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The Right Triangle as the Simplex in 2D Euclidean Space, Generalized to n Dimensions

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Abstract

The purpose of the research is to show that the general triangle can be replaced by the right-angled triangle as the 2D simplex, and this concept can be generalized to any higher dimensions. The main results are that such forms do exist in any dimensions; meet the requirements usually placed on an n-dimensional simplex; a hypotenuse and legs can be defined in these shapes; and a formula can be given to calculate the volume of the shape solely from the legs by a direct generalization of the Pythagorean Theorem, without computing the Cayley-Menger determinant.

Keywords

Cycles of Incidence, Quadrirectangular Tetrahedron, Rectangular Pentachoron, Generalization of Pythagoras Theorem, Volume of a Rectangular Simplex, Cayley-Menger Determinant

1. Introduction

Various generalizations of the Pythagoras Theorem have been known for centuries. Among these, the construction most relevant to this article investigates a trirectangular tetrahedron with three faces of right triangles. The three right angles of the triangles meet at one vertex of the tetrahedron (De Gua [1]). This extension of the Pythagoras Theorem can also be implemented in higher dimensions, but with results completely different from those described in this article.

Until recently, the subject of n-dimensional geometries was very alien for me. I could not find a perceptible model of the abstract concepts. It reminded me of my previous experience with hyperbolic geometry which I could only understand through the hemispherical Poincaré model.

The first idea that led me towards the topic came from my work with spherical geometry (Lénárt [2]). A polygon can be defined as a cycle of incidence of alter-

nating points and segments with an even number of elements, vertices and sides together. However, in spherical geometry a point on a great circle and two perpendicular great circles can both be viewed as special cases of two incident elements. It follows that a closed cycle of incidence may consist of an odd number of elements.

This conception can be extended to hyperbolic and Euclidean geometry for segments of perpendicular straight lines.

Figure 1 shows a spherical Napier pentagram, Figure 2, a hyperbolic rectangular Napier pentagon, both 5-cycles. (All pentagons on Figure 2 are regular hyperbolic Napier pentagons.) The adjacent perpendiculars represent incident elements, while their point of intersection is omitted from the cycle.

It follows that spherical or hyperbolic geometry allows 5-cycles to be simplices instead of 6-cycles of general triangles with alternating vertices and sides.

The next step is to apply the same thought to Euclidean plane geometry. The Euclidean 5-cycle is a right triangle with two vertices and three sides, including the two perpendiculars. But can we proceed to higher dimensions to find shapes

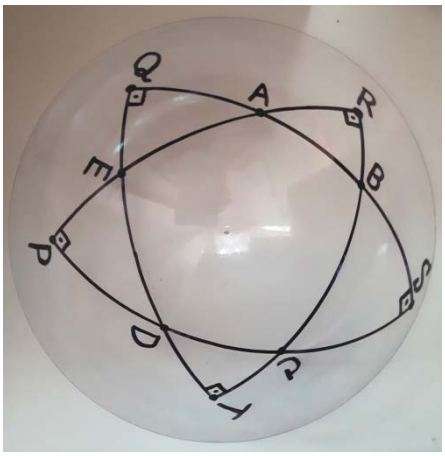
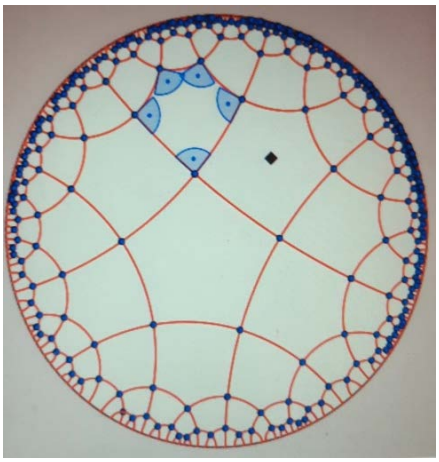


Figure 1. Spherical Napier pentagram, a 5-cycle.



Source: Lajos Szilassi, personal communication.

Figure 2. Hyperbolic Napier pentagon, a 5-cycle.

whose faces are all right triangles?

This finding inspired me to explore the shape in 3D Euclidean space and move to higher dimensions.

2. Simplices in *n*-Dimensional Euclidean Spaces

The initial idea is based on an axiom of spherical geometry: If two equators are perpendicular, the pole point of one is on the equator of the other, and vice versa. It follows that the sentences "Two great circles are perpendicular" and "A point and a great circle coincide" are equivalent, interchangeable statements about points and great circles in the same construction. Two perpendicular straight lines represent a special case of the incidence relation.

Given a system of geometry in which we define the incidence of a point and a straight line, and the perpendicular property between two straight lines. Consider both cases as special cases of incidence. A cycle of incidence is an ordered series of elements in which any two adjacent elements are incident, including the last and the first.

Any polygon represents a cycle of incidence. A triangle is a 6-cycle, a quadrilateral is an 8-cycle, and so on. However, perpendicular straight lines as incident elements allow for cycles with an odd number of elements. For example, a triangle with one right angle is a 5-cycle in Euclidean, spherical or hyperbolic geometry. The vertex at the intersection of the two legs is not counted, and the cycle consists of five elements, namely, two vertices and three sides, including the two perpendiculars.

In this sense, the right triangle is not a special case of the general triangle. On the contrary, the right triangle, the 5-cycle is the simplex, and the general triangle, the 6-cycle is a composite shape derived from the right triangle.

The main subject of this article is the Euclidean case of the 5-cycle as the simplex, which seems to be the least suitable for the purpose. Regular 5-cycles are excluded here, since regular right triangles do not exist in Euclidean geometry, in contrast with the regular spherical Napier pentagram or the regular hyperbolic Napier pentagon.

The task is as follows: We are looking for a new type of simplex in two-, three-, ... n-dimensional Euclidean geometry. Each face is a right-angled triangle, and the n-dimensional shape consists of $(n+1)$ number of $(n-1)$-dimensional shapes.

3. The Simplex in Two Dimensions, $n = 2$

The measure of the angles and sides are all correct on the picture (**Figure 3**).

Remark. Right angles in the present paper are indicated as arcs between two perpendicular sides, regardless of their apparent length. (The usual notations of a right angle proved confusing for the drawings.)

4. The Simplex in Three Dimensions, $n = 3$

While the 2D configuration is easy to construct, the 3D shape is by no means

trivial. We are looking for a tetrahedron of which all four faces are right triangles. This shape does exist and can be constructed in the Euclidean 3D space by cutting a rectangular cuboid of dimensions *a*, *b*, *c* along the plane of a space diagonal and a face diagonal (Figure 4 and Figure 5).

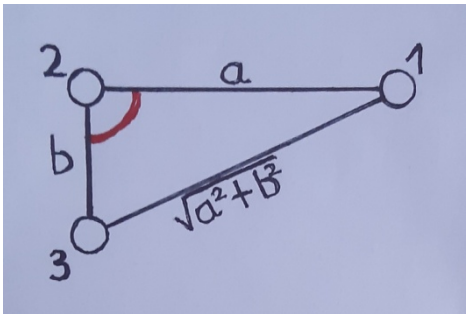


Figure 3. The Euclidean right triangle with two freely given data as the length of the two legs, and the hypotenuse calculated by the Pythagoras Theorem.

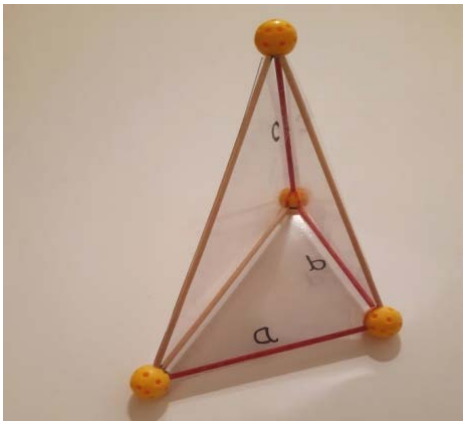


Figure 4. The initial position of the 3D model of the quadrirectangular tetrahedron. Given three non-coplanar and non-concurrent sides, *a* perpendicular to *b*, *b* to *c*, *c* to 2D subspace *ab*.

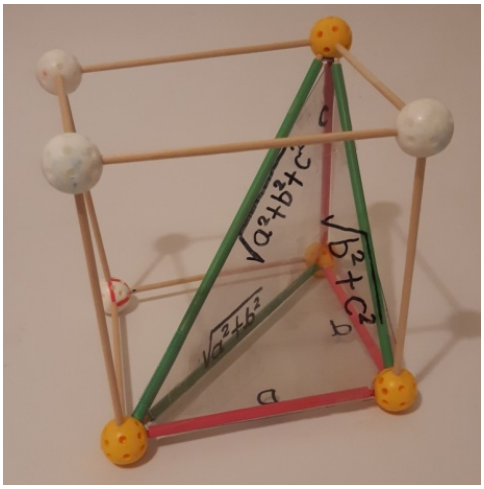


Figure 5. The completed quadrirectangular tetrahedron with six sides and four faces of right triangles, cut out of a rectangular cuboid of dimensions *a*, *b*, *c*.

In the Euclidean 3D space, the shape is a tetrahedron with six sides, (a), (b), (c), ($\sqrt{a^2+b^2}$), ($\sqrt{b^2+c^2}$), ($\sqrt{a^2+b^2+c^2}$), and four faces of right triangles, ($a,b,\sqrt{a^2+b^2}$), ($b,c,\sqrt{b^2+c^2}$), ($a,\sqrt{b^2+c^2},\sqrt{a^2+b^2+c^2}$), ($c,\sqrt{a^2+b^2},\sqrt{a^2+b^2+c^2}$).

Figure 6 shows the configuration on a flat diagram:

Starting from vertices 1, 2, 3, 4 with sides $a=12, b=23, c=34$ given, side 41 can be determined in two ways: either through triangles 123 and 341, - or triangles 234 and 412.

In triangle 123 we have legs a, b, and hypotenuse $\sqrt{a^2+b^2}$; in triangle 341, legs $\sqrt{a^2+b^2}$, c, and hypotenuse $\sqrt{a^2+b^2+c^2}$.

In triangle 234 we have legs b, c, and hypotenuse $\sqrt{b^2+c^2}$; in triangle 412, legs $\sqrt{b^2+c^2}$, a, and hypotenuse $\sqrt{a^2+b^2+c^2}$.

Both ways yield the same result $\sqrt{a^2+b^2+c^2}$.

This 3D shape cannot be realized on a 2D flat surface.

5. Displaying Right-Angled *n*-Dimensional Shapes on Regular Planar Polygons

In order to proceed to higher dimensions, it is of advantage to turn to regular Petrie polygons (cf. Coxeter [4]). The idea is to display right-angled shapes in distorted form. The vertices, faces and edges of the shape are represented by the sides, diagonals and angles of regular polygons on flat surface. Visualization is more difficult in 2D and 3D cases, but generalization is easier in higher dimensions.

$n = 2$

The right-angled 2D triangle is displayed on the sides and angles of a regular 2D triangle with sides and angles distorted (Figure 7, cf., Figure 3). This shape has 3 vertices, 1 face of a right-angled Euclidean triangle, and 3 sides. It can be realized in 2D Euclidean plane.

$n = 3$

The quadrirectangular tetrahedron is displayed on a regular 2D square with sides, diagonals and angles distorted (Figure 8, cf., Figure 6). This shape has 4 vertices, 4 faces of right-angled Euclidean triangles, and 6 edges. It can be

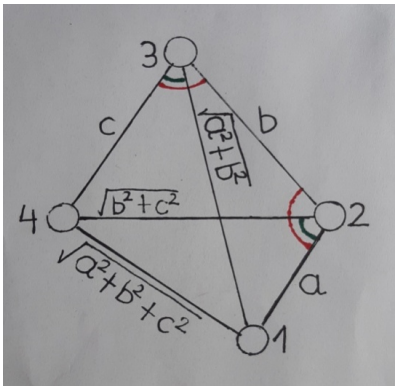


Figure 6. The quadrirectangular tetrahedron on a flat diagram.

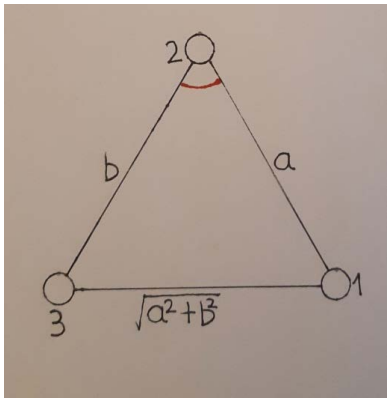


Figure 7. Right triangle displayed on a regular 2D triangle with sides and angles distorted.

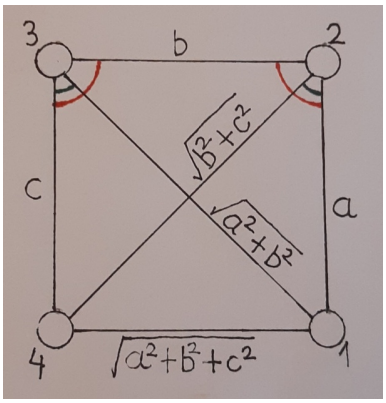


Figure 8. Quadrirectangular tetrahedron displayed on a regular 2D square with distorted sides, diagonals and angles distorted.

realized in 3D Euclidean space, but not on 2D Euclidean plane.

6. The 4D Right-Angled Pentachoron

$n = 4$

This shape has 5 vertices, 10 faces of Euclidean triangles and 10 edges, just as with the general pentachoron. Each face is a right triangle, so the entire shape has a total of 10 right angles. It requires four dimensions to construct, and cannot be realized in 3D Euclidean space.

Enter four independent data a, b, c, d for which the segment of length a is perpendicular to b, b to c, c to d. Any four vertices determine a quadrirectangular tetrahedron. Figure 9 shows the initial position, while Figure 10 shows the completed construction.

Any four vertices determine a right-angled tetrahedron. Table 1 shows the defining equations of the sides of the four right triangles for each tetrahedron, as illustrated on the completed pentachoron on Figure 10.

7. Generalizing to n Dimensions. Hypotenuse and Legs

Theorem 1:

Any n-dimensional right-angled simplex can be decomposed into $(n-1)$

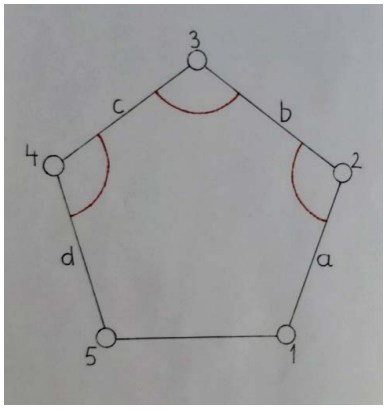


Figure 9. Initial configuration of constructing a right pentachoron.

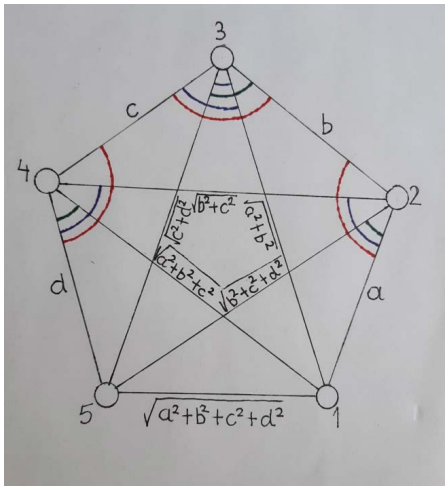


Figure 10. The completed construction of a right pentachoron with 10 right angles.

Table 1. Defining equations of the four right triangles for the five tetrahedrons of the pentachoron.

1234	
123	$\left(a^2+b^2\right)=\left(a^2\right)+\left(b^2\right)$
234	$\left(b^2+c^2\right)=\left(b^2\right)+\left(c^2\right)$
341	$\left(a^2+b^2+c^2\right)=\left(a^2+b^2\right)+\left(c^2\right)$
412	$\left(a^2+b^2+c^2\right)=\left(a^2\right)+\left(b^2+c^2\right)$
2345	
234	$\left(b^2+c^2\right)=\left(b^2\right)+\left(c^2\right)$
345	$\left(c^2+d^2\right)=\left(c^2\right)+\left(d^2\right)$
452	$\left(b^2+c^2+d^2\right)=\left(b^2+c^2\right)+\left(d^2\right)$
523	$\left(b^2+c^2+d^2\right)=\left(b^2\right)+\left(c^2+d^2\right)$

3451	
345	$\left(c^2+d^2\right)=\left(c^2\right)+\left(d^2\right)$
451	$\left(a^2+b^2+c^2+d^2\right)=\left(a^2+b^2+c^2\right)+\left(d^2\right)$
513	$\left(a^2+b^2+c^2+d^2\right)=\left(a^2+b^2\right)+\left(c^2+d^2\right)$
134	$\left(a^2+b^2+c^2\right)=\left(a^2+b^2\right)+\left(c^2\right)$
4512	
451	$\left(a^2+b^2+c^2+d^2\right)=\left(a^2+b^2+c^2\right)+\left(d^2\right)$
512	$\left(a^2+b^2+c^2+d^2\right)=\left(b^2+c^2+d^2\right)+\left(a^2\right)$
124	$\left(a^2+b^2+c^2\right)=\left(b^2+c^2\right)+\left(a^2\right)$
245	$\left(b^2+c^2+d^2\right)=\left(b^2+c^2\right)+\left(d^2\right)$
5123	
512	$\left(a^2+b^2+c^2+d^2\right)=\left(b^2+c^2+d^2\right)+\left(a^2\right)$
123	$\left(a^2+b^2\right)=\left(a^2\right)+\left(b^2\right)$
235	$\left(b^2+c^2+d^2\right)=\left(b^2\right)+\left(c^2+d^2\right)$
351	$\left(a^2+b^2+c^2+d^2\right)=\left(a^2+b^2\right)+\left(c^2+d^2\right)$

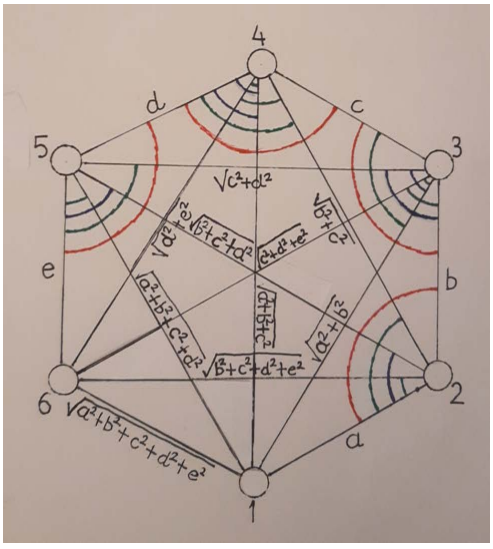


Figure 11. 5D Rectangular simplex (hexachoron).

dimensional simplices.

The proof is based on induction: Suppose that the statement is valid for all (n − 1) dimensional right-angled simplices. The detailed proof for arbitrary n is very clumsy to describe, so I illustrate the pattern for the n = 6 case (**Figure 11**).

Table 2 shows the defining equations of the sides of the five right triangles for each pentachoron, as illustrated on the completed hexachoron on Figure 12.

Definition. The element $d_{6,1} =$ can be called the *hypotenuse*, and the other sides $d_{12} = a, d_{23} = b, \cdots, d_{56} = e$ the *legs* of the *n*-dimensional right-angled simplex. By the same logic, the right-angled simplices can be called *Pythagorean shapes* in *n*-dimensional space (*Pythagorean triangle, Pythagorean tetrahedron, Pythagorean pentachoron,* etc.).

On Figure 12, side $\sqrt{a^2+b^2+c^2+d^2+e^2}$ is the 5D hypotenuse, *a*, *b*, *c*, *d*, *e* are the legs. The hypotenuses of the building pentachorons are listed in Table 3.

The same pattern can be applied to arbitrary dimension, with $d_{12}^2, d_{23}^2, \cdots, d_{n-1,n}^2, d_{n,n+1}^2, d_{n+1,1}^2$ as distances of sides between the adjacent vertices, and $d_{n+1,1}^2 = d_{12}^2 + d_{23}^2 + \cdots + d_{n-1,n}^2 + d_{n,n+1}^2$.

8. Volume of a Right-Angled Simplex in 3D Euclidean Space

Apply the Cayley-Menger determinant to determine the volume by the length of sides for the 3D tetrahedron:

Table 2. Defining equations of the five right triangles for the six pentachorons of the hexachoron.

Pentachoron	Sides
12345	$a,b,c,d,\sqrt{a^2+b^2+c^2+d^2}$
23456	$b,c,d,e,\sqrt{b^2+c^2+d^2+e^2}$
34561	$c,d,e,\sqrt{a^2+b^2+c^2+d^2+e^2},\sqrt{a^2+b^2}$
45612	$\sqrt{a^2+b^2+c^2+d^2+e^2},a,\sqrt{b^2+c^2}$
56123	$\sqrt{a^2+b^2+c^2+d^2+e^2},a,b,\sqrt{c^2+d^2}$
61234	$\sqrt{a^2+b^2+c^2+d^2+e^2},a,b,c,\sqrt{d^2+e^2}$

Table 3. Hypotenuses in the six pentachorons of the hexachoron.

Pentachoron	Hypotenuse
12345	$\sqrt{a^2+b^2+c^2+d^2}$
23456	$\sqrt{b^2+c^2+d^2+e^2}$
34561	$\sqrt{a^2+b^2+c^2+d^2+e^2}$
45612	$\sqrt{a^2+b^2+c^2+d^2+e^2}$
56123	$\sqrt{a^2+b^2+c^2+d^2+e^2}$
61234	$\sqrt{a^2+b^2+c^2+d^2+e^2}$

$$\left|D_{3,CM}\right| = \frac{2^3(3!)^2}{(-1)^{3+1}}V^2 = \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & a^2 & a^2+b^2 & a^2+b^2+c^2 \\ 1 & a^2 & 0 & b^2 & b^2+c^2 \\ 1 & a^2+b^2 & b^2 & 0 & c^2 \\ 1 & a^2+b^2+c^2 & b^2+c^2 & c^2 & 0 \end{vmatrix}$$

$$= -\begin{vmatrix} 1 & a^2 & a^2+b^2 & a^2+b^2+c^2 \\ 1 & 0 & b^2 & b^2+c^2 \\ 1 & b^2 & 0 & c^2 \\ 1 & b^2+c^2 & c^2 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 0 & a^2+b^2 & a^2+b^2+c^2 \\ 1 & a^2 & b^2 & b^2+c^2 \\ 1 & a^2+b^2 & 0 & c^2 \\ 1 & a^2+b^2+c^2 & c^2 & 0 \end{vmatrix} \quad (1)$$

$$- \begin{vmatrix} 1 & 0 & a^2 & a^2+b^2+c^2 \\ 1 & a^2 & 0 & b^2+c^2 \\ 1 & a^2+b^2 & b^2 & c^2 \\ 1 & a^2+b^2+c^2 & b^2+c^2 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 0 & a^2 & a^2+b^2 \\ 1 & a^2 & 0 & b^2 \\ 1 & a^2+b^2 & b^2 & 0 \\ 1 & a^2+b^2+c^2 & b^2+c^2 & c^2 \end{vmatrix}$$

$$= 4\times a^2\times b^2c^2 + 4\times a^2\times b^2c^2 + 0 + 0 = 2^3\times a^2\times b^2\times c^2$$

Remark. The proof can be checked by a determinant calculator with arbitrary input data.

For example, $n=3, a=4, b=3, c=2$:

Calculate the volume with the Cayley-Menger determinant:

$$\frac{2^3(3!)^2}{(-1)^{3+1}}V^2 = \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 16 & 25 & 29 \\ 1 & 16 & 0 & 9 & 13 \\ 1 & 25 & 9 & 0 & 4 \\ 1 & 29 & 13 & 4 & 0 \end{vmatrix}$$

$$= -\begin{vmatrix} 1 & 16 & 25 & 29 \\ 1 & 0 & 9 & 13 \\ 1 & 9 & 0 & 4 \\ 1 & 13 & 4 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 0 & 25 & 29 \\ 1 & 16 & 9 & 13 \\ 1 & 25 & 0 & 4 \\ 1 & 29 & 4 & 0 \end{vmatrix} - \begin{vmatrix} 1 & 0 & 16 & 29 \\ 1 & 16 & 0 & 13 \\ 1 & 25 & 9 & 4 \\ 1 & 29 & 13 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 0 & 16 & 25 \\ 1 & 16 & 0 & 9 \\ 1 & 25 & 9 & 0 \\ 1 & 29 & 13 & 4 \end{vmatrix} \quad (2)$$

$$= 2304 + 2304 + 0 + 0 = 4608 = 2^3\times(3!)^2\times V^2 = 288\times V^2$$

$$= 288\times 16, V^2 = 16, V = 4.$$

Now calculate the same volume with the generalized Pythagoras Theorem:

$$2^3\times a^2\times b^2\times c^2 = 2^3\times 4^2\times 3^2\times 2^2 = 4608 = 2^3\times(3!)^2\times V^2$$
$$= 288\times V^2 = 288\times 16, V^2 = 16, V = 4. \quad (3)$$

9. The Generalized Pythagoras Theorem in *n*-Dimensional Euclidean Spaces

Theorem 2:

Given a right-angled simplex with $n+1$ vertices of the defining Petrie polygon in an *n*-dimensional Euclidean space. Denote $d_{i,i+1}$ the distance between adjacent vertices *i* and *i*+1, and $d_{n+1,1}$ between the last and the first. Assume that $d_{n+1,1}^2 = d_{12}^2 + d_{23}^2 + \cdots + d_{n-1,n}^2 + d_{n,n+1}^2$. Define $d_{n+1,1}$ as the hypotenuse of the simplex, while the other sides are the legs (Figure 12). The volume of the simplex

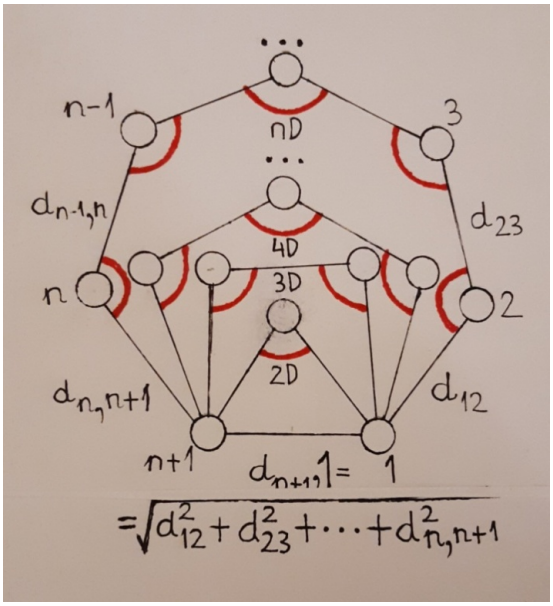


Figure 12. The initial configuration of a rectangular simplex in 2, 3, 4, ... , *n*, ... dimensions. Arcs always indicate right angles.

can be calculated by direct generalization of the Pythagoras Theorem, multiplying the squares of the legs, and dividing the product by $(n!)^2$:

$$\left|D_{n,CM}\right| = 2^n \times d_{12}^2 \times d_{23}^2 \times \cdots \times d_{n-1,n}^2 \times d_{n,n+1}^2 = 2^n \times (n!)^2 \times V^2$$

Proof:

The general case is rather clumsy to describe, so I give two numerical examples instead which can readily be extended to any *n*.

Example 1:

$n = 2, d_{12} = 5, d_{23} = 4, d_{31} = 3$ (the right triangle on the plane with the Pythagorean triplet 3, 4, 5).

The volume calculated by the Cayley-Menger determinant:

$$\left|D_{n,CM}\right| = \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 25 & 9 \\ 1 & 25 & 0 & 16 \\ 1 & 9 & 16 & 0 \end{vmatrix} = -576$$

$$= \frac{2^2 (2!)^2}{(-1)^{2+1}} V^2 = -16V^2 = -16 \times 36, V = \sqrt{36} = 6. \tag{3}$$

The volume calculated by the generalized Pythagoras Theorem:

$$V^2 = \frac{1}{(2!)^2} \times 16 \times 9 = 36, V = \sqrt{36} = 6.$$

Example 2:

$$n = 5, a = 8, b = 5, c = 4, d = 3, e = 2$$

The volume calculated by the Cayley-Menger determinant:

$$\left|D_{5,CM}\right| = \begin{vmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 64 & 89 & 105 & 114 & 118 \\ 1 & 64 & 0 & 25 & 41 & 50 & 54 \\ 1 & 89 & 25 & 0 & 16 & 25 & 29 \\ 1 & 105 & 41 & 16 & 0 & 9 & 13 \\ 1 & 114 & 50 & 25 & 9 & 0 & 4 \\ 1 & 118 & 54 & 29 & 13 & 4 & 0 \end{vmatrix} = 29491200$$

$$= \frac{2^5 (5!)^2}{(-1)^{5+1}} V^2 = 32 \times 120^2 \times V^2 = 460800 \times 64, V = 8.$$

The volume calculated by the generalized Pythagoras Theorem:

$$V^2 = \frac{1}{(5!)^2} \times (64 \times 25 \times 16 \times 9 \times 4) = \frac{1}{(5!)^2} 921600 = 64, V = 8.$$

10. Decomposing a General Tetrahedron into Pythagorean Tetrahedrons

In 2D Euclidean space, on a flat surface a triangle can be decomposed into two Pythagorean triangles by an altitude of the triangle (Figure 13).

In the 3D Euclidean space, a similar method gives six right-angled Pythagorean tetrahedra (Figure 14).

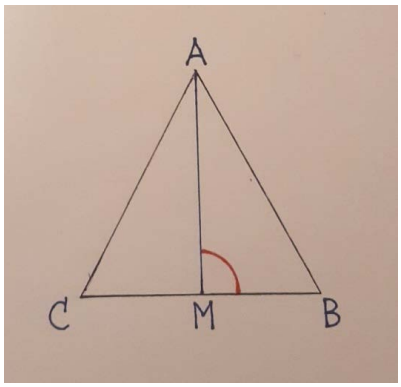


Figure 13. Decomposing a 2D triangle into two right triangles.

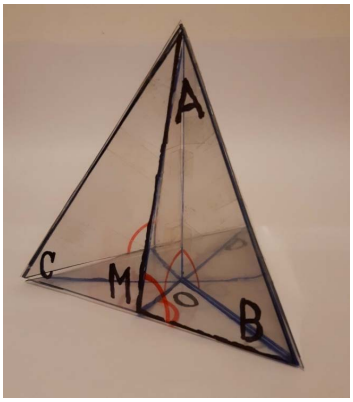


Figure 14. Decomposing a tetrahedron into six Pythagorean tetrahedra.

On Figure 14, tetrahedron $ABCD$ is divided into six Pythagorean (quadrirectangular) tetrahedra.

Drop a perpendicular from vertex A to plane BCD, and label O the foot of A on plane BCD. Drop a perpendicular from A to side BC, and label M the point of intersection. Connect foot O with vertex B and point M. Now consider, for example, tetrahedron $ABMO$. The construction gives that angles $>AOM$ and $>AOB$ are right angles regardless of whether O is the orthocentre of triangle BCD or not. By the same construction, angles $>AMB$ and $>OMB$ are also right angles. It follows that tetrahedron $ABMO$ is a Pythagorean, quadrirectangular tetrahedron with hypotenuse OM, and legs BM, MO, OA.

Example 1. Apply the method to calculate the volume of the regular tetrahedron with unit sides:

The Pythagorean tetrahedron $ABMO$ has legs

$BM=\frac{1}{\sqrt{2}}, MO=\frac{1}{\sqrt{6}}, OA=\frac{1}{\sqrt{3}}$. so its volume can be calculated by the generalized Pythagorean theorem:

$$V_P^2=\frac{1}{(3!)^2}\times\left(\left(\frac{1}{\sqrt{2}}\right)^2\times\left(\frac{1}{\sqrt{6}}\right)^2\times\left(\frac{1}{\sqrt{3}}\right)^2\right)=\frac{1}{36}\times\frac{3}{36}\times\frac{6}{9}=\frac{18}{36^3}=\frac{1}{2592}.$$

The volume of the regular tetrahedron ABCD is equal to six times the volume V_P of the Pythagorean tetrahedron ABMO:

$$V^2=(6V_P)^2=6^2\frac{1}{2592}=\frac{1}{72}=\frac{2}{144}, V=\frac{\sqrt{2}}{12}.$$

Example 2. Apply the method to calculate the volume of the trirectangular tetrahedron with three unit sides meeting at the vertex with three right angles:

$$V_P^2=\frac{1}{(3!)^2}\times\left(\left(\frac{1}{\sqrt{2}}\right)^2\times\left(\frac{1}{\sqrt{6}}\right)^2\times\left(\frac{1}{\sqrt{3}}\right)^2\right)=\frac{1}{36}\times\frac{3}{36}.$$

The volume of the regular tetrahedron $ABCD$ is equal to six times the volume V_P of the Pythagorean tetrahedron $ABMO$:

$$V^2=(6V_P)^2=6^2\frac{1}{36}\times\frac{1}{36}, V=\frac{1}{6}.$$

The decomposition can be generalized in like manner to higher dimensions.

11. Conclusions

The article offers a new kind of simplex in n-dimensional spaces and a generalization of the Pythagorean Theorem. The main results are that simplices with solely rectangular faces exist for arbitrary dimensions, and a formula is given to generalize the Pythagorean Theorem. It states that the square of the volume of a rectangular simplex with a hypotenuse and legs can be calculated in the same way as in the 2D case, multiplying the squares of legs, and dividing the product

by $\frac{1}{(n!)^2}$, without computing the Cayley-Menger determinant.

This option leads to an alternative construction of multidimensional geometries using rectangular simplexes as building blocks, instead of the traditional general triangles. It can simplify theorems and techniques in other areas of multidimensional geometries and their applications, in determinant theory or n-dimensional calculus. Moreover, it can be connected with recent tendencies of using Euclidean geometry, "euclidicity" in the four-dimensional space of the Theorem of Relativity (Machotka [5]). Another challenge is whether Pythagorean shapes can be applied in the theory of the Cayley-Menger determinant in spherical and hyperbolic spaces (cf. Tao [6] or Audet [7]).

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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