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Spectra of the Energy Operator of Two-Electron System in the Impurity Hubbard Model

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Abstract

We consider two-electron systems for the impurity Hubbard Model and investigate the spectrum of the system in a singlet state for the ν-dimensional integer valued lattice Z^{ν}. We proved the essential spectrum of the system in the singlet state is consists of union of no more then three intervals, and the discrete spectrum of the system in the singlet state is consists of no more then five eigenvalues. We show that the discrete spectrum of the system in the triplet and singlet states differ from each other. In the singlet state the appear additional two eigenvalues. In the triplet state the discrete spectrum of the system can be empty set, or is consists of one-eigenvalue, or is consists of two eigenvalues, or is consists of three eigenvalues. For investigation the structure of essential spectra and discrete spectrum of the energy operator of two-electron systems in an impurity Hubbard model, for which the momentum representation is convenient. In addition, we used the tensor products of Hilbert spaces and tensor products of operators in Hilbert spaces and described the structure of essential spectrum and discrete spectrum of the energy operator of two-electron systems in an impurity Hubbard model.

Keywords

Two-Electron System, Impurity Hubbard Model, Singlet State, Triplet State, Essential Spectra, Discrete Spectrum

1. Introduction

In the early 1970s, three papers [1] [2] [3], where a simple model of a metal was proposed that has become a fundamental model in the theory of strongly correlated electron systems, appeared almost simultaneously and independently. In that model, a single nondegenerate electron band with a local Coulomb interaction is considered. The model Hamiltonian contains only two parameters: the parameter B of electron hopping from a lattice site to a neighboring site and the parameter U of the on-site Coulomb repulsion of two electrons. In the secondary quantization representation, the Hamiltonian can be written as

$$H = B\sum_{m,\tau,\gamma} a^{+}_{m,\gamma} a_{m+\tau,\gamma} + U\sum_{m} a^{+}_{m,\uparrow} a_{m,\uparrow} a^{+}_{m,\downarrow} a_{m,\downarrow},$$

where $a^{+}_{m,\gamma}$ and $a_{m,\gamma}$ denote Fermi operators of creation and annihilation of an electron with spin γ on a site m and the summation over τ means summation over the nearest neighbors on the lattice.

Recall that the local form of Coulomb interaction was first introduced for an impurity model in a metal by Anderson [4].

The Hubbard model is currently one of the most extensively studied multielectron models of metals [5] [6]. But little is known about exact results for the spectrum and wave functions of the crystal described by the Hubbard model and impurity Hubbard model, and obtaining the corresponding statements is therefore of great interest.

In the work [7] is considered dominant correlation effects in two-electron atoms.

The spectrum and wave functions of the system of three electrons in a crystal described by the Hubbard Hamiltonian were studied in [8]. Correspondingly, the spectrum of the energy operator of system of four electrons for a crystal described by the Hubbard Hamiltonian in the triplet state was studied in [9]. For the four-electron systems are exists quintet state, and three type triplet states, and two type singlet states. The spectrum of the energy operator of four-electron systems in the Hubbard model in the quintet, and singlet states were studied in [10].

The use of films in various areas of physics and technology arouses great interest in studying a localized impurity state (LIS) of magnet. Therefore, it is important to study the spectral properties of electron systems in the impurity Hubbard model. The spectrum of the energy operator of three-electron systems in the impurity Hubbard model in the second doublet state was studied [11].

The spectrum and wave functions of the system of three electrons in a crystal described by the Impurity Hubbard Hamiltonian were studied in [12] in the quartet state of the system.

Naturally, we have analogous problem in the case of two-electron systems in the impurity Hubbard model. Here there are exist two states: triplet and singlet states. The investigation of the spectrum of Hamiltonian for this model in the case triplet state for three-dimensional lattice was given by Yu. Kh. Ishkobilov [13]. In the same time in this paper there are no exact values of Hamiltonian parameters for which exists the eigenvalues of corresponding operator.

In this paper we give a full description of the structure of the essential spectra and discrete spectrum of two-electron systems in the impurity Hubbard model for triplet and singlet states. The main result of this paper is Theorems 11 and 12, which describe the spectrum of considered model for singlet state. The results of sections 3 and 4 and 5 there are preliminary facts for the proof of Theorems 11 and 12, also Theorem 15 and 16.

2. Preliminaries

The Hamiltonian of two-electron systems in the impurity Hubbard model has the following form

$$H = A\sum_{m,\gamma} a^+_{m,\gamma} a_{m,\gamma} + B\sum_{m,\tau,\gamma} a^+_{m,\gamma} a_{m+\tau,\gamma} + U\sum_{m} a^+_{m,\uparrow} a_{m,\uparrow} a^+_{m,\downarrow} a_{m,\downarrow}$$
$$+(A_0 - A)\sum_{\gamma} a^+_{0,\gamma} a_{0,\gamma} + (B_0 - B)\sum_{\tau,\gamma}\left(a^+_{0,\gamma} a_{\tau,\gamma} + a^+_{\tau,\gamma} a_{0,\gamma}\right) \quad (1)$$
$$+(U_0 - U) a^+_{0,\uparrow} a_{0,\uparrow} a^+_{0,\downarrow} a_{0,\downarrow}.$$

Here A (A_0) is the electron energy at a regular (impurity) lattice site; B (B_0) is the transfer integral between electrons (between electron and impurity) in a neighboring sites (we assume that $B > 0$, $B_0 > 0$), $\tau = \pm e_j, j = 1,2,\cdots,\nu$, where e_j are unit mutually orthogonal vectors, which means that summation is taken over the nearest neighbors, U (U_0) is the parameter of the on-site Coulomb interaction of two electrons, correspondingly in the regular (impurity) lattice site; γ is the spin index, $\gamma = \uparrow$ or $\gamma = \downarrow$, $\uparrow$ and $\downarrow$ denote the spin values $\frac{1}{2}$ and $-\frac{1}{2}$, and $a^+_{m,\gamma}$ and $a_{m,\gamma}$ are the respective electron creation and annihilation operators at a site $m \in Z^\nu$, where Z^ν is a ν-dimensional integer lattice.

It is known that the Hamiltonian H acts in the antisymmetric complex Foc'k space $\left(\mathcal{H}_{as}, (\cdot,\cdot)_{\mathcal{H}_{as}}\right)$. Suppose that φ_0 is the vacuum vector in the space $\mathcal{H}_{as}$. The triplet state corresponds to the free motion of two electrons over the lattice with the basis functions $t_{m,n} = a^+_{m\uparrow} a^+_{n\uparrow}\varphi_0$. The linear subspace $\mathcal{H}^t_2 \subset \mathcal{H}_{as}$, corresponding to the triplet state is the set of all vectors of the form $\psi = \sum_{m,n\in Z^\nu} f(m,n) t_{m,n}$, $f \in l^{as}_2$, where l^{as}_2 is the subspace of antisymmetric functions in the Hilbert space $l_2\left(\left(Z^\nu\right)^2\right)$.

Using the standard anticommutation relations between electron creation and annihilation operators at lattice sites, $\left\{a_{m,\gamma}, a^+_{n,\beta}\right\} = \delta_{m,n}\delta_{\gamma,\beta}$, $\left\{a_{m,\gamma}, a_{n,\beta}\right\} = \left\{a^+_{m,\gamma}, a^+_{n,\beta}\right\} = \theta$, and also take into account that $a_{m,\gamma}\varphi_0 = \theta$, ($\theta$ is the zero element) we get the following

Theorem 1. *The subspace* $\mathcal{H}^t_2$ *is invariant with respect to the action of operator H, and the restriction* $H^t_2 = H|_{\mathcal{H}^t_2}$ *of operator H to the subspace* $\mathcal{H}^t_2$ *is a bounded self-adjoint operator. The operator* H^t_2 *acts on a vector* $\psi \in \mathcal{H}^t_2$ *as*

$$H^t_2\psi = \sum_{p,q}\left(\bar{H}^t_2 f\right)(p,q) t_{p,q}, \quad (2)$$

where $\bar{H}^t_2$ is a bounded self-adjoint operator acting in the Hilbert space l^{as}_2 as

$$\left(\bar{H}^t_2 f\right)(p,q) = 2Af(p,q) + 2B\sum_{\tau}\left[f(p+\tau,q) + f(p,q+\tau)\right]$$
$$+(A_0 - A)\left[\delta_{p,0} + \delta_{q,0}\right] f(p,q) + (B_0 - B)\sum_{\tau}\left[\delta_{p,\tau} f(0,q) \quad (3)\right.$$
$$\left. + \delta_{q,\tau} f(p,0) + \delta_{p,0} f(\tau,q) + \delta_{q,0} f(p,\tau)\right]$$

(here $\delta_{k,j}$ is the Kronecker symbol).

We need the following Lemma on the coincidence of the spectra of operators $\bar{H}_2^t$ and H_2^t.

Lemma 2. *The spectra of the operators* $\bar{H}_2^t$ *and* H_2^t *coincide.*

Proof. Let $\lambda \in \sigma\left(H_2^t\right)$. Since H_2^t is the bounded self-adjoint operators, it follows by the Weyl criterion (see, for example, Ch. VII, section 14 in [14]) that there exists a sequence of vectors $\{\psi_n\}_{n=1}^{\infty} \subset \mathcal{H}_2^t$ such that $\psi_n = \sum_{p,q} f_n(p,q) a_{p,\uparrow}^{+} a_{q,\uparrow}^{+} \varphi_0$, $\|\psi_n\| = 1$, and

$$\lim_{n\to\infty} \left\| \left(H_2^t - \lambda\right)\psi_n \right\| = 0. \tag{4}$$

On the other hand, setting $F_n = \left\{ f_n(p,q)_{p,q\in Z^\nu} \right\} \in l_2^{as}$, we have

$$\begin{aligned}\left\| H_2^t\psi_n - \lambda\psi_n \right\|^2 &= \left(H_2^t\psi_n - \lambda\psi_n, H_2^t\psi_n - \lambda\psi_n \right)_{\mathcal{H}_{as}} \\ &= \sum_{p,q} \left| \left(\bar{H}_2^t - \lambda\right) f_n(p,q) \right|^2 \\ &= \left\| \bar{H}_2^t F_n - \lambda F_n \right\|^2,\end{aligned}$$

and

$$\|F_n\|^2 = \sum\nolimits_{p,q} \left| f_n(p,q) \right|^2 = \|\psi_n\|^2 = 1.$$

Hence, by (4), we get that $\left\| \bar{H}_2^t F_n - \lambda F_n \right\| \to 0$ as $n\to\infty$. Using again the Weyl criterion we have $\lambda \in \sigma\left(\bar{H}_2^t\right)$. Therefore, $\sigma\left(H_2^t\right) \subset \sigma\left(\bar{H}_2^t\right)$.

Conversely, let $\hat{\lambda} \in \sigma\left(\bar{H}_2^t\right)$. By the Weyl criterion, there exists a sequence $F_n = \left\{ f_n(p,q)_{p,q\in Z^\nu} \right\} \in l_2^{as}$ such that $\|F_n\| = \sqrt{\sum\nolimits_{p,q} \left| f_n(p,q) \right|^2} = 1$ and

$$\lim_{n\to\infty} \left\| \left(\bar{H}_2^t - \hat{\lambda}\right) F_n \right\| = 0. \tag{5}$$

Setting $\psi_n = \sum_{p,q} f_n(p,q) a_{p,\uparrow}^{+} a_{q,\uparrow}^{+} \varphi_0$, we have $\|\psi_n\| = \|F_n\| = 1$ and by (5)

$$\left\| \left(H_2^t - \hat{\lambda}\right)\psi_n \right\| = \left\| \left(\bar{H}_2^t - \hat{\lambda}\right) F_n \right\| \to 0 \text{ as } n\to\infty.$$

Hence by the Weyl criterion, we obtain that $\hat{\lambda} \in \sigma\left(H_2^t\right)$. Therefore, $\sigma\left(\bar{H}_2^t\right) \subset \sigma\left(H_2^t\right)$. □

Bellow we will call the operator $\bar{H}_2^t$ as the operator two-electron triplet state operator.

Let $\mathcal{F} : l_2\left(\left(Z^\nu\right)^2\right) \to L_2\left(\left(T^\nu\right)^2\right) := \tilde{\mathcal{H}}_2^t$, be the Fourier transform, where T^ν is the ν-dimensional torus endowed with the normalized Lebesgue measure $d\lambda$, that is, $\lambda\left(T^\nu\right) = 1$. Setting $\tilde{H}_2^t = \mathcal{F}\bar{H}_2^t\mathcal{F}^{-1}$ we get that the operator $\tilde{H}_2^t$ acts in the Hilbert space $L_2^{as}\left(\left(T^\nu\right)^2\right)$, where L_2^{as} is the linear subspace of antisymmetric functions in $L_2\left(\left(T^\nu\right)^2\right)$.

Using the equality (3) and properties of the Fourier transform we have the following

Theorem 3. *The operator* $\tilde{H}_2^t$ *acting in the space* $\tilde{\mathcal{H}}_2^t$ *as*

$$\begin{aligned}\left(\tilde{H}_2^t\tilde{f}\right)(\mu,\gamma) &= 2A\tilde{f}(\mu,\gamma)+2B\sum_{i=1}^{\nu}\left[\cos\mu_i+\cos\gamma_i\right]\tilde{f}(\mu,\gamma)+\varepsilon_1\int_{T^\nu}\tilde{f}(s,\gamma)\mathrm{d}s\\ &+\varepsilon_1\int_{T^\nu}\tilde{f}(\mu,t)\mathrm{d}t+2\varepsilon_2\int_{T^\nu}\sum_{i=1}^{\nu}\left[\cos s_i+\cos\mu_i\right]\tilde{f}(s,\gamma)\mathrm{d}s\\ &+2\varepsilon_2\int_{T^\nu}\sum_{i=1}^{\nu}\left[\cos t_i+\cos\gamma_i\right]\tilde{f}(\mu,t)\mathrm{d}t,\\ \mu &= (\mu_1,\cdots,\mu_n),\gamma=(\gamma_1,\cdots,\gamma_n),s=(s_1,\cdots,s_n),t=(t_1,\cdots,t_n)\in T^\nu,\end{aligned} \tag{6}$$

where $\varepsilon_1 = A_0 - A$, $\varepsilon_2 = B_0 - B$, and $\varepsilon_3 = U_0 - U$.

It is clear that spectral properties of energy operator of two-electron systems in the impurity Hubbard model in the triplet state are closely related to the spectral properties of its one-electron subsystems in the impurity Hubbard model. First we investigate the spectrum of one-electron subsystems.

3. Spectra of the Energy Operator of One-Electron System in the Impurity Hubbard Model

The Hamiltonian H of one-electron systems in the impurity Hubbard model also has the form (1). We let $\mathcal{H}_1$ denote the Hilbert space spanned by the vectors in the form $\psi=\sum_p a_{p,\uparrow}^+\varphi_0$. It is called the space of one-electron states of the operator H. The space $\mathcal{H}_1$ is invariant with respect to action of operator H. Denote by $H_1 = H|_{\mathcal{H}_1}$ the restriction of H to the subspace $\mathcal{H}_1$.

As in the proof of Theorem 1, using the standard anticommutation relations between electron creation and annihilation operators at lattice sites, we get the following

Theorem 4. *The subspace* $\mathcal{H}_1$ *is invariant with respect to the action of the operator* H, *and the restriction* H_1 *is a linear bounded self-adjoint operator, acting in* $\mathcal{H}_1$ *as*

$$H_1\psi=\sum_p\left(\bar{H}_1 f\right)(p)a_{p,\uparrow}^+\varphi_0,\ \psi\in\mathcal{H}_1, \tag{7}$$

where $\bar{H}_1$ *is a linear bounded self-adjoint operator acting in the space* l_2 *as*

$$\begin{aligned}\left(\bar{H}_1 f\right)(p) &= Af(p)+B\sum_\tau f(p+\tau)+\varepsilon_1\delta_{p,0}f(p)\\ &+\varepsilon_2\sum_\tau\left(\delta_{p,\tau}f(0)+\delta_{p,0}f(\tau)\right).\end{aligned} \tag{8}$$

Lemma 5. *The spectra of the operators* $\bar{H}_1$ *and* H_1 *coincide.*

The proof of Lemma 5 is the same as the proof of the Lemma 2.

As in section 2 denote by $\mathcal{F}: l_2(Z^\nu)\to L_2(T^\nu) := \tilde{\mathcal{H}}_1$ the Fourier transform. Setting $\tilde{H}_1=\mathcal{F}\bar{H}_1\mathcal{F}^{-1}$ we get that the operator $\bar{H}_1$ acts in the Hilbert space $L_2(T^\nu)$. Using the equality (8) and properties of the Fourier transform we have the following

Theorem 6 *The operator* $\tilde{H}_1$ *acting in the space* $\tilde{\mathcal{H}}_1$ *as*

$$\left(\tilde{H}_1 f\right)(\mu)=\left[A+2B\sum_{i=1}^{\nu}\cos\mu_i\right]f(\mu)+\varepsilon_1\int_{T^\nu}f(s)\,\mathrm{d}s$$
$$+2\varepsilon_2\int_{T^\nu}\sum_{i=1}^{\nu}\left[\cos\mu_i+\cos s_i\right]f(s)\,\mathrm{d}s, \tag{9}$$
$$\mu=(\mu_1,\cdots,\mu_n), s=(s_1,\cdots,s_n)\in T^\nu.$$

It is clear that the continuous spectrum of operator $\tilde{H}_1$ is independent of the numbers ε_1 and ε_2, and is equal to segment $[m_\nu, M_\nu]=[A-2B\nu, A+2B\nu]$, where $m_\nu=\min_{x\in T^\nu}h(x)$, $M_\nu=\max_{x\in T^\nu}h(x)$ (here $h(x)=A+2B\sum_{i=1}^{\nu}\cos x_i$).

To find the eigenvalues and eigenfunctions of operator $\tilde{H}_1$ we rewrite (9) in following form:

$$\left\{A+2B\sum_{i=1}^{\nu}\cos\mu_i-z\right\}f(\mu)+\varepsilon_1\int_{T^\nu}f(s)\,\mathrm{d}s$$
$$+2\varepsilon_2\int_{T^\nu}\sum_{i=1}^{\nu}\left[\cos\mu_i+\cos s_i\right]f(s)\,\mathrm{d}s=0, \tag{10}$$

where $z\in\mathbb{R}$.

Suppose first that $\nu=1$ and denote $a=\int_T f(s)\,\mathrm{d}s$, $b=\int_T f(s)\cos s\,\mathrm{d}s$, $h(\mu)=A+2B\cos\mu$. From (10) it follows that

$$f(\mu)=-\frac{(\varepsilon_1+2\varepsilon_2\cos\mu)a+2\varepsilon_2 b}{h(\mu)-z}. \tag{11}$$

Now substitute (11) in expressing of *a* and *b* we get the following system of two linear homogeneous algebraic equations:

$$\left(1+\int_T\frac{\varepsilon_1+2\varepsilon_2\cos s}{h(s)-z}\mathrm{d}s\right)\cdot a+2\varepsilon_2\int_T\frac{\mathrm{d}s}{h(s)-z}\cdot b=0;$$
$$\int_T\frac{\cos s(\varepsilon_1+2\varepsilon_2\cos s)}{h(s)-z}\mathrm{d}s\cdot a+\left(1+2\varepsilon_2\int_T\frac{\cos s\,\mathrm{d}s}{h(s)-z}\right)\cdot b=0.$$

This system has a nontrivial solution if and only if the determinant $\Delta_1(z)$ of this system is equal to zero, where

$$\Delta_1(z)=\left(1+\int_T\frac{(\varepsilon_1+2\varepsilon_2\cos s)\mathrm{d}s}{h(s)-z}\right)\cdot\left(1+2\varepsilon_2\int_T\frac{\cos s\,\mathrm{d}s}{h(s)-z}\right)$$
$$-2\varepsilon_2\int_T\frac{\mathrm{d}s}{h(s)-z}\cdot\int_T\frac{\cos s(\varepsilon_1+2\varepsilon_2\cos s)}{h(s)-z}\mathrm{d}s.$$

Therefore, it is true the following

Lemma 7. *If a real number* $z\notin[m_1, M_1]$ *then z is an eigenvalue of the operator* $\tilde{H}_1$ *if and only if* $\Delta_1(z)=0$.

The following Theorem describe of the exchange of the spectrum of operator $\tilde{H}_1$ in the case $\nu=1$.

Theorem 8. *Let* $\nu=1$. *Then*

A) If $\varepsilon_2=-B$ and $\varepsilon_1<-2B$ (respectively, $\varepsilon_2=-B$ and $\varepsilon_1>2B$), then the operator $\tilde{H}_1$ has a unique eigenvalue $z=A+\varepsilon_1$, lying the below (respectively, above) of the continuous spectrum of operator $\tilde{H}_1$.

B) If $\varepsilon_1 < 0$ and $\varepsilon_2 = -2B$ or $\varepsilon_2 = 0$ (respectively, $\varepsilon_1 > 0$ and $\varepsilon_2 = -2B$ or $\varepsilon_2 = 0$), then the operator $\tilde{H}_1$ has a unique eigenvalue $z = A - \sqrt{4B^2 + \varepsilon_1^2}$ (respectively, $z = A + \sqrt{4B^2 + \varepsilon_1^2}$), lying the below (respectively, above) of the continuous spectrum of operator $\tilde{H}_1$.

C) If $\varepsilon_1 = 0$ and $\varepsilon_2 > 0$ or $\varepsilon_1 = 0$ and $\varepsilon_2 < -2B$, then the operator $\tilde{H}_1$ has a two eigenvalues $z_1 = A - \frac{2BE}{\sqrt{E^2 - 1}}$ and $z_2 = A + \frac{2BE}{\sqrt{E^2 - 1}}$, where $E = \frac{(B + \varepsilon_2)^2}{\varepsilon_2^2 + 2B\varepsilon_2}$, lying the below and above of the continuous spectrum of operator $\tilde{H}_1$.

D) If $\varepsilon_1 = \frac{2(\varepsilon_2^2 + 2B\varepsilon_2)}{B}$ (respectively, $\varepsilon_1 = -\frac{2(\varepsilon_2^2 + 2B\varepsilon_2)}{B}$), then the operator $\tilde{H}_1$ has a unique eigenvalue $z = A + \frac{2B(E^2 + 1)}{E^2 - 1}$ (respectively, $z = A - \frac{2B(E^2 + 1)}{E^2 - 1}$), where $E = \frac{(B + \varepsilon_2)^2}{\varepsilon_2^2 + 2B\varepsilon_2}$, lying the above (respectively, below) of the continuous spectrum of operator $\tilde{H}_1$.

E) If $\varepsilon_2 > 0$ and $\varepsilon_1 > \frac{2(\varepsilon_2^2 + 2B\varepsilon_2)}{B}$ (respectively, $\varepsilon_2 < -2B$ and $\varepsilon_1 > \frac{2(\varepsilon_2^2 + 2B\varepsilon_2)}{B}$), then the operator $\tilde{H}_1$ has a unique eigenvalue $z = A + \frac{2B\left(\alpha + E\sqrt{E^2 - 1 + \alpha^2}\right)}{E^2 - 1}$, where $E = \frac{(B + \varepsilon_2)^2}{\varepsilon_2^2 + 2B\varepsilon_2}$, and the real number $\alpha > 1$, lying the above of the continuous spectrum of operator $\tilde{H}_1$.

F) If $\varepsilon_2 > 0$ and $\varepsilon_1 < -\frac{2(\varepsilon_2^2 + 2B\varepsilon_2)}{B}$ (respectively, $\varepsilon_2 < -2B$ and $\varepsilon_1 < -\frac{2(\varepsilon_2^2 + 2B\varepsilon_2)}{B}$), then the operator $\tilde{H}_1$ has a unique eigenvalue $z = A - \frac{2B\left(\alpha + E\sqrt{E^2 - 1 + \alpha^2}\right)}{E^2 - 1} < m_1$, where $E = \frac{(B + \varepsilon_2)^2}{\varepsilon_2^2 + 2B\varepsilon_2}$, and the real number $\alpha > 1$, lying the below of the continuous spectrum of operator $\tilde{H}_1$.

K) If $\varepsilon_2 > 0$ and $0 < \varepsilon_1 < \frac{2(\varepsilon_2^2 + 2B\varepsilon_2)}{B}$ (respectively, $\varepsilon_2 < -2B$ and $0 < \varepsilon_1 < \frac{2(\varepsilon_2^2 + 2B\varepsilon_2)}{B}$), then the operator $\tilde{H}_1$ has a exactly two eigenvalues $z_1 = A + \frac{2B\left(\alpha - E\sqrt{E^2 - 1 + \alpha^2}\right)}{E^2 - 1} < m_1$ and $z_2 = A + \frac{2B\left(\alpha + E\sqrt{E^2 - 1 + \alpha^2}\right)}{E^2 - 1} > M_1$, where $E = \frac{(B + \varepsilon_2)^2}{\varepsilon_2^2 + 2B\varepsilon_2}$, and the real number $0<\alpha<1$, lying correspondingly, the below and above of the continuous spectrum of operator $\tilde{H}_1$.

M) If $\varepsilon_2>0$ and $-\frac{2\left(\varepsilon_2^2+2B\varepsilon_2\right)}{B}<\varepsilon_1<0$ (respectively, $\varepsilon_2<-2B$ and $-\frac{2\left(\varepsilon_2^2+2B\varepsilon_2\right)}{B}<\varepsilon_1<0$), then the operator $\tilde{H}_1$ has a exactly two eigenvalues $z_1=A-\frac{2B\left(\alpha+E\sqrt{E^2-1+\alpha^2}\right)}{E^2-1}<m_1$ and

$z_2=A-\frac{2B\left(\alpha-E\sqrt{E^2-1+\alpha^2}\right)}{E^2-1}>M_1$, where $E=\frac{\left(B+\varepsilon_2\right)^2}{\varepsilon_2^2+2B\varepsilon_2}$, and the real number $0<\alpha<1$, lying correspondingly the below and above of the continuous spectrum of operator $\tilde{H}_1$.

N) If $-2B<\varepsilon_2<0$, then the operator $\tilde{H}_1$ has no eigenvalues lying the outside of the continuous spectrum of operator $\tilde{H}_1$.

Proof. In the case $\nu=1$, the continuous spectrum of the operator $\tilde{H}_1$ coincide with segment $[m_1,M_1]=[A-2B,A+2B]$. Expressing all integrals in the equation $\Delta_1(z)=0$ through the integral $J(z)=\int_T\frac{\mathrm{d}s}{A+2B\cos s-z}$, we find that the equation $\Delta_1(z)=0$ is equivalent to the equation

$$\left[\varepsilon_1B^2+\left(\varepsilon_2^2+2B\varepsilon_2\right)(z-A)\right]J(z)+\left(B+\varepsilon_2\right)^2=0. \tag{12}$$

Moreover, the function $J(z)=\int_T\frac{\mathrm{d}s}{A+2B\cos s-z}$ is a differentiable function on the set $\mathbb{R}\setminus[m_1,M_1]$, in addition, $J'(z)=\int_T\frac{\mathrm{d}s}{\left[A+2B\cos s-z\right]^2}>0$, $z\notin[m_1,M_1]$. Thus the function $J(z)$ is an monotone increasing function on $(-\infty,m_1)$ and on $(M_1,+\infty)$. Furthermore, $J(z)\to+0$ as $z\to-\infty$, $J(z)\to+\infty$ as $z\to m_1-0$, $J(z)\to-\infty$ as $z\to M_1+0$, and $J(z)\to-0$ as $z\to+\infty$.

If $\varepsilon_1B^2+\left(\varepsilon_2^2+2B\varepsilon_2\right)(z-A)\neq0$ then from (12) follows that

$$J(z)=-\frac{\left(B+\varepsilon_2\right)^2}{\varepsilon_1B^2+\left(\varepsilon_2^2+2B\varepsilon_2\right)(z-A)}.$$

The function $\psi(z)=-\frac{\left(B+\varepsilon_2\right)^2}{\varepsilon_1B^2+\left(\varepsilon_2^2+2B\varepsilon_2\right)(z-A)}$ has a point of asymptotic discontinuity $z_0=A-\frac{B^2\varepsilon_1}{\varepsilon_2^2+2B\varepsilon_2}$. Since $\psi'(z)=\frac{\left(B+\varepsilon_2\right)^2\left(\varepsilon_2^2+2B\varepsilon_2\right)}{\left[\varepsilon_1B^2+\left(\varepsilon_2^2+2B\varepsilon_2\right)(z-A)\right]^2}$ for all $z\neq z_0$ it follows that the function $\psi(z)$ is an monotone increasing (decreasing) function on $(-\infty,z_0)$ and on $(z_0,+\infty)$ in the case $\varepsilon_2^2+2B\varepsilon_2>0$ (respectively, $\varepsilon_2^2+2B\varepsilon_2<0$), in addition, and if $\varepsilon_2>0$, or $\varepsilon_2<-2B$, then

$\psi(z)\to+0$ as $z\to-\infty$, $\psi(z)\to+\infty$ as $z\to z_0-0$, $\psi(z)\to-\infty$ as $z\to z_0+0$, $\psi(z)\to-0$ as $z\to+\infty$ (respectively, if $-2B<\varepsilon_2<0$, then $\psi(z)\to-0$ as $z\to-\infty$, $\psi(z)\to-\infty$ as $z\to z_0-0$, $\psi(z)\to+\infty$ as $z\to z_0+0$, $\psi(z)\to+0$ as $z\to+\infty$).

A) If $\varepsilon_2=-B$ and $\varepsilon_1<-2B$ (respectively, $\varepsilon_2=-B$ and $\varepsilon_1>2B$), then the equation for eigenvalues and eigenfunctions (12) has the form

$$\left\{\varepsilon_1B^2-B^2(z-A)\right\}J(z)=0. \tag{13}$$

It is clear, that $J(z)\neq0$ for the values $z\notin\sigma_{cont}(\tilde{H}_1)$. Therefore, $\varepsilon_1-z+A=0$, *i.e.*, $z=A+\varepsilon_1$. If $\varepsilon_1<-2B$ ($\varepsilon_1>2B$), then this eigenvalue lying the below (the above) of the continuous spectrum of operator $\tilde{H}_1$.

B) If $\varepsilon_1<0$ and $\varepsilon_2=-2B$ or $\varepsilon_2=0$ (respectively, $\varepsilon_1>0$ and $\varepsilon_2=-2B$ or $\varepsilon_2=0$), then the equation for the eigenvalues and eigenfunctions has the form $J(z)=-\frac{1}{\varepsilon_1}$. It is clear, what the integral $J(z)$ calculated in a quadrature, of the below (above) of continuous spectrum of operator $\tilde{H}_1$, the integral $J(z)>0$ ($J(z)<0$), consequently, $\varepsilon_1<0$ ($\varepsilon_1>0$). The calculated the integral $J(z)=\int_{T^\nu}\frac{\mathrm{d}s}{A+2B\cos s-z}$, the below of the continuous spectrum of operator $\tilde{H}_1$, we have the equation of the form

$$\frac{1}{\sqrt{(A-z)^2-4B^2}}=-\frac{1}{\varepsilon_1}.$$

This equation has a solution $z=A-\sqrt{\varepsilon_1^2+4B^2}$, lying the below of the continuous spectrum of operator $\tilde{H}_1$. In the above of continuous spectrum of operator $\tilde{H}_1$, the equation take the form

$$-\frac{1}{\sqrt{(z-A)^2-4B^2}}=-\frac{1}{\varepsilon_1}.$$

This equation has a solution of the form $z=A+\sqrt{\varepsilon_1^2+4B^2}$, lying the above of the continuous spectrum of operator $\tilde{H}_1$.

C) If $\varepsilon_1=0$ and $\varepsilon_2>0$ or $\varepsilon_1=0$ and $\varepsilon_2<-2B$, then the equation for the eigenvalues and eigenfunctions take in the form

$$\left(\varepsilon_2^2+2B\varepsilon_2\right)(z-A)J(z)=-(B+\varepsilon_2)^2,$$

or

$$J(z)=-\frac{(B+\varepsilon_2)^2}{\left(\varepsilon_2^2+2B\varepsilon_2\right)(z-A)}.$$

Denote $E=\frac{(B+\varepsilon_2)^2}{\varepsilon_2^2+2B\varepsilon_2}$. Then $J(z)=-\frac{E}{z-A}$, or $J(z)=\frac{E}{A-z}$. In the below of the continuous spectrum of the operator $\tilde{H}_1$, we have the equation of the form

$$\frac{1}{\sqrt{(A-z)^2-4B^2}}=\frac{E}{A-z}.$$

This equation has a solution $z=A-\frac{2BE}{\sqrt{E^2-1}}$. It is obviously, that $E^2>1$. This eigenvalue lying the below of the continuous spectrum of operator $\tilde{H}_1$. In the above of the continuous spectrum of operator $\tilde{H}_1$, the equation for the eigenvalues and eigenfunctions has the form

$$-\frac{1}{\sqrt{(z-A)^2-4B^2}}=-\frac{E}{z-A}.$$

From here, we find $z=A+\frac{2BE}{\sqrt{E^2-1}}$. This eigenvalue lying the above of the continuous spectrum of operator $\tilde{H}_1$.

D) If $\varepsilon_1=\frac{2(\varepsilon_2^2+2B\varepsilon_2)}{B}$, then the equation for eigenvalues and eigenfunctions has the form $(\varepsilon_2^2+2B\varepsilon_2)(z-A+2B)J(z)=-(B+\varepsilon_2)^2$, from this

$$J(z)=-\frac{(B+\varepsilon_2)^2}{(\varepsilon_2^2+2B\varepsilon_2)(z-A+2B)}. \tag{14}$$

We denote $E=\frac{(B+\varepsilon_2)^2}{\varepsilon_2^2+2B\varepsilon_2}$. In the first we consider Equation (14) in the below of continuous spectrum of $\tilde{H}_1$. In Equation (14) we find the equation of the form

$$\frac{1}{\sqrt{(A-z)^2-4B^2}}=\frac{E}{A-z-2B}.$$

From this, we find $z_1=A+\frac{2B(E^2+1)}{E^2-1}$ and $z_2=A-2B$. Now we verify the conditions $z_i<A-2B, i=1,2$. The inequality $z_1<A-2B$, is incorrectly, and inequality $z_2<A-2B$, also is incorrectly. We now consider Equation (14) in the above of continuous spectrum of operator $\tilde{H}_1$. We have

$$-\frac{1}{\sqrt{(z-A)^2-4B^2}}=-\frac{E}{z-A+2B}.$$

In this equation we find the solutions above of continuous spectrum of operator $\tilde{H}_1$. Now we verify the conditions $z_i>A+2B, i=1,2$. The inequality $z_1>A+2B$, is correctly, and inequality $z_2>A+2B$, is incorrectly. Consequently, in this case the operator $\tilde{H}_1$ has a unique eigenvalue $z_1=A+\frac{2B(E^2+1)}{E^2-1}$, lying the above of continuous spectrum of operator $\tilde{H}_1$.

Let $\varepsilon_1=-\frac{2(\varepsilon_2^2+2B\varepsilon_2)}{B}$, then the equation of eigenvalues and eigenfunctions take in the form $J(z) = -\frac{E}{z - A - 2B}$, where $E = \frac{(B+\varepsilon_2)^2}{\varepsilon_2^2 + 2B\varepsilon_2}$.

In the below of continuous spectrum of $\tilde{H}_1$, we have equation of the form

$$\frac{1}{\sqrt{(A-z)^2 - 4B^2}} = \frac{E}{A - z + 2B}.$$

From here we find $z_1 = A - \frac{2B(E^2+1)}{E^2-1}$ and $z_2 = A + 2B$. The appear inequalities $z_1 < A - 2B$, is correct, and $z_2 < A - 2B$, is incorrect. In the above of continuous spectrum of operator $\tilde{H}_1$, we have equation of the form

$$-\frac{1}{\sqrt{(z-A)^2 - 4B^2}} = -\frac{E}{z - A - 2B}.$$

It follows that, what $z_1 = A - \frac{2B(E^2+1)}{E^2-1}$ and $z_2 = A + 2B$. The inequality $z_1 > A + 2B$ and $z_2 > A + 2B$, are incorrectly. Therefore, in this case the operator $\tilde{H}_1$ has a unique eigenvalue $z_1 = A - \frac{2B(E^2+1)}{E^2-1}$, lying the below of continuous spectrum of operator $\tilde{H}_1$.

E) If $\varepsilon_2 > 0$ and $\varepsilon_1 > \frac{2(\varepsilon_2^2 + 2B\varepsilon_2)}{B}$ (respectively, $\varepsilon_2 < -2B$ and $\varepsilon_1 > \frac{2(\varepsilon_2^2 + 2B\varepsilon_2)}{B}$), then consider necessary, that $\varepsilon_1 = \alpha \times \frac{2(\varepsilon_2^2 + 2B\varepsilon_2)}{B}$, where $\alpha > 1-$ real number. Then the equation for eigenvalues and eigenfunctions has the form $(\varepsilon_2^2 + 2B\varepsilon_2)(z - A + 2\alpha B)J(z) + (B+\varepsilon_2)^2 = 0$. From this $J(z) = -\frac{(B+\varepsilon_2)^2}{(\varepsilon_2^2 + 2B\varepsilon_2)(z - A + 2\alpha B)}$. We denote $E = \frac{(B+\varepsilon_2)^2}{\varepsilon_2^2 + 2B\varepsilon_2}$, then $J(z) = -\frac{E}{z - A + 2\alpha B}$. In the first we consider this equation in the below of the continuous spectrum of operator $\tilde{H}_1$. Then $\frac{1}{\sqrt{(A-z)^2 - 4B^2}} = \frac{E}{A - z - 2\alpha B}$.

This equation has the solutions $z_1 = A + \frac{2B\left(\alpha + E\sqrt{E^2 - 1 + \alpha^2}\right)}{E^2 - 1}$ and $z_2 = A + \frac{2B\left(\alpha - E\sqrt{E^2 - 1 + \alpha^2}\right)}{E^2 - 1}$. Now, we verify the condition $z_i < A - 2B, i = 1,2$. The solution z_1 no satisfy the condition $z_1 < A - 2B$, but z_2 satisfy the condition $z_2 < A - 2B$. We now verify the conditions $z_2 < A - 2\alpha B$. The appear, this inequality is incorrectly. The appear inequalities $z_1 > A + 2B$ is correct, and $z_2 > A + 2B$, is incorrect. We now verify the conditions $z_1 > A - 2\alpha B$. So far as, $A - 2\alpha B < A + 2B$, the appear, this inequality is correctly. Consequently, in this case, the operator $\tilde{H}_1$ has a unique eigenvalue $z_1 = A + \frac{2B\left(\alpha + E\sqrt{E^2 - 1 + \alpha^2}\right)}{E^2 - 1}$, above of continuous spectrum of $\tilde{H}_1$.

F) If $\varepsilon_2 > 0$, and $\varepsilon_1 < -\frac{2\left(\varepsilon_2^2 + 2B\varepsilon_2\right)}{B}$ (respectively, $\varepsilon_2 < -2B$, and $\varepsilon_1 < -\frac{2\left(\varepsilon_2^2 + 2B\varepsilon_2\right)}{B}$), then we assume that $\varepsilon_1 = -\alpha \times \frac{2\left(\varepsilon_2^2 + 2B\varepsilon_2\right)}{B}$, where $\alpha > 1-$ real number. The equation for eigenvalues and eigenfunctions take in the form $\left(\varepsilon_2^2 + 2B\varepsilon_2\right)(z - A - 2\alpha B)J(z) = -(B + \varepsilon_2)^2$. From here $J(z) = -\frac{(B + \varepsilon_2)^2}{\left(\varepsilon_2^2 + 2B\varepsilon_2\right)(z - A - 2\alpha B)}$. The introduce notation $E = \frac{(B + \varepsilon_2)^2}{\varepsilon_2^2 + 2B\varepsilon_2}$. Then

$$J(z) = -\frac{E}{z - A - 2\alpha B}. \tag{15}$$

In the below of the continuous spectrum of operator $\tilde{H}_1$, we have the equation $J(z) = \frac{E}{A - z + 2\alpha B}$, from here $\frac{1}{\sqrt{(A - z)^2 - 4B^2}} = \frac{E}{A - z + 2\alpha B}$; this equation take the form $\left(E^2 - 1\right)(A - z)^2 - 4\alpha B(A - z) - 4B^2\left(E^2 + \alpha^2\right) = 0$. We find $z_1 = A - \frac{2B\left(\alpha + E\sqrt{E^2 - 1 + \alpha^2}\right)}{E^2 - 1}$ and $z_2 = A - \frac{2B\left(\alpha - E\sqrt{E^2 - 1 + \alpha^2}\right)}{E^2 - 1}$. We now verify the conditions $z_i < m_1 = A - 2B, i = 1, 2$. The appear, that $z_1 < A - 2B$, is correctly and $z_2 < A - 2B$, is incorrectly. Now we consider Equation (15) in the above of the continuous spectrum of operator $\tilde{H}_1$. Then $J(z) = -\frac{E}{z - A - 2\alpha B}$. From this

$$-\frac{1}{\sqrt{(z - A)^2 - 4B^2}} = -\frac{E}{z - A - 2\alpha B}.$$

We find $z_1 = A - \frac{2B\left(\alpha + E\sqrt{E^2 - 1 + \alpha^2}\right)}{E^2 - 1}$ and

$z_2 = A + \frac{2B\left(-\alpha + E\sqrt{E^2 - 1 + \alpha^2}\right)}{E^2 - 1}$. We verify the conditions $z_i > A + 2B, i = 1, 2$. The appear $z_1 > A + 2B$, it is not true, and the $z_2 > A + 2B$, is true. We now verify the conditions $z_2 > A + 2\alpha B$. The appear, this inequality is incorrectly. Consequently, in this case, the operator $\tilde{H}_1$ have unique eigenvalue $z_1 = A - \frac{2B\left(\alpha + E\sqrt{E^2 - 1 + \alpha^2}\right)}{E^2 - 1} < m_1$, *i.e.*, lying the below of the continuous spectrum of operator $\tilde{H}_1$.

K) If $\varepsilon_2 > 0$ and $0 < \varepsilon_1 < \frac{2(\varepsilon_2^2 + 2B\varepsilon_2)}{B}$ (respectively, $\varepsilon_2 < -2B$ and $0 < \varepsilon_1 < \frac{2(\varepsilon_2^2 + 2B\varepsilon_2)}{B}$), then we assume that $\varepsilon_1 = \alpha \times \frac{2(\varepsilon_2^2 + 2B\varepsilon_2)}{B}$, where $0 < \alpha < 1$ real number. The equation for eigenvalues and eigenfunctions take in the form

$$\left(\varepsilon_2^2 + 2B\varepsilon_2\right)\left(z - A + 2\alpha B\right)J(z) = -\left(B + \varepsilon_2\right)^2, 0 < \alpha < 1. \qquad (16)$$

We denote $E = \frac{(B + \varepsilon_2)^2}{\varepsilon_2^2 + 2B\varepsilon_2}$. Then the Equation (16) receive the form

$$J(z) = -\frac{E}{z - A + 2\alpha B}.$$

In the below of the continuous spectrum of $\tilde{H}_1$ we have equation of the form

$$\frac{1}{\sqrt{(A-z)^2 - 4B^2}} = \frac{E}{A - z - 2\alpha B}, 0 < \alpha < 1.$$

This equation has a solutions $z_1 = A + \frac{2B\left(\alpha + E\sqrt{E^2 - 1 + \alpha^2}\right)}{E^2 - 1}$ and $z_2 = A + \frac{2B\left(\alpha - E\sqrt{E^2 - 1 + \alpha^2}\right)}{E^2 - 1}$. The inequalities $z_1 < A - 2B$ and $z_1 < A - 2\alpha B$, is implements. The inequalities $z_2 < A - 2B$, is correctly, and the inequality $z_1 < A - 2B$, is incorrectly. We now verify the conditions $z_2 < A - 2\alpha B$, since $A - 2B < A - 2\alpha B$, this inequality is true. We now consider Equation (16) in the above of the continuous spectrum of the operator $\tilde{H}_1$. We have the equation of the form

$$-\frac{1}{\sqrt{(z-A)^2 - 4B^2}} = -\frac{E}{z - A + 2\alpha B}.$$

This equation has a solutions $z_1 = A + \frac{2B\left(\alpha + E\sqrt{E^2 - 1 + \alpha^2}\right)}{E^2 - 1}$ and $z_2 = A + \frac{2B\left(\alpha - E\sqrt{E^2 - 1 + \alpha^2}\right)}{E^2 - 1}$. The inequalities $z_1 > A + 2B$ and $z_1 > A - 2\alpha B$ is true, as $A + 2B > A - 2\alpha B$, that the inequality $z_1 > A - 2\alpha B$ is correctly. The inequalities $z_2 > A + 2B$ and $z_2 > A + 2\alpha B$ is incorrectly. Consequently, in this case the operator $\tilde{H}_1$ has a exactly two eigenvalues $z_1 = A + \frac{2B\left(\alpha - E\sqrt{E^2 - 1 + \alpha^2}\right)}{E^2 - 1}$ and $z_2 = A + \frac{2B\left(\alpha + E\sqrt{E^2 - 1 + \alpha^2}\right)}{E^2 - 1}$, lying the below and above of the continuous spectrum of the operator $\tilde{H}_1$.

M) If $-\frac{2(\varepsilon_2^2 + 2B\varepsilon_2)}{B} < \varepsilon_1 < 0$ (respectively, $\varepsilon_2 < -2B$ and

$-\frac{2\left(\varepsilon_2^2+2B\varepsilon_2\right)}{B}<\varepsilon_1<0$), the we take $\varepsilon_1=-\alpha\times\frac{2\left(\varepsilon_2^2+2B\varepsilon_2\right)}{B}$, where $0<\alpha<1-$ real number. Then the equation for eigenvalues and eigenfunctions has the form

$$\left(\varepsilon_2^2+2B\varepsilon_2\right)(z-A-2\alpha B)J(z)=-(B+\varepsilon_2)^2, 0<\alpha<1.$$

We denote $E=\frac{(B+\varepsilon_2)^2}{\varepsilon_2^2+2B\varepsilon_2}$. Then Equation (16) receive the form $J(z)=-\frac{E}{z-A-2\alpha B}$. In the below of the continuous spectrum of the operator $\tilde{H}_1$ we have equation of the form

$$\frac{1}{\sqrt{(A-z)^2-4B^2}}=\frac{E}{A-z+2\alpha B}, 0<\alpha<1.$$

This equation has a solutions $z_1=A-\frac{2B\left(\alpha+E\sqrt{E^2-1+\alpha^2}\right)}{E^2-1}$ and $z_2=A-\frac{2B\left(\alpha-E\sqrt{E^2-1+\alpha^2}\right)}{E^2-1}$. The inequalities $z_1<A-2B$ and $z_1<A-2\alpha B$, is implements. The inequalities $z_2<A-2B$, is correctly, and the inequality $z_1<A-2B$, is correctly. We now verify the conditions $z_2<A-2\alpha B$, since $A-2B<A-2\alpha B$, this inequality it is not true. We now consider Equation (16) in the above of the continuous spectrum of the operator $\tilde{H}_1$. We have the equation of the form

$$-\frac{1}{\sqrt{(z-A)^2 4B^2}}=-\frac{E}{z-A+2\alpha B}.$$

This equation has a solutions $z_1=A-\frac{2B\left(\alpha+E\sqrt{E^2-1+\alpha^2}\right)}{E^2-1}$ and $z_2=A-\frac{2B\left(\alpha-E\sqrt{E^2-1+\alpha^2}\right)}{E^2-1}$. The inequalities $z_1>A+2B$ and $z_1>A-2\alpha B$ it is not true, as $A+2B>A-2\alpha B$, that the inequality $z_1>A-2\alpha B$ is incorrectly. The inequalities $z_2>A+2B$, and $z_2>A+2\alpha B$ is correctly. Consequently, in this case the operator $\tilde{H}_1$ has a exactly two eigenvalues $z_1=A-\frac{2B\left(\alpha+E\sqrt{E^2-1+\alpha^2}\right)}{E^2-1}$, and $z_2=A-\frac{2B\left(\alpha-E\sqrt{E^2-1+\alpha^2}\right)}{E^2-1}$, lying the below and above of the continuous spectrum of operator $\tilde{H}_1$.

N). If $-2B<\varepsilon_2<0$, then $\varepsilon_2^2+2B\varepsilon_2<0$, and the function $\psi(z)=-\frac{(B+\varepsilon_2)^2}{\varepsilon_1 B+\left(\varepsilon_2^2+2B\varepsilon_2\right)(z-A)}$ is a decreasing function in the intervals $(-\infty,z_0)$ and $(z_0,+\infty)$; by, $z\to-\infty$ the function $\psi(z)\to-0$, and by

$z \to z_0 - 0$, the function $\psi(z) \to -\infty$, and by $z \to +\infty$, $\psi(z) \to +0$, and by $z \to z_0 + 0$, $\psi(z) \to +\infty$. The function $J(z) \to 0$, by $z \to -\infty$, and by $z \to m_1 - 0$, the function $J(z) \to +\infty$, and by $z \to M_1 + 0$, the function $J(z) \to -\infty$, by $z \to +\infty$, the function $J(z) \to -0$. Therefore, the equation $\psi(z) = J(z)$, that's impossible the solutions in the outside the continuous spectrum of operator $\tilde{H}_1$. Therefore, in this case, the operator $\tilde{H}_1$ has no eigenvalues lying the outside of the continuous spectrum of the operator $\tilde{H}_1$. □

Now we consider the two-dimensional case. In two-dimensional case, we have, what the equation $\Delta_2(z) = 0$, is equivalent to the equation of the form $(\varepsilon_2 + B)^2 + \{\varepsilon_1 B^2 + (\varepsilon_2^2 + 2B\varepsilon_2)(z - A)\} J(z) = 0$, where $J(z) = \int_{T^2} \frac{ds_1 ds_2}{A + 2B(\cos s_1 + \cos s_2) - z}$. In this case, also $J(z) \to +0$, as $z \to -\infty$, and $J(z) \to +\infty$, as $z \to m_2 - 0$, and $J(z) \to -\infty$, as $z \to M_2 + 0$, and $J(z) \to -0$, as $z \to +\infty$. In one- and two-dimensional case the behavior of function $J(z)$ be similarly. Therefore, we have the analogously results, what is find the one-dimensional case.

We consider the three-dimensional case. In the first We consider the Watson integral [15] $W = \frac{1}{\pi^3} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{3 dx dy dz}{3 - \cos x - \cos y - \cos z} \simeq 1,516$.

Theorem 9. *Let* $\nu = 3$. *Then*

A). 1) If $\varepsilon_2 = -B$ and $\varepsilon_1 < -6B$ (respectively, $\varepsilon_2 = -B$ and $\varepsilon_1 > 6B$), then the operator $\tilde{H}_1$ has a unique eigenvalue $z = A + \varepsilon_1$, lying the below (respectively, above) of the continuous spectrum of operator $\tilde{H}_1$.

2) If $\varepsilon_2 = -B$ and $-6B \le \varepsilon_1 < -2B$ (respectively, $\varepsilon_2 = -B$ and $2B < \varepsilon_1 \le 6B$), then the operator $\tilde{H}_1$ has no eigenvalue, lying the below (respectively, above) of the continuous spectrum of operator $\tilde{H}_1$.

B) If $\varepsilon_2 = -2B$ or $\varepsilon_2 = 0$ and $\varepsilon_1 < 0$, $\varepsilon_1 \le -\frac{6B}{W}$ (respectively, $\varepsilon_2 = 0$ and $\varepsilon_1 > 0$, and $\varepsilon_1 \ge \frac{6B}{W}$), then the operator $\tilde{H}_1$ has a unique eigenvalue z_1 (respectively, z_2), lying the below (respectively, above) of the continuous spectrum of operator $\tilde{H}_1$. If $\varepsilon_2 = 0$ and $\varepsilon_1 < 0$, and $-\frac{6B}{W} \le \varepsilon_1 < 0$ (respectively, $\varepsilon_2 = 0$ and $\varepsilon_1 > 0$, and $0 < \varepsilon_1 \le \frac{6B}{W}$), then the operator $\tilde{H}_1$ has no eigenvalue the outside of the continuous spectrum of operator $\tilde{H}_1$.

C) If $\varepsilon_1 = 0$ and $\varepsilon_2 > 0, E < W$ (respectively, $\varepsilon_1 = 0$ and $\varepsilon_2 < -2B, E < W$), then the operator $\tilde{H}_1$ has a unique eigenvalue z ($\tilde{z}$), where $E = \frac{(B + \varepsilon_2)^2}{\varepsilon_2^2 + 2B\varepsilon_2}$, lying the below (above) of the continuous spectrum of operator $\tilde{H}_1$. If $\varepsilon_1 = 0$ and $\varepsilon_2 > 0$, $E > W$ (respectively, $\varepsilon_1 = 0$ and $\varepsilon_2 < -2B$,

$E > W$), then the operator $\tilde{H}_1$ has no eigenvalues the outside of the continuous spectrum of operator $\tilde{H}_1$.

D) If $\varepsilon_1 = \frac{2(\varepsilon_2^2 + 2B\varepsilon_2)}{B}$ and $E < \frac{4}{3}W$ (respectively, $\varepsilon_1 = -\frac{2(\varepsilon_2^2 + 2B\varepsilon_2)}{B}$, and $E < \frac{4}{3}W$), then the operator $\tilde{H}_1$ has a unique eigenvalue *z*, (respectively, $\tilde{z}$), lying the above (respectively, below) of the continuous spectrum of operator $\tilde{H}_1$.

E) If $\varepsilon_2 > 0$ and $\varepsilon_1 > \frac{2(\varepsilon_2^2 + 2B\varepsilon_2)}{B}$ and $E < \left(1 + \frac{\alpha}{3}\right)W$ (respectively, $\varepsilon_2 < -2B$ and $\varepsilon_1 > \frac{2(\varepsilon_2^2 + 2B\varepsilon_2)}{B}$ and $E < \left(1 + \frac{\alpha}{3}\right)W$), then the operator $\tilde{H}_1$ has a unique eigenvalue *z*, lying the above of the continuous spectrum of operator $\tilde{H}_1$.

F) If $\varepsilon_2 > 0$ and $\varepsilon_1 < -\frac{2(\varepsilon_2^2 + 2B\varepsilon_2)}{B}$ and $E < \left(1 + \frac{\alpha}{3}\right)W$ (respectively, $\varepsilon_2 < -2B$ and $\varepsilon_1 < -\frac{2(\varepsilon_2^2 + 2B\varepsilon_2)}{B}$ and $E < \left(1 + \frac{\alpha}{3}\right)W$), then the operator $\tilde{H}_1$ has a unique eigenvalue z_1, lying the below of the continuous spectrum of operator $\tilde{H}_1$.

K) If $\varepsilon_2 > 0$ and $0 < \varepsilon_1 < \frac{2(\varepsilon_2^2 + 2B\varepsilon_2)}{B}$ and $\left(1 - \frac{\alpha}{3}\right)W < E < \left(1 + \frac{\alpha}{3}\right)W$ (respectively, $\varepsilon_2 < -2B$ and $0 < \varepsilon_1 < \frac{2(\varepsilon_2^2 + 2B\varepsilon_2)}{B}$ and $\left(1 - \frac{\alpha}{3}\right)W < E < \left(1 + \frac{\alpha}{3}\right)W$), then the operator $\tilde{H}_1$ has a exactly two eigenvalues z_1 and z_2, lying the above and below of the continuous spectrum of operator $\tilde{H}_1$.

M) If $\varepsilon_2 > 0$ and $-\frac{2(\varepsilon_2^2 + 2B\varepsilon_2)}{B} < \varepsilon_1 < 0$ and $\left(1 - \frac{\alpha}{3}\right)W < E < \left(1 + \frac{\alpha}{3}\right)W$ (respectively, $\varepsilon_2 < -2B$ and $-\frac{2(\varepsilon_2^2 + 2B\varepsilon_2)}{B} < \varepsilon_1 < 0$ and $\left(1 - \frac{\alpha}{3}\right)W < E < \left(1 + \frac{\alpha}{3}\right)W$), then the operator $\tilde{H}_1$ has a exactly two eigenvalues z_1 and z_2, lying the above and below of the continuous spectrum of operator $\tilde{H}_1$.

N) If $-2B < \varepsilon_2 < 0$, then the operator $\tilde{H}_1$ has no eigenvalues lying the outside of the continuous spectrum of operator $\tilde{H}_1$.

Proof. In the case $\nu = 3$, the continuous spectrum of the operator $\tilde{H}_1$ coincide with segment $[A - 6B, A + 6B]$. Expressing all integrals in the equation

$$\Delta_3(z)=\left(1+\int_{T^3}\frac{\left(\varepsilon_1+2\varepsilon_2\sum_{i=1}^{3}\cos s_i\right)\mathrm{d}s_1\mathrm{d}s_2\mathrm{d}s_3}{A+2B\sum_{i=1}^{3}\cos s_i-z}\right)$$

$$\times\left(1+6\varepsilon_2\int_{T^3}\frac{\cos s_i\mathrm{d}s_1\mathrm{d}s_2\mathrm{d}s_3}{A+2B\sum_{i=1}^{3}\cos s_i-z}\right)$$

through the

$$-6\varepsilon_2\int_{T^3}\frac{\mathrm{d}s_1\mathrm{d}s_2\mathrm{d}s_3}{A+2B\sum_{i=1}^{3}\cos s_i-z}\int_{T^3}\frac{\left(\varepsilon_1+2\varepsilon_2\sum_{i=1}^{3}\cos s_i\right)\cos s_1\mathrm{d}s_1\mathrm{d}s_2\mathrm{d}s_3}{A+2B\sum_{i=1}^{3}\cos s_i-z}$$

$$=0$$

integral $J(z)=\int_{T^3}\frac{\mathrm{d}s_1\mathrm{d}s_2\mathrm{d}s_3}{A+2B\sum_{i=1}^{3}\cos s_i-z}$, we find that the equation $\Delta_3(z)=0$ is equivalent to the equation $\left[\varepsilon_1B^2+\left(\varepsilon_2^2+2B\varepsilon_2\right)(z-A)\right]J(z)+(B+\varepsilon_2)^2=0$.

Moreover, the function $J(z)=\int_{T^3}\frac{\mathrm{d}s_1\mathrm{d}s_2\mathrm{d}s_3}{A+2B\sum_{i=1}^{3}\cos s_i-z}$ is a differentiable function on the set $\mathbb{R}\setminus[m_3,M_3]$, in addition,

$$J'(z)=\int_{T^3}\frac{\mathrm{d}s_1\mathrm{d}s_2\mathrm{d}s_3}{\left[A+2B\sum_{i=1}^{3}\cos s_i-z\right]^2}>0, z\notin[m_3,M_3].$$

In the three-dimensional case, the integral $\int_{T^3}\frac{\mathrm{d}s_1\mathrm{d}s_2\mathrm{d}s_3}{3+\cos s_1+\cos s_2+\cos s_2}=\int_{T^3}\frac{\mathrm{d}s_1\mathrm{d}s_2\mathrm{d}s_3}{3-\cos s_1-\cos s_2-\cos s_2}$ have the finite value. Expressing these integral via Watson integral W, and taking into account, what the measure is normalized, we have, that $J(z)=\frac{W}{6B}$. Thus the function $J(z)$ is an monotone increasing function on $(-\infty,m_3)$ and on $(M_3,+\infty)$. Furthermore, in the three-dimensional case $J(z)\to+0$ at $z\to-\infty$, and $J(z)=\frac{W}{6B}$ as $z=A-6B$, and $J(z)\to-0$ as $z\to+\infty$, and $J(z)=-\frac{W}{6B}$ as $z=A+6B$.

If $\varepsilon_1B^2+\left(\varepsilon_2^2+2B\varepsilon_2\right)(z-A)\neq0$ then from (12) follows that

$$J(z)=-\frac{(B+\varepsilon_2)^2}{\varepsilon_1B^2+\left(\varepsilon_2^2+2B\varepsilon_2\right)(z-A)}.$$

The function $\psi(z)=-\frac{(B+\varepsilon_2)^2}{\varepsilon_1B^2+\left(\varepsilon_2^2+2B\varepsilon_2\right)(z-A)}$ has a point of asymptotic discontinuity $z_0=A-\frac{B^2\varepsilon_1}{\varepsilon_2^2+2B\varepsilon_2}$. Since $\psi'(z)=\frac{(B+\varepsilon_2)^2\left(\varepsilon_2^2+2B\varepsilon_2\right)}{\left[\varepsilon_1B^2+\left(\varepsilon_2^2+2B\varepsilon_2\right)(z-A)\right]^2}$ for all $z\neq z_0$ it follows that the function $\psi(z)$ is an monotone increasing (decreasing) function on $(-\infty,z_0)$ and on $(z_0,+\infty)$ in the case $\varepsilon_2^2+2B\varepsilon_2>0$ (respectively, $\varepsilon_2^2+2B\varepsilon_2<0$), in addition, and if $\varepsilon_2>0$, or $\varepsilon_2<-2B$, then $\psi(z)\to+0$ as $z\to-\infty$, $\psi(z)\to+\infty$ as $z\to z_0-0$, $\psi(z)\to-\infty$ as

$z \to z_0 + 0$, $\psi(z) \to -0$ as $z \to +\infty$ (respectively, if $-2B < \varepsilon_2 < 0$, then $\psi(z) \to -0$ as $z \to -\infty$, $\psi(z) \to -\infty$ as $z \to z_0 - 0$, $\psi(z) \to +\infty$ as $z \to z_0 + 0$, $\psi(z) \to +0$ as $z \to +\infty$).

A) If $\varepsilon_2 = -B$ and $\varepsilon_1 < -6B$ (respectively, $\varepsilon_2 = -B$ and $\varepsilon_1 > 6B$), then the equation for eigenvalues and eigenfunctions (12) has the form: $\{\varepsilon_1 B^2 - B^2(z - A)\} J(z) = 0$. It is clear, that $J(z) \neq 0$ for the values $z \notin \sigma_{cont}(\tilde{H}_1)$. Therefore, $\varepsilon_1 - z + A = 0$, *i.e.*, $z = A + \varepsilon_1$. If $\varepsilon_1 < -6B$, then this eigenvalue lying the below of the continuous spectrum of operator $\tilde{H}_1$, if $\varepsilon_1 > 6B$, then this eigenvalue lying the above of the continuous spectrum of operator $\tilde{H}_1$. If $-6B \leq \varepsilon_1 < -2B$ (respectively, $2B < \varepsilon_1 \leq 6B$,) then this eigenvalue not lying in the outside of the continuous spectrum of operator $\tilde{H}_1$.

B) If $\varepsilon_2 = -2B$ or $\varepsilon_2 = 0$ and $\varepsilon_1 < 0$ (respectively, $\varepsilon_2 = -2B$ or $\varepsilon_2 = 0$ and $\varepsilon_1 > 0$), then the equation for the eigenvalues and eigenfunctions has the form $\varepsilon_1 B^2 J(z) + B^2 = 0$, that is, $J(z) = -\frac{1}{\varepsilon_1}$. The equation $J(z) = -\frac{1}{\varepsilon_1}$ in the below (respectively, above) of continuous spectrum of operator $\tilde{H}_1$ have the solution, one should implements the inequality $-\frac{1}{\varepsilon_1} < \frac{W}{6B}$ (respectively, $-\frac{1}{\varepsilon_1} > -\frac{W}{6B}$), *i.e.*, $\varepsilon_1 < -\frac{6B}{W}$, $\varepsilon_1 < 0$ (respectively, $\varepsilon_1 > \frac{6B}{W}$, $\varepsilon_1 > 0$). If $-\frac{6B}{W} < \varepsilon_1 < 0$ (respectively, $0 < \varepsilon_1 < \frac{6B}{W}$), then the operator $\tilde{H}_1$ has no eigenvalues the outside the continuous spectrum of operator $\tilde{H}_1$.

C) If $\varepsilon_1 = 0$ and $\varepsilon_2 > 0$ (respectively, $\varepsilon_1 = 0$ and $\varepsilon_2 < -2B$), then the equation for the eigenvalues and eigenfunctions take in the form

$(\varepsilon_2^2 + 2B\varepsilon_2)(z - A) J(z) = -(B + \varepsilon_2)^2$, or $J(z) = -\frac{(B + \varepsilon_2)^2}{(\varepsilon_2^2 + 2B\varepsilon_2)(z - A)}$. Denote $E = \frac{(B + \varepsilon_2)^2}{\varepsilon_2^2 + 2B\varepsilon_2}$. Then $J(z) = -\frac{E}{z - A}$, or $J(z) = \frac{E}{A - z}$. The equation $J(z) = -\frac{E}{z - A}$ in the below (respectively, above) of continuous spectrum of operator $\tilde{H}_1$ have the solution, one should implements the inequality $\frac{E}{6B} < \frac{W}{6B}$ (respectively, $-\frac{E}{6B} > -\frac{W}{6B}$), *i.e.*, $E < W$. If $\varepsilon_1 = 0$ and $\varepsilon_2 > 0$, $E > W$ (respectively, $\varepsilon_1 = 0$ and $\varepsilon_2 < -2B$, $E > W$), then the operator $\tilde{H}_1$ has no eigenvalues the outside the continuous spectrum of $\tilde{H}_1$.

D) If $\varepsilon_1 = \frac{2(\varepsilon_2^2 + 2B\varepsilon_2)}{B}$, then the equation for eigenvalues and eigenfunctions has the form $(\varepsilon_2^2 + 2B\varepsilon_2)(z - A + 2B) J(z) = -(B + \varepsilon_2)^2$, from this we have equation in the form (14): $J(z) = -\frac{(B + \varepsilon_2)^2}{(\varepsilon_2^2 + 2B\varepsilon_2)(z - A + 2B)}$. We denote

$E=\frac{(B+\varepsilon_2)^2}{\varepsilon_2^2+2B\varepsilon_2}$. In the first we consider Equation (14) in the below of continuous spectrum of operator $\tilde{H}_1$. In the below of continuous spectrum of operator $\tilde{H}_1$, the function $\frac{E}{A-z-2B}\to+0$, as $z\to-\infty$, $\frac{E}{A-z-2B}=\frac{E}{4B}$, as $z=A-6B$. Therefore, the below of continuous spectrum of operator $\tilde{H}_1$, the equation $J(z)=\frac{E}{A-z-2B}$ has a unique solution, if $\frac{E}{4B}>\frac{W}{6B}$, *i.e.*, $E>\frac{2W}{3}$. This inequality incorrectly. Therefore, the below of continuous spectrum of operator $\tilde{H}_1$, this equation has no solution.

We now consider the equation for eigenvalues and eigenfunctions $J(z)=-\frac{E}{z-A+2B}$, in the above of continuous spectrum of operator $\tilde{H}_1$. In the above of continuous spectrum of operator $\tilde{H}_1$, the function $\frac{E}{A-z-2B}\to-0$, as $z\to+\infty$, $\frac{E}{A-z-2B}=-\frac{E}{8B}$, as $z=A+6B$. Therefore, the above of continuous spectrum of operator $\tilde{H}_1$, the equation $J(z)=\frac{E}{A-z-2B}$ has a unique solution, if $-\frac{E}{8B}>-\frac{W}{6B}$, *i.e.*, $E<\frac{4W}{3}$. This inequality correctly. Therefore, the above of continuous spectrum of operator $\tilde{H}_1$, this equation has a unique solution *z*.

If $\varepsilon_1=-\frac{2(\varepsilon_2^2+2B\varepsilon_2)}{B}$, then the equation for eigenvalues and eigenfunctions has the form $(\varepsilon_2^2+2B\varepsilon_2)(z-A-2B)J(z)=-(B+\varepsilon_2)^2$, from this we have the equation in the form (14): $J(z)=-\frac{(B+\varepsilon_2)^2}{(\varepsilon_2^2+2B\varepsilon_2)(z-A+2B)}$. We denote $E=\frac{(B+\varepsilon_2)^2}{\varepsilon_2^2+2B\varepsilon_2}$. In the first we consider Equation (14) in the below of continuous spectrum of operator $\tilde{H}_1$. In the below of continuous spectrum of operator $\tilde{H}_1$, the function $\frac{E}{A-z+2B}\to+0$, as $z\to-\infty$, $\frac{E}{A-z+2B}=\frac{E}{8B}$, as $z=A-6B$. Therefore, the below of continuous spectrum of operator $\tilde{H}_1$, the equation $J(z)=\frac{E}{A-z+2B}$ has a unique solution, if $\frac{E}{8B}<\frac{W}{6B}$, *i.e.*, $E<\frac{4W}{3}$. This inequality correctly. Therefore, the below of continuous spectrum of operator $\tilde{H}_1$, this equation has a unique solution.

We now consider the equation for eigenvalues and eigenfunctions $J(z)=-\frac{E}{z-A-2B}$, in the above of continuous spectrum of operator $\tilde{H}_1$. In the above of continuous spectrum of operator $\tilde{H}_1$, the function $\frac{E}{A-z+2B}\to-0$, as $z\to+\infty$, $\frac{E}{A-z+2B}=-\frac{E}{4B}$, as $z=A+6B$. Therefore, the above of continuous spectrum of operator $\tilde{H}_1$, the equation $J(z)=\frac{E}{A-z+2B}$ has a unique solution, if $-\frac{E}{4B}>-\frac{W}{6B}$, *i.e.*, $E<\frac{2W}{3}$. This inequality incorrectly. Therefore, the above of continuous spectrum of operator $\tilde{H}_1$, this equation has no solution.

E) If $\varepsilon_2>0$ and $\varepsilon_1>\frac{2(\varepsilon_2^2+2B\varepsilon_2)}{B}$ (respectively, $\varepsilon_2<-2B$ and $\varepsilon_1>\frac{2(\varepsilon_2^2+2B\varepsilon_2)}{B}$), then consider necessary, that $\varepsilon_1=\alpha\times\frac{2(\varepsilon_2^2+2B\varepsilon_2)}{B}$, where $\alpha>1-$ real number. Then the equation for eigenvalues and eigenfunctions has the form $(\varepsilon_2^2+2B\varepsilon_2)(z-A+2\alpha B)J(z)+(B+\varepsilon_2)^2=0$. From this $J(z)=-\frac{(B+\varepsilon_2)^2}{(\varepsilon_2^2+2B\varepsilon_2)(z-A+2\alpha B)}$. We denote $E=\frac{(B+\varepsilon_2)^2}{\varepsilon_2^2+2B\varepsilon_2}$, then $J(z)=-\frac{E}{z-A+2\alpha B}$. In the first we consider this equation in the below of the continuous spectrum of operator $\tilde{H}_1$. Then $J(z)\to+0$, as $z\to-\infty$, $J(z)=\frac{W}{6B}$, as $z=A-6B$, $-\frac{E}{z-A+2\alpha B}\to+0$, as $z\to-\infty$, and $-\frac{E}{z-A+2\alpha B}=\frac{E}{(6-2\alpha)B}$, as $z=A-6B$. The equation $J(z)=-\frac{E}{z-A+2\alpha B}$ have a unique solution, if $\frac{E}{(6-2\alpha)B}<\frac{W}{6B}$. From here $E<\frac{(3-\alpha)W}{3}$. This inequality is incorrect. Therefore, the below of continuous spectrum of operator $\tilde{H}_1$, the operator $\tilde{H}_1$ has no eigenvalues.

The above of continuous spectrum of operator $\tilde{H}_1$, we have the $J(z)\to-0$, if $z\to+\infty$, $J(z)=-\frac{W}{6B}$, if $z=A-6B$. Besides, $-\frac{E}{z-A+2\alpha B}\to-0$, as $z\to+\infty$, $-\frac{E}{z-A+2\alpha B}=-\frac{E}{6B+2\alpha B}$, if $z=A+6B$.

The equation $J(z)=-\frac{E}{z-A+2\alpha B}$ have a unique solution, if $-\frac{E}{(6+2\alpha)B}>-\frac{W}{6B}$. From here $E<\frac{(3+\alpha)W}{3}$. This inequality is correctly. Therefore, the above of continuous spectrum of operator $\tilde{H}_1$, the operator $\tilde{H}_1$ has a unique eigenvalues z_1.

F) If $\varepsilon_2>0$, and $\varepsilon_1<-\frac{2(\varepsilon_2^2+2B\varepsilon_2)}{B}$ (respectively, $\varepsilon_2<-2B$, and $\varepsilon_1<-\frac{2(\varepsilon_2^2+2B\varepsilon_2)}{B}$), then we assume that $\varepsilon_1=-\alpha\times\frac{2(\varepsilon_2^2+2B\varepsilon_2)}{B}$, where $\alpha>1-$ real number. The equation for eigenvalues and eigenfunctions take in the form $(\varepsilon_2^2+2B\varepsilon_2)(z-A-2\alpha B)J(z)=-(B+\varepsilon_2)^2$. From here

$$J(z) = -\frac{(B+\varepsilon_2)^2}{(\varepsilon_2^2 + 2B\varepsilon_2)(z - A - 2\alpha B)}.$$

The introduce notation $E = \frac{(B+\varepsilon_2)^2}{\varepsilon_2^2 + 2B\varepsilon_2}$. Then we have the equation in the form (15): $J(z) = -\frac{E}{z - A - 2\alpha B}$. In the below of the continuous spectrum of operator $\tilde{H}_1$, we have the equation $J(z) = \frac{E}{A - z + 2\alpha B}$. In the below of continuous spectrum of operator $\tilde{H}_1$, $-\frac{E}{z - A - 2\alpha B} \to +0$, as $z \to -\infty$, $-\frac{E}{z - A - 2\alpha B} = \frac{E}{6B + 2\alpha B}$, as $z = A - 6B$.

The equation $J(z) = -\frac{E}{z - A + 2\alpha B}$ have a unique solution, if $\frac{E}{(6+2\alpha)B} < \frac{W}{6B}$. From here $E < \frac{(3+\alpha)W}{3}$. This inequality is correctly. Therefore, the below of continuous spectrum of operator $\tilde{H}_1$, the operator $\tilde{H}_1$ has a unique eigenvalues.

In the above of continuous spectrum of operator $\tilde{H}_1$, $-\frac{E}{z - A - 2\alpha B} \to -0$, as $z \to +\infty$, $-\frac{E}{z - A - 2\alpha B} = -\frac{E}{6B - 2\alpha B}$, as $z = A + 6B$. Therefore, the above of continuous spectrum of operator $\tilde{H}_1$, the operator $\tilde{H}_1$ has a unique eigenvalues, if $-\frac{E}{6B - 2\alpha B} > -\frac{W}{6B}$. From here $E < \frac{(3-\alpha)W}{3}$, what is incorrectly. Therefore, the above of continuous spectrum of operator $\tilde{H}_1$, the operator $\tilde{H}_1$ has no eigenvalues.

K) If $\varepsilon_2 > 0$ and $0 < \varepsilon_1 < \frac{2(\varepsilon_2^2 + 2B\varepsilon_2)}{B}$ (respectively, $\varepsilon_2 < -2B$ and $0 < \varepsilon_1 < \frac{2(\varepsilon_2^2 + 2B\varepsilon_2)}{B}$), the we take $\varepsilon_1 = \alpha \times \frac{2(\varepsilon_2^2 + 2B\varepsilon_2)}{B}$, where $0 < \alpha < 1-$ positive real number. Then the equation for eigenvalues and eigenfunctions has the form (16): $(\varepsilon_2^2 + 2B\varepsilon_2)(z - A + 2\alpha B)J(z) = -(B+\varepsilon_2)^2$, $0 < \alpha < 1$.

We denote $E = \frac{(B+\varepsilon_2)^2}{\varepsilon_2^2 + 2B\varepsilon_2}$. Then Equation (16) receive the form

$$J(z) = -\frac{E}{z - A + 2\alpha B}.$$

In the below of continuous spectrum of operator $\tilde{H}_1$, we have $-\frac{E}{z - A + 2\alpha B} \to +0$, as $z \to -\infty$, and $-\frac{E}{z - A + 2\alpha B} = \frac{E}{2B(3-\alpha)}$, as $z = A - 6B$. The equation $J(z) = -\frac{E}{z - A + 2\alpha B}$ have a unique solution the below of continuous spectrum of operator $\tilde{H}_1$, if $\frac{E}{(6-2\alpha)B} > \frac{W}{6B}$. From here $E > \frac{(3-\alpha)W}{3}$. This inequality is correctly. Therefore, the below of continuous spectrum of operator $\tilde{H}_1$, the operator $\tilde{H}_1$ has a unique eigenvalues z_1.

The above of continuous spectrum of operator $\tilde{H}_1$, we have $-\frac{E}{z-A+2\alpha B} \to -0$, as $z \to +\infty$, and $-\frac{E}{z-A+2\alpha B} = -\frac{E}{2B(3+\alpha)}$, as $z = A+6B$. The equation $J(z) = -\frac{E}{z-A+2\alpha B}$ have a unique solution the above of operator $\tilde{H}_1$, if $-\frac{E}{2B(3+\alpha)} > -\frac{W}{6B}$, *i.e.*, $E < \frac{(3+\alpha)W}{3}$. This inequality is correctly.

Consequently, in this case the operator $\tilde{H}_1$ have two eigenvalues z_1 and z_2, lying the below and above of continuous spectrum of operator $\tilde{H}_1$.

M) If $\varepsilon_2 > 0$ and $-\frac{2(\varepsilon_2^2 + 2B\varepsilon_2)}{B} < \varepsilon_1 < 0$ (respectively, $\varepsilon_2 < -2B$ and $-\frac{2(\varepsilon_2^2 + 2B\varepsilon_2)}{B} < \varepsilon_1 < 0$), the we take $\varepsilon_1 = -\alpha \times \frac{2(\varepsilon_2^2 + 2B\varepsilon_2)}{B}$, where $0 < \alpha < 1-$ positive real number. Then the equation for eigenvalues and eigenfunctions has the form (16): $(\varepsilon_2^2 + 2B\varepsilon_2)(z-A-2\alpha B)J(z) = -(B+\varepsilon_2)^2$, $0 < \alpha < 1$.

We denote $E = \frac{(B+\varepsilon_2)^2}{\varepsilon_2^2 + 2B\varepsilon_2}$. Then the Equation (16) receive the form

$$J(z) = -\frac{E}{z-A-2\alpha B}.$$

In the below of continuous spectrum of operator $\tilde{H}_1$, we have $-\frac{E}{z-A-2\alpha B} \to +0$, as $z \to -\infty$, and $-\frac{E}{z-A-2\alpha B} = \frac{E}{2B(3+\alpha)}$, as $z = A-6B$. The equation $J(z) = -\frac{E}{z-A-2\alpha B}$ have a unique solution the below of continuous spectrum of operator $\tilde{H}_1$, if $\frac{E}{(6+2\alpha)B} < \frac{W}{6B}$. From here $E < \frac{(3+\alpha)W}{3}$. This inequality is correctly. Therefore, the below of continuous spectrum of operator $\tilde{H}_1$, the operator $\tilde{H}_1$ has a unique eigenvalues z_1.

The above of continuous spectrum of operator $\tilde{H}_1$, we have $-\frac{E}{z-A-2\alpha B} \to -0$, as $z \to +\infty$, and $-\frac{E}{z-A-2\alpha B} = -\frac{E}{2B(3-\alpha)}$, as $z = A+6B$. The equation $J(z) = -\frac{E}{z-A-2\alpha B}$ have a unique solution the above of continuous spectrum of operator $\tilde{H}_1$, if $-\frac{E}{2B(3-\alpha)} < -\frac{W}{6B}$, *i.e.*, $E > \frac{(3-\alpha)W}{3}$. This inequality is correctly.

Consequently, in this case the operator $\tilde{H}_1$ have two eigenvalues z_1 and z_2, lying the below and above of continuous spectrum of operator $\tilde{H}_1$.

N) If $-2B < \varepsilon_2 < 0$, then $\varepsilon_2^2 + 2B\varepsilon_2 < 0$, and the function $\psi(z) = -\frac{(B+\varepsilon_2)^2}{\varepsilon_1 B + (\varepsilon_2^2 + 2B\varepsilon_2)(z-A)}$ is a decreasing function in the intervals $(-\infty, z_0)$ and $(z_0, +\infty)$; by, $z \to -\infty$ the function $\psi(z) \to -0$, and by $z \to z_0 - 0$, the function $\psi(z) \to -\infty$, and by $z \to +\infty$, $\psi(z) \to +0$, and by $z \to z_0 + 0$, $\psi(z) \to +\infty$. The function $J(z) \to +0$, by $z \to -\infty$, and by $z = A - 6B$, the function $J(z) = \frac{W}{6B}$, and by $z = A + 6B$, the function $J(z) = -\frac{W}{6B}$, by $z \to +\infty$, the function $J(z) \to -0$. Therefore, the equation $\psi(z) = J(z)$, that's impossible the solutions in the outside the continuous spectrum of operator $\tilde{H}_1$. Therefore, in this case, the operator $\tilde{H}_1$ has no eigenvalues lying the outside of the continuous spectrum of the operator $\tilde{H}_1$. □

From obtaining results is obviously, that the spectrum of operator $\tilde{H}_1$ is consists from continuous spectrum and no more than two eigenvalues. In turn the operator $\tilde{H}_2^t$ is represented in the form

$$\tilde{H}_2^t = \tilde{H}_1 \otimes I + I \otimes \tilde{H}_1, \tag{17}$$

where I is the unit operator in the space $\tilde{\mathcal{H}}_1$

The spectrum of the operator $A \otimes I + I \otimes B$, where A and B are densely defined bounded linear operators, was studied in [16] [17] [18]. In this work explicit formulas were given there that express the essential spectrum $\sigma_{ess}(A \otimes I + I \otimes B)$ of $A \otimes I + I \otimes B$ and the discrete spectrum $\sigma_{disc}(A \otimes I + I \otimes B)$ in terms of the spectrum $\sigma(A)$ of A and the discrete spectrum $\sigma_{disc}(A)$ of A and in terms of the spectrum $\sigma(B)$ of B and the discrete spectrum $\sigma_{disc}(B)$ of B.

4. Structure of the Essential Spectrum and Discrete Spectrum of Operator $\tilde{H}_2^t$

Now, using the obtained results (Theorem 8) and representation (17), we describe the structure of the essential spectrum and the discrete spectrum of the operator $\tilde{H}_2^t$.

Theorem 10. *Let* $\nu = 1$. *Then*

A) If $\varepsilon_2 = -B$ and $\varepsilon_1 < -2B$ (respectively, $\varepsilon_2 = -B$ and $\varepsilon_1 > 2B$), then the essential spectrum of the operator $\tilde{H}_2^t$ is consists of the union of two segments: $\sigma_{ess}(\tilde{H}_2^t) = [2A-4B, 2A+4B] \cup [A-2B+z, A+2B+z]$, and discrete spectrum of the operator $\tilde{H}_2^t$ is consists of a single point: $\sigma_{disc}\left(\tilde{H}_2^t\right)=\{2z\}$, where $z=A+\varepsilon_1$, lying the below (above) of the essential spectrum of operator $\tilde{H}_2^t$.

B) If $\varepsilon_2=-2B$ or $\varepsilon_2=0$ and $\varepsilon_1<0$ (respectively, $\varepsilon_2=-2B$ or $\varepsilon_2=0$ and $\varepsilon_1>0$), then the essential spectrum of the operator $\tilde{H}_2^t$ is consists of the union of two segments: $\sigma_{ess}\left(\tilde{H}_2^s\right)=[2A-4B,2A+4B]\cup[A-2B+z,A+2B+z]$, and discrete spectrum of the operator $\tilde{H}_2^t$ is consists of a single point: $\sigma_{disc}\left(\tilde{H}_2^t\right)=\{2z\}$, where $z=A-\sqrt{4B^2+\varepsilon_1^2}$ (respectively, $z=A+\sqrt{4B^2+\varepsilon_1^2}$), lying the below (above) of the essential spectrum of operator $\tilde{H}_2^t$.

C) If $\varepsilon_1=0$ and $\varepsilon_2>0$ or $\varepsilon_1=0$ and $\varepsilon_2<-2B$, then the essential spectrum of the operator $\tilde{H}_2^t$ is consists of the union of three segments:

$$\sigma_{ess}\left(\tilde{H}_2^t\right)=[2A-4B,2A+4B]\cup[A-2B+z_1,A+2B+z_1]\cup[A-2B+z_2,A+2B+z_2]$$

, and discrete spectrum of the operator $\tilde{H}_2^t$ is consists of a two point: $\sigma_{disc}\left(\tilde{H}_2^t\right)=\{2z_1,2z_2\}$, where $z_1=A-\frac{2BE}{\sqrt{E^2-1}}$, and $z_2=A+\frac{2BE}{\sqrt{E^2-1}}$, and $E=\frac{(B+\varepsilon_2)^2}{\varepsilon_2^2+2B\varepsilon_2}$, in this case the eigenvalue z_1+z_2, lying the essential spectrum of operator $\tilde{H}_2^t$.

D) If $\varepsilon_1=\frac{2\left(\varepsilon_2^2+2B\varepsilon_2\right)}{B}$ (respectively, $\varepsilon_1=-\frac{2\left(\varepsilon_2^2+2B\varepsilon_2\right)}{B}$), then the essential spectrum of the operator $\tilde{H}_2^t$ is consists of the union of two segments: $\sigma_{ess}\left(\tilde{H}_2^s\right)=[2A-4B,2A+4B]\cup[A-2B+z,A+2B+z]$, and discrete spectrum of the operator $\tilde{H}_2^t$ is consists of a single point: $\sigma_{disc}\left(\tilde{H}_2^t\right)=\{2z\}$, where $z=A+\frac{2B\left(E^2+1\right)}{E^2-1}$ (respectively, $z=A-\frac{2B\left(E^2+1\right)}{E^2-1}$), and $E=\frac{(B+\varepsilon_2)^2}{\varepsilon_2^2+2B\varepsilon_2}$, lying the above (below) of the essential spectrum of operator $\tilde{H}_2^t$.

E) If $\varepsilon_2>0$ and $\varepsilon_1>\frac{2\left(\varepsilon_2^2+2B\varepsilon_2\right)}{B}$ (respectively, $\varepsilon_2<-2B$ and $\varepsilon_1>\frac{2\left(\varepsilon_2^2+2B\varepsilon_2\right)}{B}$), then the essential spectrum of the operator $\tilde{H}_2^t$ is consists of the union of two segments:
$\sigma_{ess}\left(\tilde{H}_2^s\right)=[2A-4B,2A+4B]\cup[A-2B+z,A+2B+z]$, and discrete spectrum of the operator $\tilde{H}_2^t$ is consists of a single point: $\sigma_{disc}\left(\tilde{H}_2^t\right)=\{2z\}$, where

$$z=A+\frac{2B\left(\alpha+E\sqrt{E^2-1+\alpha^2}\right)}{E^2-1}$$ and $E=\frac{(B+\varepsilon_2)^2}{\varepsilon_2^2+2B\varepsilon_2}$, and real number $\alpha>1$, lying the above of the essential spectrum of operator $\tilde{H}_2^t$.

F) If $\varepsilon_2>0$ and $\varepsilon_1<-\frac{2\left(\varepsilon_2^2+2B\varepsilon_2\right)}{B}$ (respectively, $\varepsilon_2<-2B$ and $\varepsilon_1<-\frac{2\left(\varepsilon_2^2+2B\varepsilon_2\right)}{B}$), then the essential spectrum of the operator $\tilde{H}_2^t$ is con-

sists of the union of two segments:
$\sigma_{ess}\left(\tilde{H}_2^s\right)=[2A-4B,2A+4B]\cup[A-2B+z,A+2B+z]$, and discrete spectrum of the operator $\tilde{H}_2^t$ is consists of a single point: $\sigma_{disc}\left(\tilde{H}_2^t\right)=\{2z\}$, where $z=A-\frac{2B\left(\alpha+E\sqrt{E^2-1+\alpha^2}\right)}{E^2-1}$ and $E=\frac{(B+\varepsilon_2)^2}{\varepsilon_2^2+2B\varepsilon_2}$, and the real number $\alpha>1$, lying the below of the essential spectrum of operator $\tilde{H}_2^t$.

K) If $\varepsilon_2>0$ and $0<\varepsilon_1<\frac{2\left(\varepsilon_2^2+2B\varepsilon_2\right)}{B}$ (respectively, $\varepsilon_2<-2B$ and $0<\varepsilon_1<\frac{2\left(\varepsilon_2^2+2B\varepsilon_2\right)}{B}$), then the essential spectrum of the operator $\tilde{H}_2^t$ is consists of the union of three segments:
$\sigma_{ess}\left(\tilde{H}_2^s\right)=[2A-4B,2A+4B]\cup[A-2B+z_1,A+2B+z_1]\cup[A-2B+z_2,A+2B+z_2]$, and discrete spectrum of the operator $\tilde{H}_2^t$ is consists of a three points: $\sigma_{disc}\left(\tilde{H}_2^t\right)=\{2z_1,z_1+z_2,2z_2\}$, where $z_1=A+\frac{2B\left(\alpha+E\sqrt{E^2-1+\alpha^2}\right)}{E^2-1}$ and $z_2=A+\frac{2B\left(\alpha-E\sqrt{E^2-1+\alpha^2}\right)}{E^2-1}$, and $E=\frac{(B+\varepsilon_2)^2}{\varepsilon_2^2+2B\varepsilon_2}$, and the real number $0<\alpha<1$, lying the outside the essential spectrum of operator $\tilde{H}_2^t$.

M) If $\varepsilon_2>0$ and $-\frac{2\left(\varepsilon_2^2+2B\varepsilon_2\right)}{B}<\varepsilon_1<0$ (respectively, $\varepsilon_2<-2B$ and $-\frac{2\left(\varepsilon_2^2+2B\varepsilon_2\right)}{B}<\varepsilon_1<0$), then the essential spectrum of the operator $\tilde{H}_2^t$ is consists of the union of three segments:
$\sigma_{ess}\left(\tilde{H}_2^s\right)=[2A-4B,2A+4B]\cup[A-2B+z_1,A+2B+z_1]\cup[A-2B+z_2,A+2B+z_2]$, and discrete spectrum of the operator $\tilde{H}_2^t$ is consists of a three points: $\sigma_{disc}\left(\tilde{H}_2^t\right)=\{2z_1,z_1+z_2,2z_2\}$, where $z_1=A+\frac{2B\left(\alpha+E\sqrt{E^2-1+\alpha^2}\right)}{E^2-1}$ and $z_2=A+\frac{2B\left(\alpha-E\sqrt{E^2-1+\alpha^2}\right)}{E^2-1}$, and $E=\frac{(B+\varepsilon_2)^2}{\varepsilon_2^2+2B\varepsilon_2}$, and the real number $0<\alpha<1$, lying the outside the essential spectrum of $\tilde{H}_2^t$.

N) If $-2B<\varepsilon_2<0$, then the the essential spectrum of the operator $\tilde{H}_2^t$ is consists is a single segment: $\sigma_{ess}\left(\tilde{H}_2^s\right)=[2A-4B,2A+4B]$, and discrete spectrum of the operator $\tilde{H}_2^t$ is empty set: $\sigma_{disc}\left(\tilde{H}_2^t\right)=\varnothing$.

Proof. 1) It follows from representation (17), and from Theorem 8, that in one-dimensional case, the continuous spectrum of the operator $\tilde{H}_1$ is consists from $\sigma_{cont}\left(\tilde{H}_1\right)=[A-2B,A+2B]$, and discrete spectrum of the operator $\tilde{H}_1$

is consists of unique eigenvalue z. Therefore, the essential spectrum of the operator $\tilde{H}_1 \otimes I + I \otimes \tilde{H}_1$ is consists of the union of two segments $[2A-4B, 2A+4B]$ and $[A-2B+z, A+2B+z]$, and discrete spectrum of the operator $\tilde{H}_1 \otimes I + I \otimes \tilde{H}_1$ consists is the unique point $2z$. These is given to the proof of statement A) from Theorem 10.

The statements B), C), D), E), F), K) from Theorem 10 are proved similarly.

We now is proved the statement M) from Theorem 10. It can be seen from Theorem 8 (statement M) in one-dimensional case the operator $\tilde{H}_1$ has exactly two eigenvalues z_1 and z_2 outside the domain of the continuous spectrum of operator $\tilde{H}_1$. Therefore,

$$\sigma_{ess}\left(\tilde{H}_2^t\right) = [2A-4B, 2A+4B] \cup [A-2B+z_1, A+2B+z_1] \cup [A-2B+z_2, A+2B+z_2]$$

. The numbers $2z_1$, $2z_2$ and z_1+z_2, are eigenvalues of operator $\tilde{H}_1 \otimes I + I \otimes \tilde{H}_1$. These is given to the proof of statement M).

We now is proved the statement N) from Theorem 10. It can be seen from Theorem 8 (statement N) in one-dimensional case the operator $\tilde{H}_1$ has no eigenvalues the outside the continuous spectrum of operator $\tilde{H}_1$. Therefore, $\sigma_{ess}\left(\tilde{H}_2^t\right) = [2A-4B, 2A+4B]$, and the operator also has no eigenvalues, *i.e.*, $\sigma_{disc}\left(\tilde{H}_1 \otimes I + I \otimes \tilde{H}_1\right) = \varnothing$. □

The next theorems described the structure of essential spectrum of the operator $\tilde{H}_2^t$ in a three-dimensional case.

Theorem 11. *Let* $\nu = 3$. *Then*

A) 1) If $\varepsilon_2 = -B$ and $\varepsilon_1 < -6B$ (respectively, $\varepsilon_2 = -B$ and $\varepsilon_1 > 6B$), then the essential spectrum of the operator $\tilde{H}_2^t$ is consists of the union of two segments: $\sigma_{ess}\left(\tilde{H}_2^t\right) = [2A-4B, 2A+4B] \cup [A-2B+z, A+2B+z]$, and discrete spectrum of the operator $\tilde{H}_2^t$ is consists of a single point: $\sigma_{disc}\left(\tilde{H}_2^t\right) = \{2z\}$, where $z = A + \varepsilon_1$.

2) If $\varepsilon_2 = -B$ and $-6B \le \varepsilon_1 < -2B$ (respectively, $\varepsilon_2 = -B$ and $2B < \varepsilon_1 \le 6B$), then the essential spectrum of the operator $\tilde{H}_2^t$ is consists of a single segment: $\sigma_{ess}\left(\tilde{H}_2^t\right) = [2A-4B, 2A+4B]$, and discrete spectrum of the operator $\tilde{H}_2^t$ is empty set.

B) If $\varepsilon_2 = -2B$ or $\varepsilon_2 = 0$ and $\varepsilon_1 < 0$, $\varepsilon_1 \le -\frac{6B}{W}$ (respectively, $\varepsilon_2 = -2B$ or $\varepsilon_2 = 0$ and $\varepsilon_1 > 0$, $\varepsilon_1 \ge \frac{6B}{W}$), then the essential spectrum of the operator $\tilde{H}_2^t$ is consists of the union of two segments:

$\sigma_{ess}\left(\tilde{H}_2^t\right) = [2A-4B, 2A+4B] \cup [A-2B+z, A+2B+z]$, and discrete spectrum of the operator $\tilde{H}_2^t$ is consists of a single point: $\sigma_{disc}\left(\tilde{H}_2^t\right) = \{2z\}$, where z are the eigenvalue of operator $\tilde{H}_1$.

If $\varepsilon_2 = -2B$ or $\varepsilon_2 = 0$ and $\varepsilon_1 < 0$, and $-\frac{6B}{W} \le \varepsilon_1 < 0$ (respectively,

$\varepsilon_2 = -2B$ or $\varepsilon_2 = 0$ and $\varepsilon_1 > 0$, and $0 < \varepsilon_1 \le \frac{6B}{W}$), then the essential spectrum of the operator $\tilde{H}_2^t$ is consists of a single segment:

$\sigma_{ess}\left(\tilde{H}_2^t\right) = \left[2A-4B, 2A+4B\right]$, and discrete spectrum of the operator $\tilde{H}_2^t$ is empty set: $\sigma_{disc}\left(\tilde{H}_2^t\right) = \varnothing$.

C) If $\varepsilon_1 = 0$ and $\varepsilon_2 > 0$, $E < W$ (respectively, $\varepsilon_1 = 0$ and $\varepsilon_2 < -2B$, $E < W$), then the essential spectrum of the operator $\tilde{H}_2^t$ is consists of the union of two segments: $\sigma_{ess}\left(\tilde{H}_2^t\right) = \left[2A-4B, 2A+4B\right] \cup \left[A-2B+z, A+2B+z\right]$, and discrete spectrum of the operator $\tilde{H}_2^t$ is consists of a single point: $\sigma_{disc}\left(\tilde{H}_2^t\right) = \{2z\}$, where z is the eigenvalue of operator $\tilde{H}_1$, and $E = \frac{(B+\varepsilon_2)^2}{\varepsilon_2^2 + 2B\varepsilon_2}$. If $\varepsilon_1 = 0$ and $\varepsilon_2 > 0$, $E > W$, (respectively, $\varepsilon_1 = 0$ and $\varepsilon_2 < -2B$, $E > W$), then the essential spectrum of the operator $\tilde{H}_2^t$ is consists of a single segment: $\sigma_{ess}\left(\tilde{H}_2^t\right) = \left[2A-4B, 2A+4B\right]$, and discrete spectrum of the operator $\tilde{H}_2^t$ is empty set: $\sigma_{disc}\left(\tilde{H}_2^t\right) = \varnothing$.

D) If $\varepsilon_1 = \frac{2\left(\varepsilon_2^2 + 2B\varepsilon_2\right)}{B}$ and $E < \frac{4}{3}W$ (respectively, $\varepsilon_1 = -\frac{2\left(\varepsilon_2^2 + 2B\varepsilon_2\right)}{B}$ and $E < \frac{4}{3}W$), then the essential spectrum of the operator $\tilde{H}_2^t$ is consists of the union of two segments:

$\sigma_{ess}\left(\tilde{H}_2^t\right) = \left[2A-4B, 2A+4B\right] \cup \left[A-2B+z, A+2B+z\right]$, and discrete spectrum of the operator $\tilde{H}_2^t$ is consists of a single point: $\sigma_{disc}\left(\tilde{H}_2^t\right) = \{2z\}$, where z is the eigenvalue of operator $\tilde{H}_1$.

E) If $\varepsilon_2 > 0$ and $\varepsilon_1 > \frac{2\left(\varepsilon_2^2 + 2B\varepsilon_2\right)}{B}$ and $E < \left(1 + \frac{\alpha}{3}\right)W$ (respectively, $\varepsilon_2 < -2B$ and $\varepsilon_1 > \frac{2\left(\varepsilon_2^2 + 2B\varepsilon_2\right)}{B}$ and $E < \left(1 + \frac{\alpha}{3}\right)W$), then the essential spectrum of the operator $\tilde{H}_2^t$ is consists of the union of two segments:

$\sigma_{ess}\left(\tilde{H}_2^t\right) = \left[2A-4B, 2A+4B\right] \cup \left[A-2B+z, A+2B+z\right]$, and discrete spectrum of the operator $\tilde{H}_2^t$ is consists of a single point: $\sigma_{disc}\left(\tilde{H}_2^t\right) = \{2z\}$, where z is the eigenvalue of $\tilde{H}_1$.

F) If $\varepsilon_2 > 0$ and $\varepsilon_1 < -\frac{2\left(\varepsilon_2^2 + 2B\varepsilon_2\right)}{B}$ and $E < \left(1 + \frac{\alpha}{3}\right)W$ (respectively, $\varepsilon_2 < -2B$ and $\varepsilon_1 < -\frac{2\left(\varepsilon_2^2 + 2B\varepsilon_2\right)}{B}$ and $E < \left(1 + \frac{\alpha}{3}\right)W$), then the essential spectrum of the operator $\tilde{H}_2^t$ is consists of the union of two segments:

$\sigma_{ess}\left(\tilde{H}_2^t\right) = \left[2A-4B, 2A+4B\right] \cup \left[A-2B+z_1, A+2B+z_1\right]$, and discrete spectrum of the operator $\tilde{H}_2^t$ is consists of a single point: $\sigma_{disc}\left(\tilde{H}_2^t\right)=\{2z_1\}$, where z_1 is the eigenvalue of $\tilde{H}_1$.

K) If $\varepsilon_2>0$ and $0<\varepsilon_1<\frac{2\left(\varepsilon_2^2+2B\varepsilon_2\right)}{B}$, and $\left(1-\frac{\alpha}{3}\right)W<E<\left(1+\frac{\alpha}{3}\right)W$ (respectively, $\varepsilon_2<-2B$ and $0<\varepsilon_1<\frac{2\left(\varepsilon_2^2+2B\varepsilon_2\right)}{B}$, and $\left(1-\frac{\alpha}{3}\right)W<E<\left(1+\frac{\alpha}{3}\right)W$), then the essential spectrum of the operator $\tilde{H}_2^t$ is consists of the union of three segments:

$$\sigma_{ess}\left(\tilde{H}_2^t\right)=[2A-4B,2A+4B]\cup[A-2B+z_1,A+2B+z_1]\cup[A-2B+z_2,A+2B+z_2]$$

, and discrete spectrum of $\tilde{H}_2^t$ is consists of a three point: $\sigma_{disc}\left(\tilde{H}_2^t\right)=\{2z_1,z_1+z_2,2z_2\}$, where z_1 and z_2, are the eigenvalues of $\tilde{H}_1$.

M) If $\varepsilon_2>0$ and $-\frac{2\left(\varepsilon_2^2+2B\varepsilon_2\right)}{B}<\varepsilon_1<0$ and $\left(1-\frac{\alpha}{3}\right)W<E<\left(1+\frac{\alpha}{3}\right)W$ (respectively, $\varepsilon_2<-2B$ and $-\frac{2\left(\varepsilon_2^2+2B\varepsilon_2\right)}{B}<\varepsilon_1<0$, and $\left(1-\frac{\alpha}{3}\right)W<E<\left(1+\frac{\alpha}{3}\right)W$), then the essential spectrum of the operator $\tilde{H}_2^t$ is consists of the union of three segments:

$$\sigma_{ess}\left(\tilde{H}_2^t\right)=[2A-4B,2A+4B]\cup[A-2B+z_1,A+2B+z_1]\cup[A-2B+z_2,A+2B+z_2]$$

, and discrete spectrum of the operator $\tilde{H}_2^t$ is consists of a three point: $\sigma_{disc}\left(\tilde{H}_2^t\right)=\{2z_1,z_1+z_2,2z_2\}$, where z_1 and z_2, are the eigenvalues of operator $\tilde{H}_1$.

N) If $-2B<\varepsilon_2<0$, then the essential spectrum of the operator $\tilde{H}_2^t$ is consists of a single segment $\sigma_{ess}\left(\tilde{H}_2^t\right)=[2A-4B,2A+4B]$, and discrete spectrum of the operator $\tilde{H}_2^t$ is empty set: $\sigma_{disc}\left(\tilde{H}_2^t\right)=\varnothing$.

Proof. The proof of Theorems 11 is similar to the proof of Theorems 10.

5. Structure of Essential Spectrum and Discrete Spectrum of the Operator $\tilde{H}_2^s$

The singlet state corresponds two-electron bound states (or antibound states) to the basis functions: $s_{m,n}=\frac{1}{\sqrt{2}}\left\{a_{m\uparrow}^{+}a_{n\downarrow}^{+}-a_{m\downarrow}^{+}a_{n\uparrow}^{+}\right\}\varphi_0$. The subspace $\tilde{H}_2^s$, corresponding to the singlet state is the set of all vectors of the form $\psi=\sum_{m,n\in Z^\nu}\tilde{f}(m,n)s_{m,n}$, $\tilde{f}\in l_2^s$, where l_2^s is the subspace of symmetric functions in $l_2\left(\left(Z^\nu\right)^2\right)$.

In this case, the Hamiltonian H acts in the symmetric Fock space $\mathcal{H}_s$. Let φ_0

be the vacuum vector in the symmetric Fock space $\mathcal{H}_s$. The singlet state corresponds the free motions of two-electrons in the lattice and their interactions.

Theorem 12. *The subspace $\mathcal{H}_2^s$ is invariant under the operator H, and the restriction H_2^s of operator H to the subspace $\mathcal{H}_2^s$ is a bounded self-adjoint operator. It generates a bounded self-adjoint operator $\bar{H}_2^s$, acting in the space l_2^s as*

$$\begin{aligned}\left(\bar{H}_2^s f\right)(p,q) &= 2Af(p,q) + 2B\sum_{\tau}\left[f(p+\tau,q) + f(p,q+\tau)\right] \\ &+ U\delta_{p,q}f(p,q) + (A_0 - A)\left[\delta_{p,0} + \delta_{q,0}\right]f(p,q) \\ &+ (B_0 - B)\sum_{\tau}\left[\delta_{p,\tau}f(0,q) + \delta_{q,\tau}f(p,0) + \delta_{p,0}f(\tau,q)\right. \\ &\left.+ \delta_{q,0}f(p,\tau)\right] + (U_0 - U)\delta_{p,0}\delta_{q,0}f(p,q),\end{aligned} \tag{18}$$

where $\delta_{k,j}$ Kronecker symbol. The operator H_2^s acts on a vector $\psi \in \mathcal{H}_2^s$ as

$$H_2^s\psi = \sum_{p,q}\left(\bar{H}_2^s f\right)(p,q)s_{p,q}. \tag{19}$$

Proof. The proof follows from describe of the acts Hamiltonian H on vectors $\psi \in \mathcal{H}_2^s$, using the standard anticommutation relations between electron creation and annihilation operators at lattice sites:
$\left\{a_{m,\gamma}, a_{n,\beta}^+\right\} = \delta_{m,n}\delta_{\gamma,\beta}$, $\left\{a_{m,\gamma}, a_{n,\beta}\right\} = \left\{a_{m,\gamma}^+, a_{n,\beta}^+\right\} = \theta$, as well as the property $a_{m,\gamma}\varphi_0 = \theta$, where θ is the zero element of $\mathcal{H}_2^s$. □

Lemma 13. *The spectra of the operators $\bar{H}_2^s$ and H_2^s coincide.*

Proof. The proof of Lemma 13 is similar to the proof of Lemma 2.

We call the operator $\bar{H}_2^s$ the two-electron singlet state operator.

We let $\mathcal{F}$ denote the Fourier transformation:
$\mathcal{F}: l_2\left(\left(Z^\nu\right)^2\right) \to L_2\left(\left(T^\nu\right)^2\right) \equiv \tilde{\mathcal{H}}_2^s$, where T^ν is the ν-dimensional torus endowed with the normalized Lebesgue measure $d\lambda$, $\lambda\left(T^\nu\right) = 1$.

We set $\tilde{H}_2^s = \mathcal{F}\bar{H}_2^s\mathcal{F}^{-1}$. In the quasimomentum representation, the operator $\bar{H}_2^s$ acts in the Hilbert space $L_2^s\left(\left(T^\nu\right)^2\right)$ as

$$\begin{aligned}\left(\tilde{H}_2^s\tilde{f}\right)(\lambda,\mu) &= 2A\tilde{f}(\lambda,\mu) + 2B\sum_{i=1}^{\nu}\left[\cos\lambda_i + \cos\mu_i\right]\tilde{f}(\lambda,\mu) \\ &+ U\int_{T^\nu} f(s,\lambda+\mu-s)\mathrm{d}s + \varepsilon_1\int_{T^\nu}\tilde{f}(s,\mu)\mathrm{d}s + \varepsilon_1\int_{T^\nu}\tilde{f}(\lambda,t)\mathrm{d}t \\ &+ 2\varepsilon_2\int_{T^\nu}\sum_{i=1}^{\nu}\left[\cos s_i + \cos\lambda_i\right]\tilde{f}(s,\mu)\mathrm{d}s \\ &+ 2\varepsilon_2\int_{T^\nu}\sum_{i=1}^{\nu}\left[\cos t_i + \cos\mu_i\right]\tilde{f}(\lambda,t)\mathrm{d}t + \varepsilon_3\int_{T^\nu}\int_{T^\nu} f(s,t)\mathrm{d}s\mathrm{d}t,\end{aligned} \tag{20}$$

where L_2^s is the subspace of symmetric functions in $L_2\left(\left(T^\nu\right)^2\right)$.

In turn, the operator $\tilde{H}_2^s$ can be represented in the form

$$\tilde{H}_2^s = \tilde{H}_1 \otimes I + I \otimes \tilde{H}_1 + K, \tag{21}$$

where $\begin{aligned}\left(K\tilde{f}\right)(\lambda,\mu) &= \left(K_1\tilde{f}\right)(\lambda,\mu) + \left(K_2\tilde{f}\right)(\lambda,\mu) \\ &= U\int_{T^\nu} f(s,\lambda+\mu-s)\mathrm{d}s + \varepsilon_3\int_{T^\nu}\int_{T^\nu} f(s,t)\mathrm{d}s\mathrm{d}t\end{aligned}$.

For the fixed value of total quasi-momentum of the two-electron system $x + y = \Lambda$ the operator K is the finite-rank operator, *i.e.*, the finite-dimensional operator. The rank of the operator K is equal to 2. Therefore, the essential spectrum of operators $\tilde{H}_2^s$ and $\tilde{H}_1 \otimes I + I \otimes \tilde{H}_1$ coincide (chapter XIII, paragraph 4, in [19]).

We now, using the obtained results and representation (17) and (21), we describe the structure of essential spectrum and discrete spectrum of the operator $\tilde{H}_2^s$.

From the beginning, we consider the operator $\tilde{H}(U) = \tilde{H}_2^t + K_1$.

Since, the family of the operators $\tilde{H}(U)$ are the family of bounded operators, that the $\tilde{H}(U)$ are the family of bounded operator valued analytical functions.

Therefore, in these family, one can the apply the Kato-Rellix theorem.

Theorem 14. (*Kato-Rellix theorem*) [19].

Let $T(\beta)$ is the analytical family in the terms of Kato. Let E_0 is a nondegenerate eigenvalue of $T(\beta_0)$. Then as β, near to β_0, the exist exactly one point $E(\beta) \in \sigma(T(\beta))$ the near E_0 and this point is isolated and nondegenerated. $E(\beta)$ is an analytical function of β as β, the near to β_0, and exist the analytical eigenvector $\Omega(\beta)$ as β the near to β_0. If the as real $\beta - \beta_0$ the operator $T(\beta)$ is a self-adjoint operator, then $\Omega(\beta)$ can selected thus, that it will be normalized of real $\beta - \beta_0$.

Since, the operator $\tilde{H}_2^t$ has a nondegenerate eigenvalue, such as, the near of eigenvalue $2z_1$ of the operator $\tilde{H}_2^t$, the operator $\tilde{H}(U)$ as *U*, near $U_0 = 0$, has a exactly one eigenvalue $E(U) \in \sigma(\tilde{H}(U))$ the near $2z_1$ and this point is isolated and nondegenerated. The $E(U)$ is a analytical function of *U* as *U*, the near to $U_0 = 0$.

As the large values the existence no more one additional eigenvalue of the operator $\tilde{H}(U)$ is following from the same, what the perturbation $(K_1\tilde{f})(\lambda,\mu) = U\int_{T^\nu} f(s, \lambda + \mu - s)\mathrm{d}s$ is the one-dimensional operator.

A new we consider the family of operators $\tilde{H}(\varepsilon_3) = \tilde{H}(U) + K_2$.

As, the operator $\tilde{H}(U)$ has a nondegenerate eigenvalue, consequently, the near of eigenvalue $E(U)$ the operator $\tilde{H}(U)$, operator $\tilde{H}(\varepsilon_3)$ as ε_3, the near of $\varepsilon_3 = 0$, has a exactly one eigenvalue $E(\varepsilon_3) \in \sigma(\tilde{H}(\varepsilon_3))$ the near $E(U)$ and this point is the isolated and nondegenerated. The $E(\varepsilon_3)$ is a analytical function of ε_3, as ε_3, the near to $\varepsilon_3 = 0$.

Later on via z_3, and z_4 we denote the additional eigenvalues of operator $\tilde{H}_2^s$. Thus, we prove the next theorems, the described the spectra of operator $\tilde{H}_2^s$.

Theorem 15. *Let* $\nu = 1$. *Then*

A) If $\varepsilon_2 = -B$ and $\varepsilon_1 < -2B$ (respectively, $\varepsilon_2 = -B$ and $\varepsilon_1 > 2B$), then the essential spectrum of the operator $\tilde{H}_2^s$ is consists of the union of two segments: $\sigma_{ess}(\tilde{H}_2^s) = [2A - 4B, 2A + 4B] \cup [A - 2B + z, A + 2B + z]$, and the discrete spectrum of the operator $\tilde{H}_2^s$ is consists no more than three points

$\sigma_{disc}\left(\tilde{H}_2^s\right)=\{2z,z_3,z_4\}$, where $z=A+\varepsilon_1$, z_3 and z_4 are the additional eigenvalues of the operator $\tilde{H}_2^s$.

B) If $\varepsilon_2=-2B$ or $\varepsilon_2=0$ and $\varepsilon_1<0$ (respectively, $\varepsilon_2=-2B$ or $\varepsilon_2=0$ and $\varepsilon_1>0$), then the essential spectrum of the operator $\tilde{H}_2^s$ is consists of the union of two segments: $\sigma_{ess}\left(\tilde{H}_2^s\right)=[2A-4B,2A+4B]\cup[A-2B+z,A+2B+z]$, and discrete spectrum of the operator $\tilde{H}_2^s$ is consists no more than three points: $\sigma_{disc}\left(\tilde{H}_2^s\right)=\{2z,z_3,z_4\}$, where $z=A-\sqrt{4B^2+\varepsilon_1^2}$ (respectively, $z=A+\sqrt{4B^2+\varepsilon_1^2}$).

C) If $\varepsilon_1=0$ and $\varepsilon_2>0$ or $\varepsilon_1=0$ and $\varepsilon_2<-2B$, then the essential spectrum of the operator $\tilde{H}_2^s$ is consists of the union of three segments:

$$\sigma_{ess}\left(\tilde{H}_2^s\right)=[2A-4B,2A+4B]\cup[A-2B+z_1,A+2B+z_1]\cup[A-2B+z_2,A+2B+z_2]$$

, and discrete spectrum of the operator $\tilde{H}_2^s$ is consists of no more than four points:

$\sigma_{disc}\left(\tilde{H}_2^s\right)=\{2z_1,2z_2,z_3,z_4\}$, where $z_1=A-\frac{2BE}{\sqrt{E^2-1}}$, and $z_2=A+\frac{2BE}{\sqrt{E^2-1}}$, and $E=\frac{(B+\varepsilon_2)^2}{\varepsilon_2^2+2B\varepsilon_2}$.

D) If $\varepsilon_1=\frac{2\left(\varepsilon_2^2+2B\varepsilon_2\right)}{B}$ (respectively, $\varepsilon_1=-\frac{2\left(\varepsilon_2^2+2B\varepsilon_2\right)}{B}$), then the essential spectrum of the operator $\tilde{H}_2^s$ is consists of the union of two segments: $\sigma_{ess}\left(\tilde{H}_2^s\right)=[2A-4B,2A+4B]\cup[A-2B+z,A+2B+z]$, and discrete spectrum of the operator $\tilde{H}_2^s$ is consists of no more than three points:

$\sigma_{disc}\left(\tilde{H}_2^s\right)=\{2z,z_3,z_4\}$, where $z=A+\frac{2B\left(E^2+1\right)}{E^2-1}$ (respectively, $z=A-\frac{2B\left(E^2+1\right)}{E^2-1}$), and $E=\frac{(B+\varepsilon_2)^2}{\varepsilon_2^2+2B\varepsilon_2}$.

E) If $\varepsilon_2>0$ and $\varepsilon_1>\frac{2\left(\varepsilon_2^2+2B\varepsilon_2\right)}{B}$ (respectively, $\varepsilon_2<-2B$ and $\varepsilon_1>\frac{2\left(\varepsilon_2^2+2B\varepsilon_2\right)}{B}$), then the essential spectrum of the operator $\tilde{H}_2^s$ is consists of the union of two segments:

$\sigma_{ess}\left(\tilde{H}_2^s\right)=[2A-4B,2A+4B]\cup[A-2B+z,A+2B+z]$, and discrete spectrum of the operator $\tilde{H}_2^s$ is consists of no more than three points:

$\sigma_{disc}\left(\tilde{H}_2^s\right)=\{2z,z_3,z_4\}$, where $z=A+\frac{2B\left(\alpha+E\sqrt{E^2-1+\alpha^2}\right)}{E^2-1}$ and $E=\frac{(B+\varepsilon_2)^2}{\varepsilon_2^2+2B\varepsilon_2}$, and the real number $\alpha>1$.

F) If $\varepsilon_2>0$ and $\varepsilon_1<-\frac{2\left(\varepsilon_2^2+2B\varepsilon_2\right)}{B}$ (respectively, $\varepsilon_2<-2B$ and

$\varepsilon_1 < -\frac{2(\varepsilon_2^2 + 2B\varepsilon_2)}{B}$), then the essential spectrum of the operator $\tilde{H}_2^s$ is consists of the union of two segments:
$\sigma_{ess}(\tilde{H}_2^s) = [2A-4B, 2A+4B] \cup [A-2B+z, A+2B+z]$, and discrete spectrum of the operator $\tilde{H}_2^s$ is consists of no more than three points:

$\sigma_{disc}(\tilde{H}_2^s) = \{2z, z_3, z_4\}$, where $z = A - \frac{2B(\alpha + E\sqrt{E^2 - 1 + \alpha^2})}{E^2 - 1}$ and $E = \frac{(B+\varepsilon_2)^2}{\varepsilon_2^2 + 2B\varepsilon_2}$, and the real number $\alpha > 1$.

K) If $\varepsilon_2 > 0$ and $0 < \varepsilon_1 < \frac{2(\varepsilon_2^2 + 2B\varepsilon_2)}{B}$ (respectively, $\varepsilon_2 < -2B$ and $0 < \varepsilon_1 < \frac{2(\varepsilon_2^2 + 2B\varepsilon_2)}{B}$), then the essential spectrum of the operator $\tilde{H}_2^s$ is consists of the union of three segments:

$$\sigma_{ess}(\tilde{H}_2^s) = [2A-4B, 2A+4B] \cup [A-2B+z_1, A+2B+z_1] \cup [A-2B+z_2, A+2B+z_2]$$

, and discrete spectrum of the operator $\tilde{H}_2^s$ is consists of no more than five points:

$\sigma_{disc}(\tilde{H}_2^s) = \{2z_1, z_1+z_2, 2z_2, z_3, z_4\}$, where $z_1 = A + \frac{2B(\alpha + E\sqrt{E^2 - 1 + \alpha^2})}{E^2 - 1}$ and $z_2 = A + \frac{2B(\alpha - E\sqrt{E^2 - 1 + \alpha^2})}{E^2 - 1}$, and $E = \frac{(B+\varepsilon_2)^2}{\varepsilon_2^2 + 2B\varepsilon_2}$, and the real number $0 < \alpha < 1$.

M) If $\varepsilon_2 > 0$ and $-\frac{2(\varepsilon_2^2 + 2B\varepsilon_2)}{B} < \varepsilon_1 < 0$ (respectively, $\varepsilon_2 < -2B$ and $-\frac{2(\varepsilon_2^2 + 2B\varepsilon_2)}{B} < \varepsilon_1 < 0$), then the essential spectrum of the operator $\tilde{H}_2^s$ is consists of the union of three segments:

$$\sigma_{ess}(\tilde{H}_2^s) = [2A-4B, 2A+4B] \cup [A-2B+z_1, A+2B+z_1] \cup [A-2B+z_2, A+2B+z_2]$$

, and discrete spectrum of the operator $\tilde{H}_2^s$ is consists of no more than five points:

$\sigma_{disc}(\tilde{H}_2^s) = \{2z_1, z_1+z_2, 2z_2, z_3, z_4\}$, where $z_1 = A + \frac{2B(\alpha + E\sqrt{E^2 - 1 + \alpha^2})}{E^2 - 1}$ and $z_2 = A + \frac{2B(\alpha - E\sqrt{E^2 - 1 + \alpha^2})}{E^2 - 1}$, and $E = \frac{(B+\varepsilon_2)^2}{\varepsilon_2^2 + 2B\varepsilon_2}$, and the real number $0 < \alpha < 1$.

N) If $-2B < \varepsilon_2 < 0$, then the essential spectrum of the operator $\tilde{H}_2^s$ is consists is a single segment $\sigma_{ess}(\tilde{H}_2^s) = [2A-4B, 2A+4B]$, and discrete spectrum of the operator $\tilde{H}_2^s$ is consists of no more than two points:
$\sigma_{disc}(\tilde{H}_2^s) = \{z_3, z_4\}$.

Proof. A). From the representation (17), (23) and the formulas (18) and (19), and the Theorem 8, follow the in one-dimensional case, the continuous spectrum of the operator $\tilde{H}_1$ is consists $\sigma_{cont}\left(\tilde{H}_1\right)=\left[A-2B, A+2B\right]$, and the discrete spectrum of the operator $\tilde{H}_1$ is consists of unique eigenvalue $z=A+\varepsilon_1$. The operator K is a two-dimensional operator. Therefore, the essential spectrum of the operators $\tilde{H}_1\otimes I+I\otimes\tilde{H}_1$ and $\tilde{H}_2^s$ coincide (see. chapter XIII, paragraph 4, in [19]) and is consists from segments $\left[2A-4B, 2A+4B\right]$, and $\left[A-2B+z, A+2B+z\right]$. Of extension the two-dimensional operator K to the operator $\tilde{H}_1\otimes I+I\otimes\tilde{H}_1$ can appear no more then two additional eigenvalues z_3 and z_4. These give the statement A) of the Theorem 15.

B) In this case the operator $\tilde{H}_1$ has a one eigenvalue z_1, lying the outside of the continuous spectrum of operator $\tilde{H}_1$. Therefore, the essential spectrum of the operators $\tilde{H}_1\otimes I+I\otimes\tilde{H}_1$ is consists of the union of two segments and discrete spectrum of the operator $\tilde{H}_1\otimes I+I\otimes\tilde{H}_1$ is consists of single point. These give the statement B) of the Theorem 15. The other statements of the Theorem 15 the analogously is proved. □

The next theorems is described the structure of essential spectrum of the operator $\tilde{H}_2^s$ in the three-dimensional case.

Theorem 16. *Let* $\nu=3$. *Then*

A) 1) If $\varepsilon_2=-B$ and $\varepsilon_1<-6B$ (respectively, $\varepsilon_2=-B$ and $\varepsilon_1>6B$,) then the essential spectrum of the operator $\tilde{H}_2^s$ is consists of the union of two segments $\sigma_{ess}\left(\tilde{H}_2^t\right)=\left[2A-12B, 2A+12B\right]\cup\left[A-6B+z, A+6B+z\right]$, and discrete spectrum of the operator $\tilde{H}_2^s$ is consists of no more then three points: $\sigma_{disc}\left(\tilde{H}_2^s\right)=\left\{2z, z_3, z_4\right\}$, where $z=A+\varepsilon_1$, z_3 and z_4 are the additional eigenvalues of the operator $\tilde{H}_2^s$.

2) If $\varepsilon_2=-B$ and $-6B\le\varepsilon_1<-2B$ (respectively, $\varepsilon_2=-B$ and $2B<\varepsilon_1\le 6B$,) then the essential spectrum of the operator $\tilde{H}_2^s$ is consists of a single segment $\sigma_{ess}\left(\tilde{H}_2^t\right)=\left[2A-12B, 2A+12B\right]$, and discrete spectrum of the operator $\tilde{H}_2^s$ is consists of no more then two points: $\sigma_{disc}\left(\tilde{H}_2^s\right)=\left\{z_3, z_4\right\}$.

B) If $\varepsilon_2=-2B$ or $\varepsilon_2=0$, and $\varepsilon_1<0$, $\varepsilon_1\le-\frac{6B}{W}$, (respectively, $\varepsilon_2=-2B$ or $\varepsilon_2=0$, and $\varepsilon_1>0$, $\varepsilon_1\ge\frac{6B}{W}$), then the essential spectrum of the operator $\tilde{H}_2^s$ is consists of the union of two segments $\sigma_{ess}\left(\tilde{H}_2^s\right)=\left[2A-12B, 2A+12B\right]\cup\left[A-6B+z_1, A+6B+z_1\right]$ (respectively, $\sigma_{ess}\left(\tilde{H}_2^s\right)=\left[2A-12B, 2A+12B\right]\cup\left[A-6B+z_2, A+6B+z_2\right]$), and discrete spectrum of the operator $\tilde{H}_2^s$ is consists of no more then three points: $\sigma_{disc}\left(\tilde{H}_2^s\right)=\left\{2z_1, z_3, z_4\right\}$ (respectively, $\sigma_{disc}\left(\tilde{H}_2^s\right)=\left\{2z_2, z_3, z_4\right\}$), where z_1 (respectively, z_2) are the eigenvalue of operator $\tilde{H}_1$.

If $-\frac{6B}{W}\le\varepsilon_1<0$ (respectively, $0<\varepsilon_1\le\frac{6B}{W}$), then the essential spectrum of the operator $\tilde{H}_2^s$ is consists of a single segment $\sigma_{ess}\left(\tilde{H}_2^s\right)=\left[2A-12B,2A+12B\right]$, and discrete spectrum of the operator $\tilde{H}_2^s$ is consists of no more then two points: $\sigma_{disc}\left(\tilde{H}_2^s\right)=\{z_3,z_4\}$.

C) If $\varepsilon_1=0$ and $\varepsilon_2>0$, $E<W$ (respectively, $\varepsilon_1=0$ and $\varepsilon_2<-2B$, $E<W$), then the essential spectrum of the operator $\tilde{H}_2^s$ is consists of the union of two segments: $\sigma_{ess}\left(\tilde{H}_2^s\right)=\left[2A-12B,2A+12B\right]\cup\left[A-6B+z,A+6B+z\right]$ (respectively, $\sigma_{ess}\left(\tilde{H}_2^s\right)=\left[2A-12B,2A+12B\right]\cup\left[A-6B+\tilde{z},A+6B+\tilde{z}\right]$), and discrete spectrum of the operator $\tilde{H}_2^s$ is consists of no more then three points: $\sigma_{disc}\left(\tilde{H}_2^s\right)=\{2z,z_3,z_4\}$ (respectively, $\sigma_{disc}\left(\tilde{H}_2^s\right)=\{2\tilde{z},z_3,z_4\}$), where z (respectively, $\tilde{z}$), is the eigenvalue of operator $\tilde{H}_1$, and $E=\frac{(B+\varepsilon_2)^2}{\varepsilon_2^2+2B\varepsilon_2}$. If $\varepsilon_1=0$ and $\varepsilon_2>0$, $E>W$ (respectively, $\varepsilon_1=0$ and $\varepsilon_2<-2B$, $E>W$), then the essential spectrum of the operator $\tilde{H}_2^s$ is consists of a single segment: $\sigma_{ess}\left(\tilde{H}_2^s\right)=\left[2A-12B,2A+12B\right]$, and discrete spectrum of the operator $\tilde{H}_2^s$ is consists no more then two points: $\sigma_{disc}\left(\tilde{H}_2^s\right)=\{z_3,z_4\}$.

D) If $\varepsilon_1=\frac{2\left(\varepsilon_2^2+2B\varepsilon_2\right)}{B}$ (respectively, $\varepsilon_1=-\frac{2\left(\varepsilon_2^2+2B\varepsilon_2\right)}{B}$), then the essential spectrum of the operator $\tilde{H}_2^s$ is consists of the union of two segments: $\sigma_{ess}\left(\tilde{H}_2^s\right)=\left[2A-12B,2A+12B\right]\cup\left[A-6B+z,A+6B+z\right]$ (respectively, $\sigma_{ess}\left(\tilde{H}_2^s\right)=\left[2A-12B,2A+12B\right]\cup\left[A-6B+\tilde{z},A+6B+\tilde{z}\right]$), and discrete spectrum of the operator $\tilde{H}_2^s$ is consists of no more then three points: $\sigma_{disc}\left(\tilde{H}_2^s\right)=\{2z,z_3,z_4\}$ (respectively, $\sigma_{disc}\left(\tilde{H}_2^s\right)=\{2\tilde{z},z_3,z_4\}$), where z (respectively, $\tilde{z}$) is the eigenvalue of operator $\tilde{H}_1$.

E) If $\varepsilon_2>0$ and $\varepsilon_1>\frac{2\left(\varepsilon_2^2+2B\varepsilon_2\right)}{B}$ and $E<\left(1+\frac{\alpha}{3}\right)W$ (respectively, $\varepsilon_2<-2B$ and $\varepsilon_1>\frac{2\left(\varepsilon_2^2+2B\varepsilon_2\right)}{B}$ and $E<\left(1+\frac{\alpha}{3}\right)W$), then the essential spectrum of the operator $\tilde{H}_2^s$ is consists of the union of two segments $\sigma_{ess}\left(\tilde{H}_2^s\right)=\left[2A-12B,2A+12B\right]\cup\left[A-6B+z_1,A+6B+z_1\right]$, and discrete spectrum of the operator $\tilde{H}_2^s$ is consists of no more then three points: $\sigma_{disc}\left(\tilde{H}_2^s\right)=\{2z_1,z_3,z_4\}$, where z_1 is the eigenvalue of operator $\tilde{H}_1$.

F) If $\varepsilon_2>0$ and $\varepsilon_1<-\frac{2\left(\varepsilon_2^2+2B\varepsilon_2\right)}{B}$ and $E<\left(1+\frac{\alpha}{3}\right)W$ (respectively, $\varepsilon_2<-2B$ and $\varepsilon_1<-\frac{2\left(\varepsilon_2^2+2B\varepsilon_2\right)}{B}$ and $E<\left(1+\frac{\alpha}{3}\right)W$), then the essential spectrum of the operator $\tilde{H}_2^s$ is consists of the union of two segments: $\sigma_{ess}\left(\tilde{H}_2^s\right)=\left[2A-12B,2A+12B\right]\cup\left[A-6B+z_1,A+6B+z_1\right]$, and discrete spectrum of the operator $\tilde{H}_2^s$ is consists of no more then three points:

$\sigma_{disc}\left(\tilde{H}_2^s\right)=\{2z_1,z_3,z_4\}$, where z_1 is the eigenvalue of operator $\tilde{H}_1$.

K) If $\varepsilon_2>0$ and $0<\varepsilon_1<\frac{2\left(\varepsilon_2^2+2B\varepsilon_2\right)}{B}$ and $E<\left(1-\frac{\alpha}{3}\right)W$ (respectively, $\varepsilon_2<-2B$ and $0<\varepsilon_1<\frac{2\left(\varepsilon_2^2+2B\varepsilon_2\right)}{B}$ and $E<\left(1-\frac{\alpha}{3}\right)W$), then the essential spectrum of the operator $\tilde{H}_2^s$ is consists of the union of three segments:

$$\sigma_{ess}\left(\tilde{H}_2^s\right)=[2A-12B,2A+12B]\cup[A-6B+z_1,A+6B+z_1]\cup[A-6B+z_2,A+6B+z_2]$$

, and discrete spectrum of the operator $\tilde{H}_2^s$ is consists of no more then five points: $\sigma_{disc}\left(\tilde{H}_2^s\right)=\{2z_1,z_1+z_2,2z_2,z_3,z_4\}$, where z_1 and z_2 are the eigenvalues of operator $\tilde{H}_1$.

M) If $\varepsilon_2>0$ and $-\frac{2\left(\varepsilon_2^2+2B\varepsilon_2\right)}{B}<\varepsilon_1<0$ and $E<\left(1+\frac{\alpha}{3}\right)W$ (respectively, $\varepsilon_2<-2B$ and $-\frac{2\left(\varepsilon_2^2+2B\varepsilon_2\right)}{B}<\varepsilon_1<0$ and $E<\left(1+\frac{\alpha}{3}\right)W$), then the essential spectrum of the operator $\tilde{H}_2^s$ is consists of the union of three segments:

$$\sigma_{ess}\left(\tilde{H}_2^s\right)=[2A-12B,2A+12B]\cup[A-6B+z_1,A+6B+z_1]\cup[A-6B+z_2,A+6B+z_2]$$

, and discrete spectrum of the operator $\tilde{H}_2^s$ is consists of no more then five points: $\sigma_{disc}\left(\tilde{H}_2^s\right)=\{2z_1,z_1+z_2,2z_2,z_3,z_4\}$, where z_1 and z_2 are the eigenvalues of operator $\tilde{H}_1$.

N) If $-2B<\varepsilon_2<0$, then the essential spectrum of the operator $\tilde{H}_2^s$ is consists of a single segment: $\sigma_{ess}\left(\tilde{H}_2^s\right)=[2A-12B,2A+12B]$, and discrete spectrum of the operator $\tilde{H}_2^s$ is consists of no more then two points: $\sigma_{disc}\left(\tilde{H}_2^s\right)=\{z_3,z_4\}$.

Proof. A) 1) From the Theorem 9 is follows, that, if $\nu=3$ and $\varepsilon_2=-B$ and $\varepsilon_1<-6B$ (respectively, $\varepsilon_2=-B$ and $\varepsilon_1>6B$), the operator $\tilde{H}_1$ has a unique eigenvalue $z=A+\varepsilon_1$, the outside the continuous spectrum of the operator $\tilde{H}_1$. Furthermore, the continuous spectrum of the operator $\tilde{H}_1$ is consists of the segment $[A-6B,A+6B]$, therefore, the essential spectrum of the operator $\tilde{H}_2^s$ is consists of a union of two segments: $\sigma_{ess}\left(\tilde{H}_2^s\right)=[2A-12B,2A+12B]\cup[A-6B+z,A+6B+z]$. The number $2z$ is the eigenvalue for the operator $\tilde{H}_2^s$. In the representation (17) the operator K is a two-dimensional operator. Therefore, the operator $\tilde{H}_2^s$ can have no more then two additional eigenvalues z_3 and z_4. Consequently, the operator $\tilde{H}_2^s$ can have no more then three eigenvalues $2z$, z_3 and z_4.

2) From the Theorem 9 is follows, that, if $\nu=3$ and $\varepsilon_2=-B$ and

$-6B \le \varepsilon_1 < -2B$ (respectively, $\varepsilon_2 = -B$ and $2B < \varepsilon_1 \le 6B$), then the operator $\tilde{H}_1$ has no eigenvalues, the outside the continuous spectrum of the operator $\tilde{H}_1$. Furthermore, the continuous spectrum of the operator $\tilde{H}_1$ is consists of the segment $[A-6B, A+6B]$, therefore, the essential spectrum of the operator $\tilde{H}_2^s$ is consists of a single segment: $\sigma_{ess}(\tilde{H}_2^s) = [2A-12B, 2A+12B]$. In the representation (17) the operator K is a two-dimensional operator. Therefore, the operator $\tilde{H}_2^s$ can have no more then two additional eigenvalues z_3 and z_4. Consequently, the operator $\tilde{H}_2^s$ can have no more then two eigenvalues z_3 and z_4.

M) From the Theorem 9 is follows, that, if $\nu = 3$ and $\varepsilon_2 > 0$ and $-\frac{2(\varepsilon_2^2 + 2B\varepsilon_2)}{B} < \varepsilon_1 < 0$ and $E < \left(1 + \frac{\alpha}{3}\right)W$ (respectively, $\varepsilon_2 < -2B$ and $-\frac{2(\varepsilon_2^2 + 2B\varepsilon_2)}{B} < \varepsilon_1 < 0$ and $E < \left(1 + \frac{\alpha}{3}\right)W$), the operator $\tilde{H}_1$ has a exactly two eigenvalues z_1 and z_2, lying the below and above of the continuous spectrum of the operator $\tilde{H}_1$. Furthermore, the continuous spectrum of the operator $\tilde{H}_1$ is consists of the segment $[A-6B, A+6B]$, therefore, then the essential spectrum of the operator $\tilde{H}_2^s$ is consists of the union of three segments: $\sigma_{ess}(\tilde{H}_2^s) = [2A-12B, 2A+12B] \cup [A-6B+z_1, A+6B+z_1] \cup [A-6B+z_2, A+6B+z_2]$, and point $2z_1$, $2z_2$ and $z_1 + z_2$, are the eigenvalues of the operator $\tilde{H}_1 \otimes I + I \otimes \tilde{H}_1$, and in the representation (17) the operator K is a two-dimensional operator. Therefore, the operator $\tilde{H}_2^s$ can have no more then two additional eigenvalues z_3 and z_4. Consequently, the operator $\tilde{H}_2^s$ can have no more then five eigenvalues $2z_1$, $z_1 + z_2$, $2z_2$, z_3 and z_4.

The other statements of the Theorem 16 the analogously is proved. □

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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