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A Fractional Model for the Single Stokes Pulse from the Nonlinear Optics

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Abstract

In this paper we refer to equations of motion for the single Stokes pulse from the nonlinear optics, called the Stokes pulse system. A fractional-order model with Caputo derivative associated to Stokes pulse system (called the fractional Stokes pulse system) is proposed. The existence and uniqueness of solution of initial value problem for this fractional system are proved. The dynamic behavior for a special fractional Stokes pulse system is investigated, including: the fractional stability, the stabilization problem using suitable linear controls and the numerical integration based on fractional Euler method.

Keywords

Fractional Stokes Pulse System, Fractional Stability, Fractional Euler Method, Numerical Integration

1. Introduction

The theory of fractional differential equations (*i.e.* fractional calculus) and its applications are based on non-integer order of derivatives and integrals [1] [2].

The use of fractional models has received a great degree of interest in a series of works due to its applications in different fields of science and engineering. For example, these models played an important role in applied mathematics [3], mathematical physics [4], theoretical and applied physics [5], study of biological systems [6], control processing [7], chaos synchronization [8] [9] and so on. The dynamics of fractional-order systems associated to dynamical systems (in particular, Hamilton-Poisson systems) have been studied by many researchers in the recent decades [10] [11]. Another series of works deals with the study of dynamical behaviors of classical and fractional differential systems on Lie groups, Lie algebroids and Leibniz algebroids [12] [13] [14].

In this paper we consider the single Stokes pulse system [15]. It is described by the following differential equations on $\mathbf{R}^3$:

$$\begin{cases} \dot{u}^1(t) = (l_2 - l_3)u^2(t)u^3(t) + b_2u^3(t) - b_3u^2(t) \\ \dot{u}^2(t) = (l_3 - l_1)u^3(t)u^1(t) + b_3u^1(t) - b_1u^3(t) \\ \dot{u}^3(t) = (l_1 - l_2)u^1(t)u^2(t) + b_1u^2(t) - b_2u^1(t) \end{cases} \tag{1.1}$$

where u^1, u^2, u^3 are state variables, $\dot{u}^i(t) = \mathrm{d}u^i(t)/\mathrm{d}t$, $l_i, b_i \in \mathbf{R}$ for $i = \overline{1,3}$ are parameters and t is the time.

The Hamilton-Poisson system (1.1) has been studied from mechanical geometry point of view [16]. It is associated to this system, the general fractional Stokes pulse system. The aim of our paper is focused on the study of a certain type of the fractional Stokes pulse system.

This paper is structured as follows. The Stokes pulse system (2.6) is described in Section 2. In Section 3 we define the fractional Stokes pulse system (3.1). The existence and uniqueness of solutions of initial value problem for the fractional model (3.1) are discussed. Also, are proposed four types of fractional Stokes pulse systems which are physically inequivalent. From the four types of fractional Stokes pulse systems, we choose a subcase of the second type, called the special fractional Stokes pulse system (3.3). The Section 4 is dedicated to analyzing of asymptotic stability of equilibrium states for the fractional model (3.3). For stabilization problem of the system (3.3), we associate the fractional Stokes pulse system with controls, denoted by (4.2). In Propositions (4.3) - (4.6) are established sufficient conditions on parameters k and k_1 to control the chaos in the fractional system (4.2). Using the fractional Euler's method, the numerical integration of the system (4.2) is presented in Section 5.

2. The Single Stokes Pulse as Hamilton-Poisson System

For details on Hamiltonian dynamics, see e.g. [17] [18] [19].

The equations of motion for the Stokes polarization parameters of a single optical beam propagating as a traveling wave in a nonlinear medium (the single Stokes pulse) from the nonlinear optics are described by using the Stokes vector u and a Hamiltonian function H.

The Stokes vector is defined using the Pauli spin matrices and is called *the polarization parameters of the single Stokes pulse*. The vector $u \in \mathbf{R}^3$ is assumed to be expressed in a linear polarization basis [15]. Let be A the transition matrix from the canonical basis of $\mathbf{R}^3$ to the polarization basis. Since A is symmetric, then it one can always transform to a polarization basis in which A has the diagonal form $W = diag(l_1, l_2, l_3)$, where $l_i, i = \overline{1,3}$ are the eigenvalues of A.

The Hamiltonian function H is determined by the Stokes vector u, the diagonal matrix W, and the constant vectors $a = (a_1, a_2, a_3)$ and $c = (c_1, c_2, c_3)$. Using the vectors a and c, we define the vector $b = (b_1, b_2, b_3)$ by:

$$b = a + r \cdot c, \quad r = |c|. \tag{2.1}$$

The matrix W describes the self-induced ellipse rotation. The vectors a and c describes the effects of linear and nonlinear anisotropy, respectively.

In terms of the Stokes parameters (the components of the vector u), the Hamiltonian function $H \in C^{\infty}\left(\mathbf{R}^3, \mathbf{R}\right)$, $u \to H(u)$, is defined by [15]:

$$H(u) = u \cdot \left(\mathbf{b} + \frac{1}{2} W \cdot \mathbf{u}\right), \tag{2.2}$$

where $\mathbf{b} = \left(b_1, b_2, b_3\right)^{\mathrm{T}}$ and $\mathbf{u} = u^{\mathrm{T}}$.

The diagonal matrix W and the choice of the vectors a and c generates the dynamics of the Stokes vector u with the frequence b.

In the coordinate system $Ou^1u^2u^3$, the Hamiltonian function H defined by (2.2), is written as:

$$H(u) = \frac{1}{2}\left[l_1\left(u^1\right)^2 + l_2\left(u^2\right)^2 + l_3\left(u^3\right)^2\right] + b_1u^1 + b_2u^2 + b_3u^3. \tag{2.3}$$

The dynamics of a single Stokes pulse is written as Hamilton-Poisson system. More precisely, the Stokes pulse system is defined on the Lie-Poisson manifold $so(3)^*$ the dual of Lie algebra $SO(3)$ with the following bracket:

$$\{f, g\}(u) = u \cdot \left(\frac{\partial f}{\partial u} \times \frac{\partial g}{\partial u}\right), \quad (\forall) f, g \in C^{\infty}\left(\mathbf{R}^3, \mathbf{R}\right) \tag{2.4}$$

and the Hamiltonian function $H : so(3)^* \cong \mathbf{R}^3 \to \mathbf{R}$ given by (2.3).

The dynamical system defined on $so(3)^*$ with Poisson bracket {., .} given by (2.4), enabling the equations of motion to be expressed in Hamiltonian form:

$$\dot{u} = \{u, H\}, \tag{2.5}$$

where $u \in \mathbf{R}^3$, t is the time and H is the Hamiltonian function [20].

We determine the equations $\dot{u}^i = \left\{u^i, H\right\}, i = \overline{1,3}$ of the system (2.5). We have:

$$\begin{aligned}\left\{u^1, H\right\} &= u \cdot \left(\frac{\partial u^1}{\partial u} \times \frac{\partial H}{\partial u}\right) = \begin{vmatrix} u^1 & u^2 & u^3 \\ 1 & 0 & 0 \\ l_1u^1 + b_1 & l_2u^2 + b_2 & l_3u^3 + b_3 \end{vmatrix} \\ &= \left(l_2 - l_3\right)u^2u^3 + b_2u^3 - b_3u^2.\end{aligned}$$

Then, the first equation of the system (2.5) is $\dot{u}^1 = \left(l_2 - l_3\right)u^2u^3 + b_2u^3 - b_3u^2$.

Finally one obtains the following differential system on $\mathbf{R}^3$:

$$\begin{cases} \dot{u}^1 = \left(l_2 - l_3\right)u^2u^3 + b_2u^3 - b_3u^2 \\ \dot{u}^2 = \left(l_3 - l_1\right)u^3u^1 + b_3u^1 - b_1u^3 \\ \dot{u}^3 = \left(l_1 - l_2\right)u^1u^2 + b_1u^2 - b_2u^1 \end{cases} \tag{2.6}$$

where the parameters $l_i, b_i \in \mathbf{R}$ for $i = \overline{1,3}$ are connected with the nature of the material and the medium.

The system (2.6) is called the *Stokes pulse dynamical system*.

Acording to [20] there are six types of Equation (2.6) which are physically inequivalent and which correspond to different types of optical media [16].

Proposition 2.1. *The functions* H given by (2.4) and $C \in C^{\infty}\left(\mathbf{R}^3, \mathbf{R}\right)$, defined by:

$$u \to C(u) = \frac{1}{2}\left[\left(u^1\right)^2 + \left(u^2\right)^2 + \left(u^3\right)^2\right],$$

are constants of the motion (*first integrals*) *for the dynamics* (2.6).

Proof. Indeed, we have

$$\mathrm{d}H/\mathrm{d}t = l_1 u^1 \dot{u}^1 + l_2 u^2 \dot{u}^2 + l_3 u^3 \dot{u}^3 + b_1 \dot{u}^1 + b_2 \dot{u}^2 + b_3 \dot{u}^3 = 0.$$

Also, we have

$$\begin{aligned} \mathrm{d}C/\mathrm{d}t &= u^1 \dot{u}^1 + u^2 \dot{u}^2 + u^3 \dot{u}^3 \\ &= u^1\left[\left(l_2 - l_3\right) u^2 u^3 + b_2 u^3 - b_3 u^2\right] + u^2\left[\left(l_3 - l_1\right) u^3 u^1 + b_3 u^1 - b_1 u^3\right] \\ &\quad + u^3\left[\left(l_1 - l_2\right) u^1 u^2 + b_1 u^2 - b_2 u^1\right] \\ &= 0. \end{aligned}$$

Remark 2.1. By Proposition 2.1, it follows that *the trajectories of motion of Stokes pulse dynamical system* (2.6) are intersections of the surfaces:

$$H = \text{constant} \quad \text{and} \quad C = \text{constant}.$$

3. The Fractional Stokes Pulse System

For basic knowledge on fractional calculus, one may refer to [21] [22].

In this paper we consider the fractional derivative operator D_t^q with $q \in (0,1)$ to be *Caputo's derivative*. This fractional derivative operator is often used in concrete applications.

Let $f \in C^{\infty}(\mathbf{R})$ and $q \in \mathbf{R}, q > 0$. The q-order Caputo differential operator [21], is described by $D_t^q f(t) = I^{m-q} f^{(m)}(t), q > 0$, where $f^{(m)}(t)$ represents the m-order derivative of the function f, $m \in \mathbf{N}^*$ is an integer such that

$$m - 1 \le q \le m \quad \text{and} \quad I^q \quad \text{is} \quad I^q f(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s) \mathrm{d}s, \quad q > 0,$$

where Γ is the Euler Gamma function. If $q = 1$, then $D_t^1 f(t) = \mathrm{d}f/\mathrm{d}t$.

The Hamilton-Poisson system (2.6) is modeled by the following fractional differential equations:

$$\begin{cases} D_q^t u^1(t) = \left(l_2 - l_3\right) u^2(t) u^3(t) + b_2 u^3(t) - b_3 u^2(t) \\ D_q^t u^2(t) = \left(l_3 - l_1\right) u^3(t) u^1(t) + b_3 u^1(t) - b_1 u^3(t), \qquad q \in (0,1) \\ D_q^t u^3(t) = \left(l_1 - l_2\right) u^1(t) u^2(t) + b_1 u^2(t) - b_2 u^1(t) \end{cases} \tag{3.1}$$

where u^i are the Stokes polarization parameters and $l_i, b_i \in \mathbf{R}$ for $i = \overline{1,3}$.

The system (3.1) is called the *fractional Stokes pulse system* associated to (2.6).

The initial value problem of fractional model (3.1) can be represented in the following matrix form:

$$D_t^q u(t) = Au(t) + u^1(t) A_1 u(t) + u^2(t) A_2 u(t) + u^3(t) A_3 u(t), \quad u(0) = u_0, \tag{3.2}$$

where $0 < q < 1$, $u(t) = \left(u^1(t), u^2(t), u^3(t)\right)^{\mathrm{T}}$, $t \in (0, \tau)$ and

$$A=\begin{pmatrix}0&-b_3&b_2\\b_3&0&-b_1\\-b_2&b_1&0\end{pmatrix},\ A_1=\begin{pmatrix}0&0&0\\0&0&l_3-l_1\\0&0&0\end{pmatrix},$$
$$A_2=\begin{pmatrix}0&0&0\\0&0&0\\l_1-l_2&0&0\end{pmatrix},\ A_3=\begin{pmatrix}0&l_2-l_3&0\\0&0&0\\0&0&0\end{pmatrix}.$$

Proposition **3.1.** *The initial value problem of the fractional Stokes pulse system* (3.1) *has a unique solution.*

Proof. Let $f(u(t))=Au(t)+u^1(t)A_1u(t)+u^2(t)A_2u(t)+u^3(t)A_3u(t)$. It is obviously continuous and bounded on
$D=\left\{u\in\mathbf{R}^3 \mid u^i\in\left[u_0^i-\delta,u_0^i+\delta\right], i=\overline{1,3}\right\}$ for any $\delta>0$. We have
$f(u(t))-f(u_1(t))=A(u(t)-u_1(t))+x(t)+y(t)+z(t)$, where
$x(t)=u^1(t)A_1u(t)-u_1^1(t)A_1u_1(t)$, $y(t)=u^2(t)A_2u(t)-u_1^2(t)A_2u_1(t)$ and
$z(t)=u^3(t)A_3u(t)-u_1^3(t)A_3u_1(t)$,

Then

(1) $\left|f(u(t))-f(u_1(t))\right|\le\|A\|\cdot|u(t)-u_1(t)|+|x(t)|+|y(t)|+|z(t)|$, where $\|\cdot\|$ and denote $|\cdot|$ matrix norm and vector norm, respectively.

It is easy to see that $x(t)=\left(u^1(t)-u_1^1(t)\right)A_1u(t)+u_1^1(t)A_1\left(u(t)-u_1(t)\right)$. Then

(2) $|x(t)|\le\|A_1\|\left(|u(t)|+|u_1^1(t)|\right)|u(t)-u_1(t)|$.

Similarly, we prove that

(3) $|y(t)|\le\|A_2\|\left(|u(t)|+|u_1^2(t)|\right)|u(t)-u_1(t)|$.

(4) $|z(t)|\le\|A_3\|\left(|u(t)|+|u_1^3(t)|\right)|u(t)-u_1(t)|$.

According to (2)-(4), the relation (1) becomes

(5) $$\begin{aligned}\left|f(u(t))-f(u_1(t))\right|\le&\left(\|A\|+\|A_1\|\left(|u(t)|+|u_1^1(t)|\right)+\|A_2\|\left(|u(t)|+|u_1^2(t)|\right)\right.\\&\left.+\|A_3\|\left(|u(t)|+|u_1^3(t)|\right)\right)\cdot|u(t)-u_1(t)|\end{aligned}.$$

We have $\|A\|=\sqrt{2\left(b_1^2+b_2^2+b_3^2\right)}:=\theta$, $\|A_1\|=|l_3-l_1|$, $\|A_2\|=|l_1-l_2|$, $\|A_3\|=|l_2-l_3|$. If $\omega=\max\left\{|l_3-l_1|,|l_1-l_2|,|l_2-l_3|\right\}$, then $\|A_i\|\le\omega$ for $i=\overline{1,3}$. From the relation (5) we deduce that

(6) $$\begin{aligned}\left|f(u(t))-f(u_1(t))\right|\le&\left(\theta+\omega\left(|u(t)|+|u_1^1(t)|\right)+\omega\left(|u(t)|+|u_1^2(t)|\right)\right.\\&\left.+\omega\left(|u(t)|+|u_1^3(t)|\right)\right)\cdot|u(t)-u_1(t)|\end{aligned}.$$

Given that inequalities $|u(t)|+|u_1^i(t)|\le2|u_0|+\delta$, $i=\overline{1,3}$, are valid, the relation (6) becomes

(7) $\left|f(u(t))-f(u_1(t))\right|\le L\cdot|u(t)-u_1(t)|$,

where $L=\theta+3\omega\left(2|u_0|+\delta\right)>0$.

The inequality (7) shows that $f(u(t))$ satisfies a Lipschitz condition. Using Theorems 1 and 2 in [23], it follows that the system (3.1) has a unique solution.

As with the nonlinear dynamics generated by the Stokes pulse system (2.6) there are six types of fractional Equation (3.1) which are physically inequivalent and which correspond to different types of optical media.

In this section we will refer to the types of fractional Stokes pulse systems for which the parameters $l_i \in \mathbf{R}, i=\overline{1,3}$ meet the following condition $l_1 \neq l_2 \neq l_3 \neq l_1$.

In this context there are the following *four types* of fractional Stokes pulse systems:

Type 1. $b=(0,0,0)$ and $l_1 \neq l_2 \neq l_3 \neq l_1$;

Type 2. $b=(0,b_2,0)$, $b_2 \neq 0$ and $l_1 \neq l_2 \neq l_3 \neq l_1$;

Type 3. $b=(b_1,0,b_3)$, $b_1 b_3 \neq 0$ and $l_1 \neq l_2 \neq l_3 \neq l_1$;

Type 4. $b=(b_1,b_2,b_3)$, $b_1 b_2 b_3 \neq 0$ and $l_1 \neq l_2 \neq l_3 \neq l_1$.

As an example, the fractional system corresponding to type 2, is given by:

$$\begin{cases} D_q^t u^1(t) = (l_2 - l_3) u^2(t) u^3(t) + b_2 u^3(t) \\ D_q^t u^2(t) = (l_3 - l_1) u^3(t) u^1(t), & q \in (0,1) \\ D_q^t u^3(t) = (l_1 - l_2) u^1(t) u^2(t) - b_2 u^1(t) \end{cases} \tag{3.3}$$

where $b_2 \in \mathbf{R}^*$ and $l_i \in \mathbf{R}, i=\overline{1,3}$ such that $l_1 \neq l_2 \neq l_3 \neq l_1$.

The system (3.3) is called the *special fractional Stokes pulse system*. It is determined by a single nonzero component of vector b and contains the three nonlinear terms of the system (3.1) (given in general form).

For the system (3.3) we introduce the following notations:

$$f_1(u) = (l_2 - l_3) u^2 u^3 + b_2 u^3,\ f_2(u) = (l_3 - l_1) u^3 u^1,\ f_3(u) = (l_1 - l_2) u^1 u^2 - b_2 u^1. \tag{3.4}$$

Proposition 3.2. *The equilibrium states of the special fractional Stokes pulse system* (3.3) *are given as the union of the following three families*:

$$E_1 := \left\{ e_1^m = (0,m,0) \in \mathbf{R}^3 \mid m \in \mathbf{R} \right\},$$

$$E_2 := \left\{ e_2^m = \left(m, \frac{b_2}{l_1 - l_2}, 0 \right) \in \mathbf{R}^3 \mid m \in \mathbf{R} \right\},$$

$$E_3 := \left\{ e_3^m = \left(0, \frac{b_2}{l_3 - l_2}, m \right) \in \mathbf{R}^3 \mid m \in \mathbf{R} \right\}.$$

Proof. The equilibrium states are solutions of the equations $f_i(u) = 0, i=\overline{1,3}$, where $f_i, i=\overline{1,3}$ are given by (3.4).

Remark 3.1. If in the fractional model (3.3) we take $q=1$, then one obtains the system for integer-order derivative which corresponds to type 4 $ of the dynamics (1.1). For this dynamical system, the nonlinear stability and the problem of existence of periodic solutions are studied, see Theorems 2.4, 3.5-3.7 [16].

4. Asymptotic Stability of the Special Fractional Stokes Pulse System (3.3)

Let us we present the study of asymptotic stability of equilibria for the fractional system (3.3). Finally, we will discuss how to stabilize the unstable equilibrium states of the system (3.3) via fractional order derivative. For this study we apply the Matignon's test [7].

In the follows we will use the notations:

$$\alpha := l_2 - l_3, \quad \beta := l_3 - l_1, \quad \gamma := l_1 - l_2. \tag{4.1}$$

With the notations (4.1), the *Jacobian matrix associated to system* (3.3) is:

$$J(u)=\begin{pmatrix} 0 & \alpha u^3 & \alpha u^2+b_2 \\ \beta u^3 & 0 & \beta u^1 \\ \gamma u^2-b_2 & \gamma u^1 & 0 \end{pmatrix}.$$

Proposition 4.1. ([7]) *Let* u_e *be an equilibrium state of system* (3.3) *and* $J(u_e)$ *be the Jacobian matrix* $J(u)$ *evaluated at* u_e.

(*i*) u_e *is locally asymptotically stable, iff all eigenvalues of the matrix* $J(u_e)$ *satisfy:*

$$\left|\arg\left(\lambda\left(J\left(u_e\right)\right)\right)\right|>\frac{q\pi}{2}.$$

(*ii*) u_e *is locally stable, iff either it is asymptotically stable, or the critical eigenvalues of* $J(u_e)$ *which satisfy* $\left|\arg\left(\lambda\left(J\left(u_e\right)\right)\right)\right|=\frac{q\pi}{2}$ *have geometric multiplicity one.*

Proposition 4.2. *The equilibrium states* $e_i^m, i=\overline{1,3}$ *are unstable* $(\forall)q\in(0,1)$.

Proof. The characteristic polynomial of the matrix

$$J\left(e_1^m\right)=\begin{pmatrix} 0 & 0 & m\alpha+b_2 \\ 0 & 0 & 0 \\ m\gamma-b_2 & 0 & 0 \end{pmatrix} \quad \textit{is}$$

$p_1(\lambda)=\det\left(J\left(e_1^m\right)-\lambda I\right)=-\lambda\left[\lambda^2-\left(\alpha\gamma m^2-b_2(\gamma-\alpha)m-b_2^2\right)\right]$. For $m=0$, the characteristic polynomials of the matrix $J(e_0)$ is $p_0(\lambda)=-\lambda\left(\lambda^2+b_2^2\right)$.

The characteristic polynomials of matrices $J\left(e_2^m\right)$ and $J\left(e_3^m\right)$ are the following:

$$p_2(\lambda)=-\lambda\left(\lambda^2-\beta\gamma m^2\right) \quad \textit{and} \quad p_3(\lambda)=-\lambda\left(\lambda^2-\alpha\beta m^2\right).$$

The equations $p_0(\lambda)=0$ and $p_i(\lambda)=0, i=\overline{1,3}$ have the root $\lambda_1=0$. Since $\arg(\lambda_1)=0<\frac{q\pi}{2}$ for all $q\in(0,1)$, by Proposition 4.1 follows that the equilibrium states e_0 and $e_i^m, i=\overline{1,3}$ are unstable for all $q\in(0,1)$.

In the case when u_e is a unstable equilibrium state of the fractional system (3.3), we associate to (3.3) a new fractional system, called the *special fractional Stokes pulse system with* (*external*) *controls* and given by:

$$\begin{cases} D_q^t u^1=\alpha u^2u^3+b_2u^3+k_1u^1 \\ D_q^t u^2=\beta u^3u^1+ku^2, \qquad\qquad q\in(0,1) \\ D_q^t u^3=\gamma u^1u^2-b_2u^1+k_1u^3 \end{cases} \tag{4.2}$$

where α,β,γ are given in (4.1) and $k,k_1\in\mathbf{R}$ are controls.

If one selects the parameters k,k_1 which then make the eigenvalues of the Jacobian matrix of fractional model (3.3) satisfy one of the conditions from Proposition 3.1, then its trajectories asymptotically approaches the unstable equilibrium state u_e in the sense that $\lim_{t\to\infty}\left\|u(t)-u_e\right\|=0$, where $\|\cdot\|$ is the Euclidean norm.

The Jacobian matrix of the fractional model (4.2) with the controls k, k_1 is

$$J(u,k,k_1)=\begin{pmatrix} k_1 & \alpha u^3 & \alpha u^2+b_2 \\ \beta u^3 & k & \beta u^1 \\ \gamma u^2-b_2 & \gamma u^1 & k_1 \end{pmatrix}.$$

Proposition 4.3. *Let be the fractional Stokes pulse system* (4.2) *with the controls* $k, k_1 \in \mathbf{R}^*$.

(*i*) *If* $k<0, k_1<0$, *then* e_0 *is asymptotically stable* $(\forall) q \in (0,1)$;

(*ii*) *If* $k<0, k_1>0$ *and* $q_0=\frac{2}{\pi}\arctan\frac{|b_2|}{k_1}$, *then:*

(1) e_0 *is asymptotically stable* $(\forall) q \in (0,q_0)$ *and it is stable for* $q=q_0$.

(2) e_0 *is unstable* $(\forall) q \in (q_0,1)$.

(*iii*) *If* $k>0$ *and* $k_1 \in \mathbf{R}^*$, *then* e_0 *is unstable* $(\forall) q \in (0,1)$.

Proof. The characteristic polynomial of the Jacobian matrix $J(e_0,k,k_1)$ is $p_0(\lambda,k,k_1)=-(\lambda-k)\left[(\lambda-k_1)^2+b_2^2\right]$. The roots of the equation $p_0(\lambda,k,k_1)=0$ are $\lambda_1=k$, $\lambda_{2,3}=k_1 \pm i b_2$.

(i) We suppose $k<0$ and $k_1<0$. In this case we have $\operatorname{Re}(\lambda_i)<0$ for $i=\overline{1,3}$. Since $|\arg(\lambda_i)|=\pi>\frac{q\pi}{2}, i=\overline{1,3}$ for all $q\in(0,1)$, by Proposition 4.1(i), it implies that e_0 is asymptotically stable for all $k,k_1\in\mathbf{R}^*$.

(ii) We suppose $k<0$ and $k_1>0$. In this case we have $\lambda_1<0$ and $\operatorname{Re}(\lambda_{2,3})>0$. Applying Proposition 4.1(i), e_0 is locally asymptotically stable, for $0<q<q_0$, where $q_0=\frac{2}{\pi}\arctan\frac{|b_2|}{k_1}$. If $q=q_0$, e_0 is stable. For $q_0<q<1$, e_0 is unstable $(\forall) k,k_1\in\mathbf{R}^*$. Hence, the assertion (ii) holds.

(iii) We suppose $k>0$ and $k_1\in\mathbf{R}^*$. Since $J(e_0,k,k_1)$ has at least a positive eigenvalue, it follows that e_0 is unstable. Hence, (iii) holds, $(\forall) q\in(0,1)$.

Proposition 4.4. *Let be the fractional Stokes pulse system* (4.2) *with the controls* $k,k_1\in\mathbf{R}^*$, $e_1^m=(0,m,0)$ *and* $\Delta_1=(\alpha m+b_2)(\gamma m-b_2)$.

1. *Let* $\Delta_1<0$ *and* $q\in(0,1)$.

(*i*) *If* $k<0$ *and* $k_1<0$, *then* e_1^m *is asymptotically stable.*

(*ii*) *Let* $k<0, k_1>0$ *and* $q_1=\frac{2}{\pi}\arctan\frac{\sqrt{-\Delta_1}}{k_1}$.

(1) *If* $b_2>0, \frac{\alpha+\gamma}{\alpha\gamma}>0$ *and* $m\in\left(-\frac{b_2}{\alpha},\frac{b_2}{\gamma}\right)$, *then* e_1^m *is asymptotically stable* $(\forall) q\in(0,q_1)$, *stable for* $q=q_1$ *and unstable* $(\forall) q\in(q_1,1)$.

(2) *If* $b_2<0, \frac{\alpha+\gamma}{\alpha\gamma}>0$ *and* $m\in\left(\frac{b_2}{\gamma},-\frac{b_2}{\alpha}\right)$, *then* e_1^m *is asymptotically stable* $(\forall) q\in(0,q_1)$, *stable for* $q=q_1$ *and unstable* $(\forall) q\in(q_1,1)$.

(*iii*) *If* $k>0$ *and* $k_1\in\mathbf{R}^*$, *then* e_1^m *is unstable* $(\forall) q\in(0,1)$.

2. *Let* $\Delta_1>0$ *and* $q\in(0,1)$.

(*i*) *If* $k<0, k_1<0$ *and* $k_1^2>\Delta_1$.

(1) *If* $b_2 > 0, \frac{\alpha+\gamma}{\alpha\gamma} > 0$ *and* $m \in \left(-\infty, -\frac{b_2}{\alpha}\right) \cup \left(\frac{b_2}{\gamma}, \infty\right)$, *then* e_1^m *is asymptotically stable.*

(2) *If* $b_2 < 0, \frac{\alpha+\gamma}{\alpha\gamma} > 0$ *and* $m \in \left(-\infty, \frac{b_2}{\gamma}\right) \cup \left(-\frac{b_2}{\alpha}, \infty\right)$, *then* e_1^m *is asymptotically stable.*

(*ii*) *Let* $k < 0, k_1 < 0, k_1^2 \le \Delta_1$ *and* $\frac{\alpha+\gamma}{\alpha\gamma} > 0$. *If* $b_2 > 0$, $m \in \left(-\infty, -\frac{b_2}{\alpha}\right) \cup \left(\frac{b_2}{\gamma}, \infty\right)$ *or* $b_2 < 0$, $m \in \left(-\infty, \frac{b_2}{\gamma}\right) \cup \left(-\frac{b_2}{\alpha}, \infty\right)$, *then* e_1^m *is unstable.*

Proof. The characteristic polynomial of (4.2) at e_1^m is

$J\left(e_1^m, k, k_1\right) = \begin{pmatrix} k_1 & 0 & \alpha m + b_2 \\ 0 & k & 0 \\ \gamma m - b_2 & 0 & k_1 \end{pmatrix}$ whose characteristic polynomial is

$p_1(\lambda, k, k_1) = \det\left(J\left(e_1^m, k, k_1\right) - \lambda I\right) = -(\lambda - k)\left[(\lambda - k_1)^2 - (\alpha m + b_2)(\gamma m - b_2)\right]$.

The roots of the characteristic equation $p_1(\lambda, k, k_1) = 0$ are $\lambda_1 = k$, $\lambda_{2,3} = k_1 \pm \sqrt{\Delta_1}$, where $\Delta_1 = (\alpha m + b_2)(\gamma m - b_2)$.

1. ***Case*** $\Delta_1 < 0$ and $q \in (0,1)$. We have the following two situations:

(1) if $b_2 > 0, \frac{\alpha+\gamma}{\alpha\gamma} > 0$, then $\Delta_1 < 0$ for all $m \in \left(-\frac{b_2}{\alpha}, \frac{b_2}{\gamma}\right)$;

(2) if $b_2 < 0, \frac{\alpha+\gamma}{\alpha\gamma} > 0$, then $\Delta_1 < 0$ for all $m \in \left(\frac{b_2}{\gamma}, -\frac{b_2}{\alpha}\right)$.

In this case, $\lambda_1 = k$, $\lambda_{2,3} = k_1 \pm i\sqrt{-\Delta_1}$.

(i) We suppose $k < 0$ and $k_1 < 0$. In this case we have $\lambda_1 < 0$ and $\operatorname{Re}(\lambda_{2,3}) < 0$. Since $\left|\arg(\lambda_i)\right| = \pi > \frac{q\pi}{2}, i = \overline{1,3}$ for all $q \in (0,1)$, by Proposition 4.1(i), it implies that e_1^m is locally asymptotically stable for all $m \in \mathbf{R}^*$.

(ii) (1)-(2). For $k < 0$ and $k_1 > 0$, we have $\lambda_1 < 0$ and $\operatorname{Re}(\lambda_{2,3}) > 0$. Applying Proposition 4.1(i), e_1^m is asymptotically stable, for $0 < q < q_1$, where $q_1 = \frac{2}{\pi}\arctan\frac{\sqrt{-\Delta_1}}{k_1}$. If $q = q_1$, then e_1^m is stable. For $q_1 < q < 1$, e_1^m is unstable.

(iii) Let $k > 0$ and $k_1 \in \mathbf{R}^*$. Since $\lambda_1 > 0$, $J\left(e_1^m, k, k_1\right)$ has at least a positive eigenvalue and so e_1^m is unstable. Hence, the assertions (i)-(iii) hold.

2. ***Case*** $\Delta_1 > 0$ and $q \in (0,1)$. Then $\lambda_1 = k$, $\lambda_{2,3} = k_1 \pm \sqrt{\Delta_1}$. We have the following two situations:

(1) if $b_2 > 0, \frac{\alpha+\gamma}{\alpha\gamma} > 0$, then $\Delta_1 > 0$ for all $m \in \left(-\infty, -\frac{b_2}{\alpha}\right) \cup \left(\frac{b_2}{\gamma}, \infty\right)$;

(2) if $b_2 < 0, \frac{\alpha+\gamma}{\alpha\gamma} > 0$, then $\Delta_1 > 0$ for all $m \in \left(-\infty, \frac{b_2}{\gamma}\right) \cup \left(-\frac{b_2}{\alpha}, \infty\right)$.

(i)-(ii) The eigenvalues $\lambda_i, i=\overline{1,3}$ are all negative if and only if $k<0, k_1<0$ *and* $k_1^2>\Delta_1$. In these hypotheses it folows that e_1^m is asymptotically stable. Also, if $k_1^2 \le \Delta_1$, then e_1^m is unstable. Therefore, the assertions (i)-(ii) hold.

Proposition 4.5. *Let be the fractional Stokes pulse system* (4.2) *with the controls* $k, k_1 \in \mathbf{R}^*$, $e_2^m = \left(m, \frac{b_2}{\gamma}, 0\right)$ *and* $\Delta_2 = (k-k_1)^2 + 4\beta\gamma m^2$.

1. *Let* $\beta\gamma<0$ *and* $m \in \mathbf{R}^*$.

(*i*) *Let* $\Delta_2<0$ *and* $q\in(0,1)$.

(1) *If* $k<0$ *and* $k_1<0$, *then* e_2^m *is asymptotically stable* $(\forall) m \in \left(-\infty, \frac{k-k_1}{2\sqrt{-\beta\gamma}}\right) \cup \left(\frac{k_1-k}{2\sqrt{-\beta\gamma}}, \infty\right)$.

(2) *If* $k_1<0, k>k_1$ *and* $k+k_1<0$, *then* e_2^m *is asymptotically stable* $(\forall) m \in \left(-\infty, \frac{k_1-k}{2\sqrt{-\beta\gamma}}\right) \cup \left(\frac{k-k_1}{2\sqrt{-\beta\gamma}}, \infty\right)$.

(3) *If* $k_1<0, k>0, k+k_1>0$, $q_2 = \frac{2}{\pi}\arctan\frac{\sqrt{-\Delta_2}}{k+k_1}$ *and* $m \in \left(-\infty, \frac{k_1-k}{2\sqrt{-\beta\gamma}}\right) \cup \left(\frac{k-k_1}{2\sqrt{-\beta\gamma}}, \infty\right)$, *then* e_2^m *is asymptotically stable* $(\forall) q\in(0,q_2)$, *stable for* $q=q_2$ *and unstable* $(\forall) q \in (q_2, 1)$.

(*ii*) *Let* $\Delta_2 \ge 0$ *and* $q\in(0,1)$.

(1) *If* $k<k_1<0$ *and* $k_1<0$, *then* e_2^m *is asymptotically stable* $(\forall) m \in \left[\frac{k-k_1}{2\sqrt{-\beta\gamma}}, \frac{k_1-k}{2\sqrt{-\beta\gamma}}\right]$.

(2) *If* $k_1<0, k>k_1$ *and* $k+k_1<0$, *then* e_2^m *is asymptotically stable* $(\forall) m \in \left[\frac{k_1-k}{2\sqrt{-\beta\gamma}}, \frac{k-k_1}{2\sqrt{-\beta\gamma}}\right]$.

(3) *If* $k_1<0, k>0, k+k_1>0$ *or* $k_1>0$, $k\in\mathbf{R}^*$, *then* e_2^m *is unstable.*

2. *Let* $\beta\gamma>0$, $m\in\mathbf{R}^*$ *and* $q\in(0,1)$.

(*i*) *Let* $k<0, k_1<0$ *and* $(k+k_1)^2 > \Delta_2 > 0$.

(1) *If* $k<k_1$, $m \in \left[\frac{k-k_1}{2\sqrt{-\beta\gamma}}, \frac{k_1-k}{2\sqrt{-\beta\gamma}}\right]$, *then* e_2^m *is asymptotically stable.*

(2) *If* $k>k_1$, $m \in \left[\frac{k_1-k}{2\sqrt{-\beta\gamma}}, \frac{k-k_1}{2\sqrt{-\beta\gamma}}\right]$, *then* e_2^m *is asymptotically stable.*

(*ii*) *Let* $k_1<0, k+k_1>0, (k+k_1)^2 \le \Delta_2$ *or* $k_1>0, k\in\mathbf{R}^*$. *If* $k<k_1$, $m \in \left[\frac{k-k_1}{2\sqrt{-\beta\gamma}}, \frac{k_1-k}{2\sqrt{-\beta\gamma}}\right]$ *or* $k>k_1$, $m \in \left[\frac{k_1-k}{2\sqrt{-\beta\gamma}}, \frac{k-k_1}{2\sqrt{-\beta\gamma}}\right]$, *then* e_2^m *is unstable.*

Proof. The Jacobian matrix of (4.2) at e_2^m is

$$J\left(e_2^m,k,k_1\right)=\begin{pmatrix} k_1 & 0 & \frac{\alpha b_2}{\gamma}+b_2 \\ 0 & k & \beta m \\ 0 & \gamma m & k_1 \end{pmatrix}$$

whose characteristic polynomial is

$p_2\left(\lambda,k,k_1\right)=-\left(\lambda-k_1\right)\left[\lambda^2-\left(k+k_1\right)\lambda+kk_1-\beta\gamma m^2\right]$. The roots of the equation $p_2\left(\lambda,k,k_1\right)=0$ are $\lambda_1=k_1$, $\lambda_{2,3}=\frac{\left(k+k_1\right)\pm\sqrt{\Delta_2}}{2}$, where $\Delta_2=\left(k-k_1\right)^2+4\beta\gamma m^2$.

1. Case $\beta\gamma<0$.

(i) ***Case*** $\Delta_2<0$ and $q\in(0,1)$. Then $\lambda_1=k_1$, $\lambda_{2,3}=\frac{\left(k+k_1\right)\pm i\sqrt{-\Delta_2}}{2}$. We have $\Delta_2<0$ if and only if $m\in\left(-\infty,\frac{k-k_1}{2\sqrt{-\beta\gamma}}\right)\cup\left(\frac{k_1-k}{2\sqrt{-\beta\gamma}},\infty\right)$ when $k<k_1$ or $m\in\left(-\infty,\frac{k_1-k}{2\sqrt{-\beta\gamma}}\right)\cup\left(\frac{k-k_1}{2\sqrt{-\beta\gamma}},\infty\right)$ when $k>k_1$.

(1) We suppose $k<k_1<0$. In this case we have $\lambda_1<0$ and $\operatorname{Re}\left(\lambda_{2,3}\right)<0$. Since $\left|\arg\left(\lambda_i\right)\right|=\pi>\frac{q\pi}{2},i=\overline{1,3}$ for all $q\in(0,1)$, by Proposition 4.1(i), it implies that e_2^m is locally asymptotically stable for all $m\in\left(-\infty,\frac{k-k_1}{2\sqrt{-\beta\gamma}}\right)\cup\left(\frac{k_1-k}{2\sqrt{-\beta\gamma}},\infty\right)$.

(2) We suppose $k_1<0,k>k_1$ and $k+k_1<0$. In this case we have $\lambda_1<0$ and $\operatorname{Re}\left(\lambda_{2,3}\right)<0$. Applying the same reasoning as in the case (i)(1), one obtains that e_2^m is locally asymptotically stable for all $m\in\left(-\infty,\frac{k_1-k}{2\sqrt{-\beta\gamma}}\right)\cup\left(\frac{k-k_1}{2\sqrt{-\beta\gamma}},\infty\right)$.

(3) We suppose $k_1<0,k>0$ and $k+k_1>0$. In this case, $\Delta_2<0$ if and only if $m\in\left(-\infty,\frac{k_1-k}{2\sqrt{-\beta\gamma}}\right)\cup\left(\frac{k-k_1}{2\sqrt{-\beta\gamma}},\infty\right)$. Then $\lambda_1<0$ and $\operatorname{Re}\left(\lambda_{2,3}\right)>0$. Applying Proposition 4.1(i), e_2^m is locally asymptotically stable, for $0<q<q_2$, where $q_2=\frac{2}{\pi}\arctan\frac{\sqrt{-\Delta_2}}{k+k_1}$. If $q=q_2$, then e_2^m is stable. For $q_2<q<1$, e_2^m is unstable.

(ii) ***Case*** $\Delta_2\geq 0$ *and* $q\in(0,1)$. Then $\lambda_1=k_1$, $\lambda_{2,3}=\frac{\left(k+k_1\right)\pm\sqrt{\Delta_2}}{2}$. We have $\Delta_2\geq 0$ if and only if $m\in\left[\frac{k-k_1}{2\sqrt{-\beta\gamma}},\frac{k_1-k}{2\sqrt{-\beta\gamma}}\right]$ when $k<k_1$ or

$m \in \left[\frac{k_1-k}{2\sqrt{-\beta\gamma}}, \frac{k-k_1}{2\sqrt{-\beta\gamma}}\right]$ when $k > k_1$.

(1) and (2). In these cases, if $k < 0$ and $k_1 < 0$, then $\lambda_2 + \lambda_3 < 0$ and $\lambda_2\lambda_3 = kk_1 - \beta\gamma m^2 > 0$. It follows $\lambda_i < 0, i = \overline{1,3}$. Then, e_2^m is locally asymptotically stable.

(3) Let $k_1 < 0, k > 0$ and $k + k_1 > 0$ or $k_1 > 0$ and $k \in \mathbf{R}^*$. Then $J\left(e_2^m, k, k_1\right)$ has at least a positive eigenvalue and so e_2^m is unstable. Hence, the assertion (ii) holds.

2. Case $\beta\gamma > 0$, $m \in \mathbf{R}^*$ *and* $q \in (0,1)$. In this case $\Delta_2 \geq 0$.

(i) We suppose $k_1 < 0$. We have $\lambda_2 < 0$ and $\lambda_3 < 0$ if and only if $\lambda_2 + \lambda_3 < 0$ and $\lambda_2\lambda_3 > 0$. Then $k + k_1 < 0$ and $(k + k_1)^2 - \Delta_2 > 0$. It follows $\lambda_i < 0, i = \overline{1,3}$ for all $m \in \mathbf{R}^*$ such that $(k + k_1)^2 > \Delta_2$. Hence, e_2^m is locally asymptotically stable.

(ii) We suppose $k_1 < 0, k + k_1 > 0$ or $k_1 > 0, k \in \mathbf{R}^*$. Then, $J\left(e_2^m, k, k_1\right)$ has at least a positive eigenvalue and so e_2^m is unstable. Therefore, the assertion (ii) holds.

Proposition 4.6. *Let be the fractional Stokes pulse system* (4.2) *with the controls* $k, k_1 \in \mathbf{R}^*$, $e_3^m = \left(0, -\frac{b_2}{\alpha}, m\right)$ *and* $\Delta_3 = (k - k_1)^2 + 4\alpha\beta m^2$.

1. *Let* $\alpha\beta < 0$ *and* $m \in \mathbf{R}^*$.

(*i*) *Let* $\Delta_3 < 0$ *and* $q \in (0,1)$.

(1) *If* $k < k_1 < 0$, *then* e_3^m *is asymptotically stable*

$(\forall) m \in \left(-\infty, \frac{k-k_1}{2\sqrt{-\alpha\beta}}\right) \cup \left(\frac{k_1-k}{2\sqrt{-\alpha\beta}}, \infty\right)$.

(2) *If* $k_1 < 0, k > k_1$ *and* $k + k_1 < 0$, *then* e_3^m *is asymptotically stable*

$(\forall) m \in \left(-\infty, \frac{k_1-k}{2\sqrt{-\alpha\beta}}\right) \cup \left(\frac{k-k_1}{2\sqrt{-\alpha\beta}}, \infty\right)$.

(3) *If* $k_1 < 0, k > 0, k + k_1 > 0$, $q_3 = \frac{2}{\pi}\arctan\frac{\sqrt{-\Delta_3}}{k + k_1}$ *and*

$m \in \left(-\infty, \frac{k_1-k}{2\sqrt{-\alpha\beta}}\right) \cup \left(\frac{k-k_1}{2\sqrt{-\alpha\beta}}, \infty\right)$, *then* e_2^m *is asymptotically stable* $(\forall) q \in (0, q_3)$, *stable for* $q = q_3$ *and unstable* $(\forall) q \in (q_3, 1)$.

(*ii*) *Let* $\Delta_3 \geq 0$ *and* $q \in (0,1)$.

(1) *If* $k < k_1 < 0$ *and* $k_1 < 0$, *then* e_3^m *is asymptotically stable*

$(\forall) m \in \left[\frac{k-k_1}{2\sqrt{-\alpha\beta}}, \frac{k_1-k}{2\sqrt{-\alpha\beta}}\right]$.

(2) *If* $k_1 < 0, k > k_1$ *and* $k + k_1 < 0$, *then* e_3^m *is asymptotically stable*

$(\forall) m \in \left[\frac{k_1-k}{2\sqrt{-\alpha\beta}}, \frac{k-k_1}{2\sqrt{-\alpha\beta}}\right]$.

(3) *If* $k_1 < 0, k > 0, k + k_1 > 0$ *or* $k_1 > 0$, $k \in \mathbf{R}^*$, *then* e_3^m *is unstable.*

2. *Let* $\alpha\beta > 0$, $m \in \mathbf{R}^*$ *and* $q \in (0,1)$.

(*i*) *Let* $k < 0, k_1 < 0$ *and* $(k+k_1)^2 > \Delta_3 > 0$.

(1) *If* $k < k_1$, $m \in \left[\frac{k-k_1}{2\sqrt{-\alpha\beta}}, \frac{k_1-k}{2\sqrt{-\alpha\beta}}\right]$, *then* e_3^m *is asymptotically stable.*

(2) *If* $k > k_1$, $m \in \left[\frac{k_1-k}{2\sqrt{-\alpha\beta}}, \frac{k-k_1}{2\sqrt{-\alpha\beta}}\right]$, *then* e_3^m *is asymptotically stable.*

(*ii*) *Let* $k_1 < 0, k+k_1 > 0, (k+k_1)^2 \le \Delta_3$ *or* $k_1 > 0, k \in \mathbf{R}^*$. *If* $k < k_1$, $m \in \left[\frac{k-k_1}{2\sqrt{-\alpha\beta}}, \frac{k_1-k}{2\sqrt{-\alpha\beta}}\right]$ *or* $k > k_1$, $m \in \left[\frac{k_1-k}{2\sqrt{-\alpha\beta}}, \frac{k-k_1}{2\sqrt{-\alpha\beta}}\right]$, *then* e_3^m *is unstable.*

Proof. The Jacobian matrix of (4.2) at e_3^m is

$$J\left(e_3^m, k, k_1\right) = \begin{pmatrix} k_1 & \alpha m & 0 \\ \beta m & k & 0 \\ -\frac{\gamma b_2}{\alpha} - b_2 & 0 & k_1 \end{pmatrix}$$

Whose characteristic polynomial is

$p_3(\lambda, k, k_1) = -(\lambda - k_1)\left[\lambda^2 - (k+k_1)\lambda + kk_1 - \alpha\beta m^2\right]$. The roots of the equation $p_3(\lambda, k, k_1) = 0$ are $\lambda_1 = k_1$, $\lambda_{2,3} = \frac{(k+k_1) \pm \sqrt{\Delta_3}}{2}$, where

$\Delta_3 = (k-k_1)^2 + 4\alpha\beta m^2$.

1. Case $\alpha\beta < 0$.

(i) ***Case*** $\Delta_3 < 0$ and $q \in (0,1)$. Then $\lambda_1 = k_1$, $\lambda_{2,3} = \frac{(k+k_1) \pm i\sqrt{-\Delta_3}}{2}$. We have $\Delta_3 < 0$ if and only if $m \in \left(-\infty, \frac{k-k_1}{2\sqrt{-\alpha\beta}}\right) \cup \left(\frac{k_1-k}{2\sqrt{-\alpha\beta}}, \infty\right)$ when $k < k_1$ or $m \in \left(-\infty, \frac{k_1-k}{2\sqrt{-\alpha\beta}}\right) \cup \left(\frac{k-k_1}{2\sqrt{-\alpha\beta}}, \infty\right)$ when $k > k_1$.

(1) We suppose $k < k_1 < 0$. In this case we have $\lambda_1 < 0$ and $\mathrm{Re}(\lambda_{2,3}) < 0$. Since $|\arg(\lambda_i)| = \pi > \frac{q\pi}{2}, i = \overline{1,3}$ for all $q \in (0,1)$, by Proposition 4.1(i), it implies that e_3^m is locally asymptotically stable for all

$m \in \left(-\infty, \frac{k-k_1}{2\sqrt{-\alpha\beta}}\right) \cup \left(\frac{k_1-k}{2\sqrt{-\alpha\beta}}, \infty\right)$.

(2) We suppose $k_1 < 0, k > k_1$ and $k + k_1 < 0$. In this case we have $\lambda_1 < 0$ and $\mathrm{Re}(\lambda_{2,3}) < 0$. Applying the same reasoning as in the case (i)(1), one obtains that e_3^m is locally asymptotically stable for all

$m \in \left(-\infty, \frac{k_1-k}{2\sqrt{-\alpha\beta}}\right) \cup \left(\frac{k-k_1}{2\sqrt{-\alpha\beta}}, \infty\right)$.

(3) We suppose $k_1 < 0, k > 0$ and $k + k_1 > 0$. In this case, $\Delta_3 < 0$ if and only

if $m \in \left(-\infty, \frac{k_1 - k}{2\sqrt{-\alpha\beta}}\right) \cup \left(\frac{k - k_1}{2\sqrt{-\alpha\beta}}, \infty\right)$. Then $\lambda_1 < 0$ and $\operatorname{Re}(\lambda_{2,3}) > 0$. Applying Proposition 4.1(i), e_3^m is locally asymptotically stable, for $0 < q < q_3$, where $q_3 = \frac{2}{\pi}\arctan\frac{\sqrt{-\Delta_3}}{k + k_1}$. If $q = q_3$, then e_3^m is stable. For $q_3 < q < 1$, e_3^m is unstable.

(ii) ***Case*** $\Delta_3 \ge 0$ *and* $q \in (0,1)$. Then $\lambda_1 = k_1$, $\lambda_{2,3} = \frac{(k + k_1) \pm \sqrt{\Delta_3}}{2}$. We have $\Delta_3 \ge 0$ if and only if $m \in \left[\frac{k - k_1}{2\sqrt{-\alpha\beta}}, \frac{k_1 - k}{2\sqrt{-\alpha\beta}}\right]$ when $k < k_1$ or $m \in \left[\frac{k_1 - k}{2\sqrt{-\alpha\beta}}, \frac{k - k_1}{2\sqrt{-\alpha\beta}}\right]$ when $k > k_1$.

(1) and (2). In these cases, if $k < 0$ and $k_1 < 0$, then $\lambda_2 + \lambda_3 < 0$ and $\lambda_2\lambda_3 = kk_1 - \alpha\beta m^2 > 0$. It follows $\lambda_i < 0, i = \overline{1,3}$. Then, e_3^m is locally asymptotically stable.

(3) Let $k_1 < 0, k > 0$ and $k + k_1 > 0$ or $k_1 > 0$ and $k \in \mathbf{R}^*$. Then $J(e_3^m, k, k_1)$ has at least a positive eigenvalue and so e_3^m is unstable. Hence, the assertion (ii) holds.

2. Case $\alpha\beta > 0$, $m \in \mathbf{R}^*$ *and* $q \in (0,1)$. In this case $\Delta_3 \ge 0$.

(i) We suppose $k_1 < 0$. We have $\lambda_2 < 0$ and $\lambda_3 < 0$ if and only if $\lambda_2 + \lambda_3 < 0$ and $\lambda_2\lambda_3 > 0$. Then $k + k_1 < 0$ and $(k + k_1)^2 - \Delta_3 > 0$. It follows $\lambda_i < 0, i = \overline{1,3}$ for all $m \in \mathbf{R}^*$ such that $(k + k_1)^2 > \Delta_3$. Hence, e_3^m is locally asymptotically stable.

(ii) We suppose $k_1 < 0, k + k_1 > 0$ or $k_1 > 0, k \in \mathbf{R}^*$. Then, $J(e_3^m, k, k_1)$ has at least a positive eigenvalue and so e_3^m is unstable. Therefore, the assertion (ii) holds.

Example 4.1. (*i*) Let be the special fractional Stokes pulse system (4.2). We select $l_1 = 1, l_2 = 0.5, l_3 = 1.5$ and $b_2 = 1$. Then $\alpha = -1, \beta = \gamma = 0.5$. We have $\frac{\alpha + \gamma}{\alpha\gamma} = 1$ and $\Delta_1 = (-m + 1)(0.5m - 1)$.

(i) Chosing $k = -0.15, k_1 = -0.4$ and $m = 1.6$, it follows that $\Delta_1 = 0.12$ and $k_1^2 = 0.16 > \Delta_1$. According to Proposition 4.4, **2.**(i)(1) it follows that the equilibrium state $e_1 = (0, 1.6, 0)$ is asymptotically stable for $q = 0.8$.

(ii) For $k = -1, k_1 = -0.3$ and $m = 1.6$, follows t $\Delta_1 = 0.12$ and $k_1^2 = 0.09 < \Delta_1$.

The conditions of Proposition 4.4, **2.**(i)(2) are achieved. Then $e_1 = (0, 1.6, 0)$ is unstable for $q = 0.75$.

Using Matlab, in **Table 1** we give a set of values for the parameters $l_i, b_2, k, k_1, i = \overline{1,3}$, the equilibrium states and corresponding eigenvalues of special fractional Stokes pulse system (4.2).

Table 1. The controls k, k_1, equilibrium states e_i^m and corresponding eigenvavues.

$l_i, b_2, k, k_1, i=\overline{1,3}$	$\lambda_i, i=\overline{1,3}$	m, q	e_i^m	*Stability*
$l_1=1$, $l_2=0.5$, $l_3=-0.5$ $b_2=1$, $k=-3$, $k_1=-0.32$	-3, $-0.32 \pm \mathrm{i}$	$m=0$, $q=0.45$	$e_0=(0,0,0)$	*asym. stable*
$l_1=1$, $l_2=0.5$, $l_3=-0.5$ $b_2=1$, $k=-3$, $k_1=1.73$	-3, $1.73 \pm \mathrm{i}$	$m=0$, $q_0=0.33$, $q=0.3$	$e_0=(0,0,0)$	*asym. stable*
$l_1=1$, $l_2=0.5$, $l_3=-0.5$ $b_2=-1$, $k=-3$, $k_1=1.73$	-3, $1.73 \pm \mathrm{i}$	$m=0$, $q_0=0.33$, $q=0.7$	$e_0=(0,0,0)$	*unstable*
$l_1=1$, $l_2=0$, $l_3=0.5$ $b_2=1$, $k=-2$, $k_1=-0.8$	-2, $-0.8 \pm 0.5291\mathrm{i}$	$m=0.6$, $q=0.4$	$e_1=(0,0.6,0)$	*asym. stable*
$l_1=1$, $l_2=0.5$, $l_3=1.5$ $b_2=1$, $k=-0.15$, $k_1=-0.4$	-0.15, $-0.4 \pm 0.3464\mathrm{i}$	$m=1.6$, $q=0.8$	$e_1=(0,1.6,0)$	*asym. stable*
$l_1=1$, $l_2=0.5$, $l_3=1.5$ $b_2=1$, $k=-1$, $k_1=-0.3$	-1, $-0.3 \pm 0.3464\mathrm{i}$	$m=1.6$, $q=0.8$	$e_1=(0,1.6,0)$	*unstable*
$l_1=1$, $l_2=0.2$, $l_3=-0.25$ $b_2=-1$, $k=-0.35$, $k_1=-0.1$	-0.1, $-0.225 \pm 0.1561\mathrm{i}$	$m=0.2$, $q=0.6$	$e_2=(0.2,-1.25,0)$	*asym. stable*
$l_1=1$, $l_2=0.2$, $l_3=-0.25$ $b_2=-1$, $k=3$, $k_1=-1$	-1, $1 \pm 1.0307\mathrm{i}$	$m=2.25$, $q_2=0.5$, $q=0.45$	$e_2=(2.25,-1.25,0)$	*asym. stable*
$l_1=1$, $l_2=0.2$, $l_3=-0.25$ $b_2=-1$, $k=3$, $k_1=-1$	-1, $1 \pm 1.0307\mathrm{i}$	$m=2.25$ $q_2=0.5$, $q=0.75$	$e_2=(2.25,-1.25,0)$	*unstable*

5. Numerical Integration of the Special Fractional Stokes Pulse System (4.2)

In this section we start with some mathematical preliminaries of the fractional Euler's method for solving initial value problem for fractional differential equations.

Consider the following general form of the initial value problem (IVP) with Caputo derivative:

$$D_t^q y(t) = f(t, y(t)), \ \ y(0) = y_0, \ \ t \in I = [0, T], \ T > 0, \tag{5.1}$$

where $y: I \to \mathbf{R}^n, f: \mathbf{R}^n \to \mathbf{R}^n$ is a continuous nonlinear function and $q \in (0,1)$, represents the order of the derivative.

The right-hand side of the IVP (5.1) in considered examples is Lipschitz functions and the numerical method used in this works to integrate system (5.1) is the *Fractional Euler's method*.

Since f is assumed to be continuous function, every solution of the initial value problem given by (5.1) is also a solution of the following *Volterra fractional integral equation*:

$$y(t) = y(0) + I_t^q f(t, y(t)), \tag{5.2}$$

where I_t^q is the q-order Riemann-Liouville integral operator, which is expressed by:

$$I_t^q f(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, y(s)) \mathrm{d}s, \quad q > 0, \quad s \in [0, T]. \tag{5.3}$$

Moreover, every solution of (5.2) is a solution of the (IVP) (5.1).

To integrate the fractional Equation (5.1), means to find the solution of (5.2) over the interval $[0, T]$. In this context, a set of points $(t_j, y(t_j))$ are produced which are used as approximated values. In order to achieve this approximation, the interval $[0, T]$ is partitioned into n subintervals $[t_j, t_{j+1}]$ each equal width $h = \frac{T}{n}, t_j = jh$ for $j = 0, 1, \cdots, n$.

For the fractional-order q and $j = 0, 1, 2, \cdots$, it computes an approximation denoted as y_{j+1} for $y(t_{j+1}), j = 0, 1, 2, \cdots$.

The general formula of the fractional Euler's method for to compute the elements y_j, is:

$$y_{j+1} = y_j + \frac{h^q}{\Gamma(q+1)} f(t_j, y(t_j)), \quad t_{j+1} = t_j + h, \quad j = 0, 1, \cdots, n. \tag{5.4}$$

For more details, see [24] [25].

For the numerical integration of the special fractional Stokes pulse system (4.2), we apply the fractional Euler method (FEM). For this, consider the following fractional differential equations:

$$\begin{cases} D_t^q u^i(t) = F_i(u^1(t), u^2(t), u^3(t)), \quad i = \overline{1,3}, \quad t \in (t_0, \tau], \quad q \in (0,1) \\ u(t_0) = (u^1(t_0), u^2(t_0), u^3(t_0)), \end{cases} \tag{5.5}$$

where

$$\begin{cases} F_1(u(t)) = (l_2 - l_3) u^2(t) u^3(t) + b_2 u^3(t) + k_1 u^1(t) \\ F_2(u(t)) = (l_3 - l_1) u^3(t) u^1(t) + k u^2(t), \\ F_3(u(t)) = (l_1 - l_2) u^1(t) u^2(t) - b_2 u^1(t) + k_1 u^3(t) \end{cases} \quad q \in (0,1) \tag{5.6}$$

where $b_2, k, k_1 \in \mathbf{R}^*$ and $l_i \in \mathbf{R}, i = \overline{1,3}$ such that $l_1 \neq l_2 \neq l_3 \neq l_1$.

Since the functions $F_i(u(t)), i = \overline{1,3}$ are continuous, then the initial value problem (5.5) is equivalent to system of Volterra integral equations, which is given as follows:

$$u^i(t) = u^i(0) + I_t^q F_i(u^1(t), u^2(t), u^3(t)), \quad i = \overline{1,3}. \tag{5.7}$$

The system (5.7) is called the *Volterra integral equations associated to special Stokes pulse system* (4.2).

The problem for solving the system (5.5) is reduced to one of solving a sequence of systems of fractional equations in increasing dimension on successive intervals $[j, (j+1)]$.

For the numerical integration of the system (5.6) one can use the fractional Euler method (the formula (5.4), which is expressed as follows:

$$u^i(j+1) = u^i(j) + \frac{h^q}{\Gamma(q+1)} F_i(u^1(j), u^2(j), u^3(j)), \quad i = \overline{1,3} \tag{5.8}$$

where $j = 0,1,\cdots,N$, $h = \frac{T}{N}$, $T > 0$, $N > 0$.

More precisely, the numerical integration of the fractional system (5.5) is given by:

$$\begin{cases} u^1(j+1) = u^1(j) + h^q \dfrac{1}{\Gamma(q+1)}\left(\alpha u^2(j)u^3(j) + b_2 u^3(j) + k_1 u^1(j)\right) \\ u^2(j+1) = u^2(j) + h^q \dfrac{1}{\Gamma(q+1)}\left(\beta u^3(j)u^1(j) + k u^2(j)\right), \qquad q \in (0,1) \\ u^3(j+1) = u^3(j) + h^q \dfrac{1}{\Gamma(q+1)}\left(\gamma u^1(j)u^2(j) - b_2 u^1(j) + k_1 u^3(j)\right) \\ u^i(0) = u_e^i + \varepsilon, \quad i = \overline{1,3}, \end{cases} \tag{5.9}$$

where $\alpha := l_2 - l_3$, $\beta := l_3 - l_1$, $\gamma := l_1 - l_2$.

Using [21] [24], we have that the numerical algorithm given by (5.9) is convergent.

Example 5.1. Let us we present the numerical integration of the special fractional Stokes pulse system with controls which has considered in Example 4.1(i). For this we apply the algorithm (5.9) and software Maple. Then, in (5.9) we take: $l_1 = 1$, $l_2 = 0.5$, $l_3 = 1.5$, $b_2 = 1$, $k = -0.15$, and $k_1 = -0.4$. It is known that the equilibrium state $e_1 = (0,1.6,0)$ is asymptotically stable.

For the numerical simulation of solutions of the above fractional model we use the rutine Maple. *spec-fract-Stokes-pulse-system-with-controls*, denoted by [sp-fr.Stokes-pulse syst]. Applying this program for $h = 0.01$, $\varepsilon = 0.01$, $u^1(0) = \varepsilon$, $u^2(0) = 1.6 + \varepsilon$, $u^3(0) = \varepsilon$, $N = 100$, $t = 102$, one obtain the orbits $(n, u^1(n))$, $(n, u^2(n))$ $(n, u^3(n))$ and $(u^1(n), u^2(n), u^3(n))$, for $q = 0.8$.

Finally, we present the rutine [sp-fr.Stokes pulse syst]:

```
#   Fractional equations associated to Stokes pulse system for q = 0.8
Du1/dt=(l2-l3)*u2*u3 + b2*u3 + k1* u1;
Du2/dt=(l3-l1)*u1*u3 + k* u2;
Du3/dt=(l1-l2)*u1*u2 - b2*u1 + k1* u3;
> with (plots):
> l1:=1.; l2:=0.5; l3:=1.5; alpha:=l2-l3; beta:=l3-l1; gamma:=l1-l2; b2:=1.;
k:=-0.15; k1:= -0.4; q:=0.8; u1e:=0.; u2e:=1.6; u3e:=0.;
> with (stats):
> h:=0.01; epsilon:=0.01; n:=100:t:=n+2; u1:= array (0 .. n): u2:= array (0 .. n):
u3:= array (0 .. n): u1[0]:=epsilon + u1e; u2[0]:=epsilon + u2e; u3[0]:=epsilon +
u3e;
> for j from 1 by 1 to n do
> u1[j]:= u1[j-1] + h ∧q *(alpha* u2[j-1]*u3[j-1] + b2*u3[j-1] + k1*
u1[j-1])/GAMMA(q+1);
u2[j]:= u2[j-1] + h ∧q *(beta* u1[j-1]*u3[j-1] + k* u2[j-1])/GAMMA(q+1);
u3[j]:= u3[j-1] + h ∧q *(gamma* u1[j-1]*u2[j-1] - b2*u1[j-1] + k1*
u3[j-1])/GAMMA(q+1);
```

od:
> plot (seq([j,u1[j]], j = 0 .. n), style = point, symbol = point, scaling = UNCONSTRAINED);
plot (seq([j,u2[j]], j = 0 .. n), style = point, symbol = point, scaling = UNCONSTRAINED);
plot (seq([j,u3[j]], j = 0 .. n), style = point, symbol = point, scaling = UNCONSTRAINED);
pointplot 3d ({seq([u1[j], [u2[j], [u3[j]], j = 0 .. n)}, style = point, symbol = point, scaling =
UNCONSTRAINED, color = red);

Remark 5.1. Appyling (5.9) and Maple for the numerical simulation of solutions of fractional model (4.2) for each set of values for parameters l_1, l_2, l_3, b_2, k and k_1, given in the **Table 1**, it will be found that the results obtained are valid.

Conclusions. This paper presents the fractional Stokes pulse system (3.1) associated to system (2.6). The special fractional Stokes pulse system (3.3) was studied from fractional differential equations theory point of view: asymptotic stability, determining of sufficient conditions on parameters k, k_1 to control the chaos in the proposed fractional system and numerical integration of the fractional model (4.2). By choosing the right parameters k and k_1 in the fractional model (4.2), this work offers a series of chaotic fractional differential systems. The other types of systems mentioned in the four types of fractional models will be studied in future works.

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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