^{H} \mathrm{d}z \rho_I\left(z\right) = M_I \tag{1.4}$$

Equation (1.1) reflects the thermal equilibrium throughout the whole system, which is the standard condition implied by the Gibbs isoperimetric variation principle [6] of thermodynamics of heterogeneous systems. Equation (2) is just the standard equation of electrostatics. At last, Equation (1.4) is close to the condition of the chemical equilibrium of charged particles [6].

3. The Exact Solution of the BVP for Two Charged Liquids with the Canonical EOS

Differentiating (1.3), we get 2 equations

$$\frac{\mathrm{d}\varphi}{\mathrm{d}z} = -\frac{a_I^2}{\sigma_I}\frac{\mathrm{d}\rho_I}{\mathrm{d}z} \tag{2.1}$$

where $a_I^2\left(\rho\right) \equiv \left(\rho_I e_I\left(\rho\right)\right)_{\rho\rho}$.

We call canonical the EOS for which $a_I^2\left(\rho\right) = const$. We use the combining index I which assumes two values "e" and "i".

Inserting (2.1) in the equation of electrostatics (1.2), we arrive at the equations

$$\frac{a_e^2}{4\pi\sigma_e}\frac{\mathrm{d}^2\rho_e}{\mathrm{d}z^2} = \sigma_e\rho_e + \sigma_i\rho_i \tag{2.2}$$

and

$$\frac{a_i^2}{4\pi\sigma_i}\frac{\mathrm{d}^2\rho_i}{\mathrm{d}z^2} = \sigma_e\rho_e + \sigma_i\rho_i \tag{2.3}$$

Looking for the solutions of (2.2), (2.3) in the form

$$\begin{Vmatrix}\rho_e \\ \rho_i\end{Vmatrix} = \begin{Vmatrix}A_e \\ A_i\end{Vmatrix} \mathrm{e}^{\lambda z} \tag{2.4}$$

we get the system of 2 equations

$$\left(\sigma_e - \frac{a_e^2}{4\pi\sigma_e}\lambda^2\right)A_e + \sigma_i A_i = 0 \tag{2.5}$$

and

$$\sigma_e A_e + \left(\sigma_i - \frac{a_i^2}{4\pi\sigma_i}\lambda^2\right)A_i = 0 \tag{2.6}$$

The system (2.5), (2.6) leads to the following secular equation

$$\left(\sigma_e - \frac{a_e^2}{4\pi\sigma_e}\lambda^2\right)\left(\sigma_i - \frac{a_i^2}{4\pi\sigma_i}\lambda^2\right) - \sigma_e\sigma_i = 0 \tag{2.7}$$

which can be rewritten as

$$\frac{a_e^2 a_i^2}{4\pi\sigma_e\sigma_i}\lambda^4 - \left(\frac{a_e^2\sigma_i}{\sigma_e} + \frac{a_i^2\sigma_e}{\sigma_i}\right)\lambda^2 = 0 \tag{2.8}$$

or

$$\lambda^4 - \Delta^2\lambda^2 = 0 \tag{2.9}$$

where we use the notation

$$\Delta^2 \equiv \frac{4\pi\sigma_i^2}{a_i^2} + \frac{4\pi\sigma_e^2}{a_e^2} \tag{2.10}$$

Thus, we arrive at the following spectrum of the eigen-values:

$$\begin{gathered}\lambda^2 = 0,\ \sigma_e A_e + \sigma_i A_i = 0 \to \begin{Vmatrix} A_e \\ A_i \end{Vmatrix} = C_0 \begin{Vmatrix} \sigma_i \\ -\sigma_e \end{Vmatrix} \\ \lambda^2 = \Delta^2 \equiv \frac{4\pi\sigma_i^2}{a_i^2} + \frac{4\pi\sigma_e^2}{a_e^2},\ \frac{a_e^2}{\sigma_e}A_e - \frac{a_i^2}{\sigma_i}A_i = 0 \to \begin{Vmatrix} A_e \\ A_i \end{Vmatrix} = C_1 \begin{Vmatrix} \dfrac{a_i^2}{\sigma_i} \\ \dfrac{a_e^2}{\sigma_e} \end{Vmatrix}\end{gathered} \tag{2.11}$$

and the following general solution:

$$\begin{Vmatrix} \rho_e \\ \rho_i \end{Vmatrix} = C_0 \begin{Vmatrix} \sigma_i \\ -\sigma_e \end{Vmatrix} + C_1 \begin{Vmatrix} \dfrac{a_i^2}{\sigma_i} \\ \dfrac{a_e^2}{\sigma_e} \end{Vmatrix} \cosh(\Delta z) \tag{2.12}$$

The constants C_0 and C_1 can be determined from the mass balance Equations (1.4).

By elementary integration we get

$$\int_{-H}^{H} \mathrm{d}z \cosh(\Delta z) = \frac{2}{\Delta}\sinh(\Delta H) \tag{2.13}$$

as implied by the following chain:

$$\begin{aligned}&\int_{-H}^{H} \mathrm{d}z \cosh(\Delta z) = \frac{2}{\Delta}\sinh(\Delta H) \\ &\int_{-H}^{H} \mathrm{d}z \cosh(\Delta z) = \frac{1}{\Delta}\int_{-H}^{H} \mathrm{d}z\Delta \cosh(\Delta z) = \frac{1}{\Delta}\int_{-\Delta H}^{\Delta H} \mathrm{d}\eta \cosh\eta \\ &\qquad = \frac{1}{\Delta}\int_{-\Delta H}^{\Delta H} \mathrm{d}\eta \cosh\eta = \frac{2}{\Delta}\sinh(\Delta H)\end{aligned}$$

with the help of (2.12), (2.13), the Equations (1.4) give us the following system of linear algebraic equations:

$$\begin{aligned} C_0 2H\sigma_i + C_1 \frac{a_i^2}{\sigma_i}\frac{2}{\Delta}\sinh(\Delta H) &= M_e \\ -C_0 2H\sigma_e + C_1 \frac{a_e^2}{\sigma_e}\frac{2}{\Delta}\sinh(\Delta H) &= M_i \end{aligned} \tag{2.14}$$

The Equations (2.14) imply the following solution

$$\begin{aligned} C_0 &= \frac{1}{2H}\frac{M_e a_e^2\sigma_e^{-1} - M_i a_i^2\sigma_i^{-1}}{\sigma_i\sigma_e\left(a_i^2\sigma_i^{-2} + a_i^2\sigma_e^{-2}\right)} \\ C_1 &= \frac{1}{2H}\frac{\Delta H}{\sinh(\Delta H)}\frac{M_e\sigma_e + M_i\sigma_i}{\sigma_i\sigma_e\left(a_i^2\sigma_i^{-2} + a_e^2\sigma_e^{-2}\right)} \end{aligned} \tag{2.15}$$

Inserting the constants from the Equation (2.15) in Equation (2.12), we get eventually

$$\begin{aligned} \begin{Vmatrix} \rho_e \\ \rho_i \end{Vmatrix} &= \frac{1}{2H}\frac{M_e a_e^2\sigma_e^{-1} - M_i a_i^2\sigma_i^{-1}}{\sigma_i\sigma_e\left(a_i^2\sigma_i^{-2} + a_e^2\sigma_e^{-2}\right)}\begin{Vmatrix} \sigma_i \\ -\sigma_e \end{Vmatrix} \\ &+ \frac{1}{2H}\frac{\Delta H}{\sinh(\Delta H)}\frac{M_e\sigma_e + M_i\sigma_i}{\sigma_i\sigma_e\left(a_i^2\sigma_i^{-2} + a_e^2\sigma_e^{-2}\right)}\begin{Vmatrix} a_i^2\sigma_i^{-1} \\ a_e^2\sigma_e^{-1} \end{Vmatrix}\cosh(\Delta z) \end{aligned} \tag{2.16}$$

Using elementary transformations, we can rewrite (2.16) as

$$\begin{aligned} & 2H\sigma_i\sigma_e\left(a_i^2\sigma_i^{-2} + a_e^2\sigma_e^{-2}\right)\begin{Vmatrix} \rho_e \\ \rho_i \end{Vmatrix} \\ &= \left(M_e a_e^2\sigma_e^{-1} - M_i a_i^2\sigma_i^{-1}\right)\begin{Vmatrix} \sigma_i \\ -\sigma_e \end{Vmatrix} + \cosh(\Delta z)\frac{\Delta H}{\sinh(\Delta H)}\left(M_e\sigma_e + M_i\sigma_i\right)\begin{Vmatrix} a_i^2\sigma_i^{-1} \\ a_e^2\sigma_e^{-1} \end{Vmatrix} \end{aligned} \tag{2.17}$$

or else

$$\begin{aligned} & \sigma_i\sigma_e\left(a_i^2\sigma_i^{-2} + a_e^2\sigma_e^{-2}\right)\begin{Vmatrix} \rho_e \\ \rho_i \end{Vmatrix} \\ &= \left(\bar{\rho}_e a_e^2\sigma_e^{-1} - \bar{\rho}_i a_i^2\sigma_i^{-1}\right)\begin{Vmatrix} \sigma_i \\ -\sigma_e \end{Vmatrix} + \cosh(\Delta z)\frac{\Delta H}{\sinh(\Delta H)}\left(\bar{\rho}_e\sigma_e + \bar{\rho}_i\sigma_i\right)\begin{Vmatrix} a_i^2\sigma_i^{-1} \\ a_e^2\sigma_e^{-1} \end{Vmatrix} \end{aligned} \tag{2.18}$$

where we use the following notation

$$\Delta^2 \equiv \frac{4\pi\sigma_i^2}{a_i^2} + \frac{4\pi\sigma_e^2}{a_e^2}, \quad \bar{\rho}_e \equiv \frac{\rho_e}{2H}, \quad \bar{\rho}_i \equiv \frac{\rho_i}{2H} \tag{2.19}$$

The solution (2.16) implies

$$\sigma_e\rho_e + \sigma_i\rho_i = \frac{Q}{2H}\frac{\Delta H}{\sinh(\Delta H)}\cosh(\Delta z) \tag{2.20}$$

where Q is the full charge of the plasma

$$Q \equiv M_e\sigma_e + M_i\sigma_i \tag{2.21}$$

Equation (2.16) implies the following relationships for the spatial distributions the volumetric charge densities:

$$\begin{aligned}
\left\|\begin{matrix}\sigma_e\rho_e\\ \sigma_i\rho_i\end{matrix}\right\| &= \frac{1}{2H}\frac{M_e a_e^2\sigma_e^{-1}-M_i a_i^2\sigma_i^{-1}}{\sigma_i\sigma_e\left(a_i^2\sigma_i^{-2}+a_e^2\sigma_e^{-2}\right)}\left\|\begin{matrix}\sigma_i\sigma_e\\ -\sigma_i\sigma_e\end{matrix}\right\| \\
&\quad+\frac{1}{2H}\frac{\Delta H}{\sinh(\Delta H)}\frac{M_e\sigma_e+M_i\sigma_i}{\sigma_i\sigma_e\left(a_i^2\sigma_i^{-2}+a_e^2\sigma_e^{-2}\right)}\left\|\begin{matrix}a_i^2\sigma_e\sigma_i^{-1}\\ a_e^2\sigma_i\sigma_e^{-1}\end{matrix}\right\|\cosh(\Delta z) \\
&= \frac{\bar\rho_e a_e^2\sigma_e^{-1}-\bar\rho_i a_i^2\sigma_i^{-1}}{a_e^2\sigma_e^{-2}+a_i^2\sigma_i^{-2}}\left\|\begin{matrix}1\\ -1\end{matrix}\right\|+\frac{\Delta H}{\sinh(\Delta H)}\frac{\bar\rho_e\sigma_e+\bar\rho_i\sigma_i}{a_e^2\sigma_e^{-2}+a_i^2\sigma_i^{-2}}\left\|\begin{matrix}a_i^2\sigma_i^{-2}\\ a_e^2\sigma_e^{-2}\end{matrix}\right\|\cosh(\Delta z) \\
&= \frac{\bar\rho_e a_e^2\sigma_e^{-1}-\bar\rho_i a_i^2\sigma_i^{-1}}{a_e^2\sigma_e^{-2}+a_i^2\sigma_i^{-2}}\left\|\begin{matrix}1\\ -1\end{matrix}\right\|+\frac{\Delta H}{\sinh(\Delta H)}\frac{\bar\rho_e\sigma_e+\bar\rho_i\sigma_i}{a_e^2\sigma_e^{-2}+a_i^2\sigma_i^{-2}}\left\|\begin{matrix}a_i^2\sigma_i^{-2}\\ a_e^2\sigma_e^{-2}\end{matrix}\right\|\cosh(\Delta z)
\end{aligned} \tag{2.22}$$

where $\bar\rho_e \equiv M_e/2H$ and $\bar\rho_i \equiv M_i/2H$ are the mean densities of the charges.

It is sometimes convenient to transform (2.22) as follows:

$$\begin{aligned}
\left\|\begin{matrix}\sigma_e\rho_e\\ \sigma_i\rho_i\end{matrix}\right\| &= \frac{\bar\rho_e a_e^2\sigma_e^{-1}-\bar\rho_i a_i^2\sigma_i^{-1}}{a_e^2\sigma_e^{-2}+a_i^2\sigma_i^{-2}}\left\|\begin{matrix}1\\ -1\end{matrix}\right\|+\frac{\Delta H}{\sinh(\Delta H)}\frac{\bar\rho_e\sigma_e+\bar\rho_i\sigma_i}{a_e^2\sigma_e^{-2}+a_i^2\sigma_i^{-2}}\left\|\begin{matrix}a_i^2\sigma_i^{-2}\\ a_e^2\sigma_e^{-2}\end{matrix}\right\|\cosh(\Delta z) \\
&= \frac{\bar\rho_e\sigma_e}{a_e^2\sigma_e^{-2}+a_i^2\sigma_i^{-2}}\left(a_e^2\sigma_e^{-2}\left\|\begin{matrix}1\\ -1\end{matrix}\right\|+\frac{\Delta H}{\sinh(\Delta H)}\left\|\begin{matrix}a_i^2\sigma_i^{-2}\\ a_e^2\sigma_e^{-2}\end{matrix}\right\|\cosh(\Delta z)\right) \\
&\quad+\frac{\bar\rho_i\sigma_i}{a_e^2\sigma_e^{-2}+a_i^2\sigma_i^{-2}}\left(a_i^2\sigma_i^{-2}\left\|\begin{matrix}-1\\ 1\end{matrix}\right\|+\frac{\Delta H}{\sinh(\Delta H)}\left\|\begin{matrix}a_i^2\sigma_i^{-2}\\ a_e^2\sigma_e^{-2}\end{matrix}\right\|\cosh(\Delta z)\right)
\end{aligned} \tag{2.23}$$

4. The Asymptotic Case of "Neutral" Ionic Liquid

In the case of $\sigma_i = 0$, the solutions (2.16), (2.17) reads

$$\left\|\begin{matrix}\rho_e\\ \rho_i\end{matrix}\right\| = \bar\rho_i\left\|\begin{matrix}0\\ 1\end{matrix}\right\|+\bar\rho_e\left\|\begin{matrix}1\\ 0\end{matrix}\right\|\frac{\tilde\Delta H}{\sinh\left(\tilde\Delta H\right)}\cosh\left(\tilde\Delta z\right)=\left\|\begin{matrix}\bar\rho_e\dfrac{\tilde\Delta H}{\sinh\left(\tilde\Delta H\right)}\cosh\left(\tilde\Delta z\right)\\ \bar\rho_i\end{matrix}\right\| \tag{3.1}$$

where

$$\tilde\Delta^2 \equiv \frac{4\pi\sigma_e^2}{a_e^2} \tag{3.2}$$

5. The Case of "Neutral" Ionic Liquid

For verification purposes, it is instructive to consider the case of the overall neutral plasma. By the natural physical definition, in this case, the net charge of plasma Q vanishes:

$$M_e\sigma_e + M_i\sigma_i = 0 \tag{4.1}$$

or

$$\Upsilon \equiv -\frac{M_i\sigma_i}{M_e\sigma_e} = 1 \tag{4.2}$$

Inserting Equation (4.1) in (2.16), we arrive at the relation

$$\left\|\begin{matrix}\rho_e\\ \rho_i\end{matrix}\right\| = \frac{1}{2H}\left\|\begin{matrix}M_e\\ M_i\end{matrix}\right\| \tag{4.3}$$

as implied by the following chain:

$$\begin{Vmatrix}\rho_e\\ \rho_i\end{Vmatrix}=\frac{1}{2H}\frac{M_e a_e^2\sigma_e^{-1}-M_i a_i^2\sigma_i^{-1}}{\sigma_i\sigma_e\left(a_i^2\sigma_i^{-2}+a_e^2\sigma_e^{-2}\right)}\begin{Vmatrix}\sigma_i\\ -\sigma_e\end{Vmatrix}=\frac{1}{2H_e}\frac{M_e\sigma a_e^2\sigma_e^{-2}-M_i\sigma_i a_i^2\sigma_i^{-2}}{\sigma_i\sigma_e\left(a_i^2\sigma_i^{-2}+a_e^2\sigma_e^{-2}\right)}\begin{Vmatrix}\sigma_i\\ -\sigma_e\end{Vmatrix}$$

$$=\frac{1}{2H_e}M_e\sigma_e\frac{a_e^2\sigma_e^{-2}+a_i^2\sigma_i^{-2}}{\sigma_i\sigma_e\left(a_i^2\sigma_i^{-2}+a_e^2\sigma_e^{-2}\right)}\begin{Vmatrix}\sigma_i\\ -\sigma_e\end{Vmatrix}=\frac{1}{2H}M_e\sigma_e\frac{1}{\sigma_i\sigma_e}\begin{Vmatrix}\sigma_i\\ -\sigma_e\end{Vmatrix}$$

$$=\frac{1}{2H}\begin{Vmatrix}\sigma_i M_e\sigma_e\dfrac{1}{\sigma_i\sigma_e}\\ -\sigma_e M_e\sigma_e\dfrac{1}{\sigma_i\sigma_e}\end{Vmatrix}=\frac{1}{2H}\begin{Vmatrix}M_e\\ -\sigma_e M_e\dfrac{1}{\sigma_i}\end{Vmatrix}=\frac{1}{2H}\begin{Vmatrix}M_e\\ M_i\end{Vmatrix}$$

The solution (4.3) is in full agreement with the intuition: in the absence of external electrostatic fields, the component is uniformly distributed inside the vessel.

6. The Case of Quasi-Neutral Plasma

Consider the quasi-neutral case, *i.e.*, the case when

$$\Upsilon=1-q,\quad |q|\ll 1 \tag{5.1}$$

We, then, get

$$M_e\sigma_e+M_i\sigma_i=(1-\Upsilon)M_e\sigma_e=qM_e\sigma_e \tag{5.2}$$

In view of the Equation (5.2), we get

$$M_i\sigma_i=-(1-q)M_e\sigma_e \tag{5.3}$$

Using Equation (5.3), we can rewrite Equation (2.16) as follows

$$\begin{aligned}\begin{Vmatrix}\rho_e\\ \rho_i\end{Vmatrix}&=\frac{M_e}{2H}\begin{Vmatrix}1\\ -\sigma_e\sigma_i^{-1}\end{Vmatrix}+q\frac{M_e}{2H}\frac{1}{a_i^2\sigma_i^{-2}+a_e^2\sigma_e^{-2}}\\ &\times\left[\begin{Vmatrix}-a_i^2\sigma_i^{-2}\\ a_i^2\sigma_i^{-3}\sigma_e\end{Vmatrix}+\frac{\Delta H}{\sinh(\Delta H)}\begin{Vmatrix}a_i^2\sigma_i^{-2}\\ a_e^2\sigma_i^{-1}\sigma_e^{-1}\end{Vmatrix}\cosh(\Delta z)\right]\end{aligned} \tag{5.4}$$

Some regrouping in Equation (5.4) gives us

$$\begin{aligned}\begin{Vmatrix}\sigma_e\rho_e\\ \sigma_i\rho_i\end{Vmatrix}&=\frac{M_e\sigma_e}{2H}\begin{Vmatrix}1\\ -1\end{Vmatrix}+q\frac{M_e\sigma_e}{2H}\frac{a_i^2\sigma_i^{-2}}{a_i^2\sigma_i^{-2}+a_e^2\sigma_e^{-2}}\\ &\times\left[-\begin{Vmatrix}1\\ -1\end{Vmatrix}+\frac{\Delta H}{\sinh(\Delta H)}\begin{Vmatrix}1\\ \dfrac{a_e^2\sigma_e^{-2}}{a_i^2\sigma_i^{-2}}\end{Vmatrix}\cosh(\Delta z)\right]\end{aligned} \tag{5.5}$$

as implied by the following chain:

$$\begin{aligned}\begin{Vmatrix}\rho_e\\ \rho_i\end{Vmatrix}&=\frac{1}{2H}M_e\sigma_e\frac{a_e^2\sigma_e^{-2}+a_i^2\sigma_i^{-2}-qa_i^2\sigma_i^{-2}}{\sigma_i\sigma_e\left(a_i^2\sigma_i^{-2}+a_e^2\sigma_e^{-2}\right)}\begin{Vmatrix}\sigma_i\\ -\sigma_e\end{Vmatrix}\\ &+\frac{1}{2H}\frac{\Delta H}{\sinh(\Delta H)}\frac{qM_e\sigma_e}{\sigma_i\sigma_e\left(a_i^2\sigma_i^{-2}+a_e^2\sigma_e^{-2}\right)}\begin{Vmatrix}\dfrac{a_i^2}{\sigma_i}\\ \dfrac{a_e^2}{\sigma_e}\end{Vmatrix}\cosh(\Delta z)\rightarrow\end{aligned}$$

$$\begin{aligned}
&\frac{M_e}{2H}\left\|\begin{matrix}1\\ -\dfrac{\sigma_e}{\sigma_i}\end{matrix}\right\| - q\frac{M_e}{2H}\frac{a_i^2\sigma_i^{-3}}{a_i^2\sigma_i^{-2}+a_e^2\sigma_e^{-2}}\left\|\begin{matrix}\sigma_i\\ -\sigma_e\end{matrix}\right\| \\
&+q\frac{M_e}{2H}\frac{\Delta H}{\sinh(\Delta H)}\frac{\sigma_e}{\sigma_i\sigma_e\left(a_i^2\sigma_i^{-2}+a_e^2\sigma_e^{-2}\right)}\left\|\begin{matrix}a_i^2\sigma_i^{-1}\\ a_e^2\sigma_e^{-1}\end{matrix}\right\|\cosh(\Delta z) \\
&=\frac{M_e}{2H}\left\|\begin{matrix}1\\ -\dfrac{\sigma_e}{\sigma_i}\end{matrix}\right\| - q\frac{M_e}{2H}\frac{1}{a_i^2\sigma_i^{-2}+a_e^2\sigma_e^{-2}} \\
&\times\left[\left\|\begin{matrix}a_i^2\sigma_i^{-2}\\ -a_i^2\sigma_i^{-3}\sigma_e\end{matrix}\right\| - \frac{\Delta H}{\sinh(\Delta H)}\left\|\begin{matrix}a_i^2\sigma_i^{-2}\\ a_e^2\sigma_i^{-1}\sigma_e^{-1}\end{matrix}\right\|\cosh(\Delta z)\right]
\end{aligned}$$

At $a_i/a_e \gg 1$, the Equation (5.4) reads

$$\frac{2H}{M_e}\left\|\begin{matrix}\rho_e\\ \rho_i\end{matrix}\right\| = \left\|\begin{matrix}1\\ -\sigma_e\sigma_i^{-1}\end{matrix}\right\|(1-q)+q\frac{\Delta H}{\sinh(\Delta H)}\left\|\begin{matrix}1\\ 0\end{matrix}\right\|\cosh(\Delta z) \tag{5.6}$$

Using Equation (5.6), we get

$$\frac{2H}{M_e}\left\|\begin{matrix}\sigma_e\rho_e\\ \sigma_i\rho_i\end{matrix}\right\| = \left\|\begin{matrix}\sigma_e\\ -\sigma_e\end{matrix}\right\|(1-q)+q\frac{\Delta H}{\sinh(\Delta H)}\left\|\begin{matrix}\sigma_e\\ 0\end{matrix}\right\|\cosh(\Delta z) \tag{5.7}$$

Using Equation (5.7), we get in turn

$$\sigma_e\rho_e+\sigma_i\rho_i = \frac{qM_e\sigma_e}{2H}\frac{\Delta_\infty H}{\sinh(\Delta_\infty H)}\cosh(\Delta_\infty z) \tag{5.8}$$

7. The Asymptotics of Incompressible Ionic Liquid and Appearance of the Extinction Points of the Density

In this case the solution (2.16) reads

$$\left\|\begin{matrix}\rho_e\\ \rho_i\end{matrix}\right\| = \frac{M_i}{2H}\left\|\begin{matrix}-\dfrac{\sigma_i}{\sigma_e}\\ 1\end{matrix}\right\| + \frac{M_e}{2H}\frac{\Delta_\infty H}{\sinh(\Delta_\infty H)}\left\|\begin{matrix}1+\dfrac{M_i}{M_e}\dfrac{\sigma_i}{\sigma_e}\\ 0\end{matrix}\right\|\cosh(\Delta z) \tag{6.1}$$

and it implies

$$\begin{aligned}
\left\|\begin{matrix}\sigma_e\rho_e\\ \sigma_i\rho_i\end{matrix}\right\| &= \frac{M_i}{2H}\left\|\begin{matrix}-\sigma_i\\ \sigma_i\end{matrix}\right\| + \frac{M_e}{2H}\frac{\Delta_\infty H}{\sinh(\Delta_\infty H)}\left\|\begin{matrix}\sigma_e+\dfrac{M_i}{M_e}\sigma_i\\ 0\end{matrix}\right\|\cosh(\Delta z) \\
&= \left\|\begin{matrix}-\sigma_i\bar{\rho}_i\\ \sigma_i\bar{\rho}_i\end{matrix}\right\| + \frac{\Delta_\infty H}{\sinh(\Delta_\infty H)}\left\|\begin{matrix}\sigma_e\bar{\rho}_e+\sigma_-\bar{\rho}_-\\ 0\end{matrix}\right\|\cosh(\Delta z)
\end{aligned} \tag{6.2}$$

as well as

$$\sigma_e\rho_e+\sigma_i\rho_i = \frac{\Delta_\infty H}{\sinh(\Delta_\infty H)}\frac{M_e\sigma_e+M_i\sigma_i}{2H}\cosh(\Delta z) \tag{6.3}$$

We, then get, using (6.1)

$$\begin{aligned}
2H\rho_e(z) &= -M_i\frac{\sigma_i}{\sigma_e}+\left(M_e+M_i\frac{\sigma_i}{\sigma_e}\right)\frac{\Delta_\infty H}{\sinh(\Delta_\infty H)}\cosh(\Delta z) \\
&= M_e\frac{\Delta_\infty H}{\sinh(\Delta_\infty H)}\cosh(\Delta z) - M_i\frac{\sigma_i}{\sigma_e}\left(1-\frac{\Delta_\infty H}{\sinh(\Delta_\infty H)}\cosh(\Delta z)\right)
\end{aligned} \tag{6.4}$$

or else

$$2H\sigma_e\rho_e(z) = -M_i\sigma_i + (M_i\sigma_i + M_e\sigma_e)\frac{\Delta_\infty H}{\sinh(\Delta_\infty H)}\cosh(\Delta z) \tag{6.5}$$

We see that the local electric charge disappears at the point $z = Z_{ext}$ such that

$$\Delta Z_{ext} = \cosh^{-1}\left(\frac{M_i\sigma_i}{M_i\sigma_i + M_e\sigma_e}\frac{\sinh(\Delta_\infty H)}{\Delta_\infty H}\right) \tag{6.6}$$

In a more general case, when both components a compressible, we have to use the relationships (2.21).

Then, we arrive at the following analogies of (6.6):

$$\begin{aligned}\Delta Z_{ext}^{e} &= \cosh^{-1}\left[\frac{M_i a_i^2\sigma_i^{-1} - M_e a_e^2\sigma_e^{-1}}{(M_e\sigma_e + M_i\sigma_i)a_i^2\sigma_i^{-2}}\frac{\sinh(\Delta H)}{\Delta H}\right]\\ \Delta Z_{ext}^{i} &= \cosh^{-1}\left[\frac{M_e a_e^2\sigma_e^{-1} - M_i a_i^2\sigma_i^{-1}}{(M_e\sigma_e + M_i\sigma_i)a_e^2\sigma_e^{-2}}\frac{\sinh(\Delta H)}{\Delta H}\right]\end{aligned} \tag{6.7}$$

In terms of $Q \equiv M_e\sigma_e + M_i\sigma_i$, the pair of Equations (6.7) can be rewritten as

$$\begin{aligned}\Delta Z_{ext}^{e} &= \cosh^{-1}\left[\frac{M_i\sigma_i\left(a_i^2\sigma_i^{-2} + a_e^2\sigma_e^{-2}\right) - Qa_e^2\sigma_e^{-2}}{Qa_i^2\sigma_i^{-2}}\frac{\sinh(\Delta H)}{\Delta H}\right]\\ &= \cosh^{-1}\left[\frac{Qa_i^2\sigma_i^{-2} - M_e\sigma_e\left(a_i^2\sigma_i^{-2} + a_e^2\sigma_e^{-2}\right)}{Qa_i^2\sigma_i^{-2}}\frac{\sinh(\Delta H)}{\Delta H}\right]\\ \Delta Z_{ext}^{i} &= \cosh^{-1}\left[\frac{-M_i\sigma_i\left(a_e^2\sigma_e^{-2} + a_i^2\sigma_i^{-2}\right) + Qa_e^2\sigma_e^{-2}}{Qa_e^2\sigma_e^{-2}}\frac{\sinh(\Delta H)}{\Delta H}\right]\\ &= \cosh^{-1}\left[\frac{M_e\sigma_e\left(a_e^2\sigma_e^{-2} + a_i^2\sigma_i^{-2}\right) - Qa_i^2\sigma_i^{-2}}{Qa_e^2\sigma_e^{-2}}\frac{\sinh(\Delta H)}{\Delta H}\right]\end{aligned} \tag{6.8}$$

At small Q, the Equations (6.8) can be approximated with the following ones:

$$\begin{aligned}\Delta Z_{ext}^{e} &= \cosh^{-1}\left[\frac{M_i\sigma_i}{Q}\frac{a_i^2\sigma_i^{-2} + a_e^2\sigma_e^{-2}}{a_i^2\sigma_i^{-2}}\frac{\sinh(\Delta H)}{\Delta H}\right]\\ &= \cosh^{-1}\left[-\frac{M_e\sigma_e}{Q}\frac{a_i^2\sigma_i^{-2} + a_e^2\sigma_e^{-2}}{a_i^2\sigma_i^{-2}}\frac{\sinh(\Delta H)}{\Delta H}\right]\\ \Delta Z_{ext}^{i} &= \cosh^{-1}\left[-\frac{M_i\sigma_i}{Q}\frac{a_e^2\sigma_e^{-2} + a_i^2\sigma_i^{-2}}{a_e^2\sigma_e^{-2}}\frac{\sinh(\Delta H)}{\Delta H}\right]\\ &= \cosh^{-1}\left[\frac{M_e\sigma_e}{Q}\frac{a_e^2\sigma_e^{-2} + a_i^2\sigma_i^{-2}}{a_e^2\sigma_e^{-2}}\frac{\sinh(\Delta H)}{\Delta H}\right]\end{aligned} \tag{6.9}$$

The relationships (6.9) imply, that in the asymptotic case $Q \to 0$, the extinction point of the "*i*" component can appears if $\sigma_i Q > 0$; the extinction point of the "*e*" component can appears if $\sigma_e Q > 0$. The clause "can appear" means that corresponding values of the inverse hyperbolic cosine "$\cosh^{-1}$" are real. However, to be physically meaningful the corresponding values of Z_{ext}^{e} or Z_{ext}^{i}

should be less than H: but, in fact, they tend to infinity at $Q \to 0$. This fact implies that for sufficiently small Q the equilibrium configurations have no extinction points.

8. Calculation of the Pressure

If we consider the isothermal processes at $T = T_0$, where T_0 is the base temperature, we arrive at the relationships:

$$\begin{aligned} \psi_e(\rho_e) - \psi_{e0} &= a_e^2 \frac{1}{2\rho_e}(\rho_e - \rho_{e0})^2 + p_{e0}\left(\frac{1}{\rho_{e0}} - \frac{1}{\rho_e}\right) \\ \psi_i(\rho_i) - \psi_{i0} &= a_i^2 \frac{1}{2\rho_i}(\rho_i - \rho_{i0})^2 + p_{i0}\left(\frac{1}{\rho_{i0}} - \frac{1}{\rho_i}\right) \end{aligned} \tag{7.1}$$

and then

$$\begin{aligned} p_e(\rho_e) &= a_e^2 \frac{1}{2}\left(\rho_e^2 - \rho_{e0}^2\right) + p_{e0},\ (\rho\psi_e)_{\rho\rho} = a_e^2 = const \\ p_i(\rho_i) &= a_i^2 \frac{1}{2}\left(\rho_i^2 - \rho_{i0}^2\right) + p_{i0},\ (\rho\psi_i)_{\rho\rho} = a_i^2 = const \end{aligned} \tag{7.2}$$

Using (2.16), we get for the case of the electrically neutral ionic substance:

$$\begin{aligned} \left\| \begin{matrix} \rho_e \\ \rho_i \end{matrix} \right\| &= \frac{1}{2H} \left\| \begin{matrix} 0 \\ M_i \end{matrix} \right\| + \frac{1}{2H} \frac{\Delta H}{\sinh(\Delta H)} \left\| \begin{matrix} M_e \\ 0 \end{matrix} \right\| \cosh(\Delta z) \\ &= \frac{1}{2H} \left\| \begin{matrix} M_e \dfrac{\Delta H}{\sinh(\Delta H)} \cosh(\Delta z) \\ M_i \end{matrix} \right\| = \left\| \begin{matrix} \dfrac{M_e}{2H} \dfrac{\Delta H}{\sinh(\Delta H)} \cosh(\Delta z) \\ \dfrac{M_i}{2H} \end{matrix} \right\| \end{aligned} \tag{7.3}$$

Using (7.3), we get

$$\begin{aligned} \rho_e^2(z) &= \left(\frac{M_i}{2H}\frac{\sigma_i}{\sigma_e}\right)^2 - \frac{M_i}{2H}\frac{\sigma_i}{\sigma_e}\frac{M_e\sigma_e + M_i\sigma_i}{H\sigma_e}\frac{\Delta_\infty H \cosh(\Delta_\infty z)}{\sinh(\Delta_\infty H)} \\ &\quad + \left(\frac{1}{2}\frac{M_e\sigma_e + M_i\sigma_i}{H\sigma_e}\frac{\Delta_\infty H \cosh(\Delta_\infty z)}{\sinh(\Delta_\infty H)}\right)^2, \\ \rho_i^2(z) &= \left(\frac{M_i}{2H}\right)^2 + \frac{M_i}{2H}\frac{\sigma_i}{\sigma_e}\frac{M_e\sigma_e + M_i\sigma_i}{H\sigma_e}\frac{a_e^2}{a_i^2}\frac{\sigma_i}{\sigma_e}\frac{\Delta_\infty H \cosh(\Delta_\infty z)}{\sinh(\Delta_\infty H)} \\ &\quad + \left(\frac{1}{2}\frac{M_e\sigma_e + M_i\sigma_i}{H\sigma_e}\frac{a_e^2}{a_i^2}\frac{\sigma_i}{\sigma_e}\frac{\Delta_\infty H \cosh(\Delta_\infty z)}{\sinh(\Delta_\infty H)}\right)^2 \end{aligned} \tag{7.4}$$

Using (7.4), we get

$$\begin{aligned} & p_i(\rho_i) - p_{i0} = a_i^2 \frac{1}{2}\left(\rho_i^2 - \rho_{i0}^2\right) \\ & \approx \frac{1}{2}a_i^2\left[\left(\frac{M_i}{2H}\right)^2 - \rho_{i0}^2\right] + a_e^2 \frac{M_i}{4H}\left(\frac{\sigma_i}{\sigma_e}\right)^2 \frac{M_e\sigma_e + M_i\sigma_i}{H\sigma_e}\frac{\Delta_\infty H \cosh(\Delta_\infty z)}{\sinh(\Delta_\infty H)} \end{aligned}$$

9. Conclusion

We found an exact 1D equilibrium solution for two-component gaseous plasma situated between two parallel planes. We supposed that each of the plasma components has the EOS, postulated in the paper by Grinfeld [5]. The equilibrium densities are described by the Equations (2.16). This solution is physically meaningful if the corresponding densities are positive everywhere. This situation definitely takes place if the net charge of plasma is sufficiently small. Otherwise, the solution (2.16) should be corrected by calculating the points of extinction in the spirit of the report Grinfeld [4]. The exact solution presented above can be recommended for validation and verification of numerical code when dealing with more complex equations of state, external electrostatic field, and the geometry of the vessels.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

References

[1] Grinfeld, M. and Grinfeld, P. (2018) The Gibbs Variational Method in Thermodynamics of Equilibrium Plasma: 1. General Conditions of Equilibrium and Stability for One-Component Charged Gas. The US Army Research Laboratory, Aberdeen Proving Ground.

[2] Grinfeld, M. and Grinfeld, P. (2018) The Gibbs Variational Method in Thermodynamics of Equilibrium Plasma: 2. The Equation of State for Plasma. The US Army Research Laboratory, Aberdeen Proving Ground.

[3] Grinfeld, M. and Grinfeld, P. (2019) The Gibbs Variational Method in Thermodynamics of Equilibrium Plasma: 3. Multicomponent Plasma. The US Army Research Laboratory, Aberdeen Proving Ground.

[4] Grinfeld, M. and Grinfeld, P. (2019) The Gibbs Variational Method in Thermodynamics of Equilibrium Plasma: 4. One-Component Charged Gas in an External Electric Field. The US Army Research Laboratory, Aberdeen Proving Ground.

[5] Grinfeld, M. (2013) Operational Equations of State for Modeling High-Rate Phenomena. In: Chalivendra, V., Song, B. and Casem, D., Eds., *Dynamic Behavior of Materials*, Springer-Verlag, New York, 337-344. https://doi.org/10.1007/978-1-4614-4238-7_43

[6] Gibbs, J.W. (1874/1878) On the Equilibrium of Heterogeneous Substances. *Transactions of the Connecticut Academy of Arts and Sciences*, **3**, 108-248, 343-524.

[7] Hidalgo, B.H., Acosta, L.D.C.M. and Hinojosa, N.D.R.F. (2022) Mathematical Model to Elaborate the Table of Thermodynamic Properties. *Journal of Applied Mathematics and Physics*, **10**, 1951-1967. https://doi.org/10.4236/jamp.2022.106134

Notation

T—the absolute temperature

H—thickness of the layer

Q—the full charge of the plasma per unit cross-section

σ_e, σ_i —charge densities of the components per unit mass

ρ_e, ρ_i —mass densities of the components

M_e, M_i —total mass of the components per unit cross section

a_e^2, a_i^2 —compressibilities of the components

φ —electrostatic potential