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Green Function of Generalized Time Fractional Diffusion Equation Using Addition Formula of Mittag-Leffler Function

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Abstract

In this paper, we use the Mittag-Leffler addition formula to solve the Green function of generalized time fractional diffusion equation in the whole plane and prove the convergence of the Green function.

Keywords

Mittag-Leffler Function, Mellin Transforms, Generalized Time Fractional Diffusion Equation, Green Function, Addition Formula

1. Introduction

The time fractional diffusion equation is always a popular one of fractional calculus equations (for example [1] [2] [3]). In [4], the analytical solution of the time-fractional diffusion equation in the sense of Caputo was given by the integral representation of M-Wright functions and exponential operators. In [5], the author derived the addition formula of Wright function by using Mellin transform of Wright function, and obtained the Green function of time-fractional diffusion equation on the whole plane. Diffusion equations are partial differential equations that describe the changes in space and time of physical quantities governed by diffusion, that is, the transfer of ions, molecules and even energy in solution from regions of high concentration to regions of low concentration. In this paper, we generalize reference [5], discuss the general situation of the equation of reference [5], we use the Mittag-Leffler function addition formula to solve the Green function of generalized time fractional diffusion equation in the whole plane and prove the convergence of the Green function. We apply the Mittag-Leffler function addition formula in the process of solving the inverse integral transformation of a two-dimensional or even multidimensional function, the function is divided into several parts and solved separately, and the calculation is reduced, and the basic solution of the function is more easily obtained.

In recent decades there has been increasing interest in Wright function [6]-[11], mainly because this function plays an important role in linear partial differential equations. The Wright function is defined as [12]

$$W_{\rho,\mu}(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\rho n+\mu)n!},\ \rho>-1,\ \mu,z\in C. \tag{1}$$

The Wright functions can also be represented by contour integrals of Hankel paths in the complex plane

$$W_{\rho,\mu}(z)=\frac{1}{2\pi i}\int_{Ha}\tau^{-\mu}\mathrm{e}^{\tau+z\tau^{-\rho}}\mathrm{d}\tau,\ \rho>-1,\ \mu,z\in C. \tag{2}$$

In the same way, the Mittag-Leffler function is closely related to fractional calculus, especially to fractional order problems in application. The Wright function and the Mittag-Leffler function can be related by the Laplace transform, and by taking the Laplace transform of the Wright function, we can get the Mittag-Leffler function [13]

$$\begin{aligned} L\{W_{\rho,\mu}(z);s\}&=L\left\{\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\rho n+\mu)n!};s\right\}\\ &=\sum_{n=0}^{\infty}\frac{1}{\Gamma(\rho n+\mu)}\cdot\frac{1}{s^{n+1}}\\ &=s^{-1}E_{\rho,\mu}(s^{-1}). \end{aligned} \tag{3}$$

The Mittag-Leffler functions of one parameter and two parameters can be represented by the series expansion [12]

$$E_{\rho}(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\rho n+1)},\ \rho>0,\ \mu,z\in C, \tag{4}$$

$$E_{\rho,\mu}(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\rho n+\mu)},\ \rho>0,\ \mu,z\in C. \tag{5}$$

The convergence condition of infinite series (5) is $\Re(\rho>0)$, $\Re(\mu>0)$.

In particular, when $\rho,\mu=1$,

$$E_{1,1}(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(n+1)}=\mathrm{e}^z. \tag{6}$$

The Mittag-Leffler functions of two parameters can be represented by the following integral

$$E_{\rho,\mu}(z)=\frac{1}{2\pi i}\int_{Ha}\frac{\mathrm{e}^{\zeta}\zeta^{\rho-\mu}}{\zeta^{\alpha}-z}\mathrm{d}\zeta,\rho>0. \tag{7}$$

The plan of this paper is as follows. Section 2 introduces the auxiliary results. Section 3 uses the addition formula of Mittag-Leffler function to solve the Green function of generalized time fractional diffusion equation. Section 4 proves the convergence of the Green function.

2. Auxiliary Results

Let's firstly introduce Mellin transforms in one and two dimensions cases [14] [15] [16], the Mellin transform $F(s)$ of a function $f(t)$, which is defined in the interval $(0,\infty)$

$$M\{f(x);s\} = F(s) = \int_0^\infty x^{s-1} f(x)\mathrm{d}x,\ c_1 < \Re(s) < c_2, \tag{8}$$

$$M_2\{f(x,y);s,\tau\} = \int_0^\infty \int_0^\infty x^{s-1} y^{\tau-1} f(x,y)\mathrm{d}x\mathrm{d}y. \tag{9}$$

Integrating repeatedly by parts, we have the following relationship for the Mellin transform of an integer-order derivative

$$M\left\{f^{(n)}(x);s\right\} = (-1)^n \frac{\Gamma(s)}{\Gamma(s-n)} F(s-n),\ n \in N. \tag{10}$$

Proof.

$$\begin{aligned} M\left\{f^{(n)}(x);s\right\} &= \int_0^\infty x^{s-1} f^{(n)}(x)\mathrm{d}x \\ &= x^{s-1} f^{(n-1)}(x)\Big|_0^\infty - \int_0^\infty \frac{1}{s-1} x^{s-2} f^{(n-1)}(x)\mathrm{d}x \\ &= x^{s-1} f^{(n-1)}(x)\Big|_0^\infty - \frac{1}{s-1} F(s-1) \\ &= \cdots \\ &= (-1)^n \frac{\Gamma(s)}{\Gamma(s-n)} F(s-n), \end{aligned} \tag{11}$$

where $f(t)$ and $Re(s)$ are such that all substitutions of the limits $t=0$ and $t=\infty$ give zero.

Here, we consider the definitions of Weyl fractional integral and derivative and the related Mellin transforms.

Definition 2.1. The Weyl fractional integral and derivative of order α are defined as [17] [18]

$$W_+^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_x^\infty (\xi - x)^{\alpha-1} f(\xi)\mathrm{d}\xi, \tag{12}$$

$$W_+^{\alpha} = \frac{(-1)^n}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}x^n} \int_x^\infty (\xi - x)^{n-\alpha-1} f(\xi)\mathrm{d}\xi \tag{13}$$

where $n-1 < \Re(\alpha) \le n$, $n \in N$.

Lemma 2.2. The Mellin transforms of any derivative, Weyl fractional integral and Weyl fractional derivative of function $f(x)$ are given by [14]

$$M\left\{f^{(-\alpha)}(x);s\right\} = \frac{\Gamma(s)}{\Gamma(s+\alpha)} F(s+\alpha), \tag{14}$$

where the superscript $(-\alpha)$ denotes the αth-order Weyl fractional integral of function $f(x)$.

$$M\left\{f^{(\alpha)}(x);s\right\}=\frac{\Gamma(s)}{\Gamma(s-\alpha)}F(s-\alpha) \tag{15}$$

where the superscript (α) denotes the αth-order Weyl fractional derivative of function $f(x)$.

Lemma 2.3. The Mellin transform of Wright function $W_{\rho,\mu}(x)$ is given by [19]

$$M\left\{W_{\rho,\mu}(-x);s\right\}=\frac{\Gamma(s)}{\Gamma(\mu-\rho s)},\Re(s)>0. \tag{16}$$

Proof. By the relation,

$$\int_0^\infty t^n f(t)\mathrm{d}t=\lim_{s\to 0}(-1)^n\frac{\mathrm{d}^n}{\mathrm{d}s^n}L\left\{f(t);s\right\}, \tag{17}$$

thus, we have

$$\begin{aligned}
M\left\{W_{\rho,\mu}(-x);s\right\}&=\int_0^\infty x^{s-1}W_{\rho,\mu}(-x)\mathrm{d}x\\
&=\lim_{t\to 0}(-1)^{s-1}\frac{\mathrm{d}^{s-1}}{\mathrm{d}t^{s-1}}L\left\{W_{\rho,\mu}(-x);t\right\}\\
&=\lim_{t\to 0}(-1)^{s-1}\frac{\mathrm{d}^{s-1}}{\mathrm{d}t^{s-1}}\int_0^\infty \mathrm{e}^{-xt}\frac{1}{2\pi i}\int_{Ha}\tau^{-\mu}\mathrm{e}^{\tau-x\tau^{-\rho}}\mathrm{d}\tau\mathrm{d}t\\
&=\lim_{t\to 0}(-1)^{s-1}\frac{\mathrm{d}^{s-1}}{\mathrm{d}t^{s-1}}\frac{1}{2\pi i}\int_{Ha}\tau^{-\mu}\mathrm{e}^{\tau}\int_0^\infty \mathrm{e}^{-xt-x\tau^{-\rho}}\mathrm{d}\tau\\
&=\lim_{t\to 0}(-1)^{s-1}\frac{\mathrm{d}^{s-1}}{\mathrm{d}t^{s-1}}\frac{1}{2\pi i}\int_{Ha}\frac{\tau^{-\mu}\mathrm{e}^{\tau}}{t+\tau^{-\rho}}\mathrm{d}\tau\\
&=\lim_{t\to 0}(-1)^{s-1}\frac{\mathrm{d}^{s-1}}{\mathrm{d}t^{s-1}}\sum_{k=0}^{\infty}\frac{(-t)^k}{\Gamma(-\rho k+\mu-\rho)}\\
&=\frac{\Gamma(s)}{\Gamma(\mu-\rho s)}.
\end{aligned} \tag{18}$$

Lemma 2.4. For $n\in N$, $|arg(a)|<|c|\pi$ and $|arg(b)|<|c|\pi$, [5] it gives rise to an addition formula for the Mittag-Leffler function

$$\begin{aligned}
&E_{c,-2cv}(-(a+b))\\
&=\int_0^1 x^{(-1-cv)}(1-x)^{(-1-cv)}L\left\{W_{c,-cv}(-atx^c)\times W_{c,-cv}\left(-bt(1-x)^c\right);s\to 1\right\}\mathrm{d}x.
\end{aligned} \tag{19}$$

3. Application to Generalized Time Fractional Diffusion Equation

In this section, we mainly analyze the solution of the generalized time fractional diffusion equation [20] [21] under Wright function, therefore, we need to determine the fundamental solution of the equation, namely Green function

$$\frac{\partial^\alpha u(x,y,t)}{\partial t^\alpha}=\frac{\partial^2 u(x,y,t)}{\partial x^2}+\frac{\partial^2 u(x,y,t)}{\partial y^2}+u(x,y,t),\ t>0,\ x,y\in R,\ 0<\alpha\le 1, \tag{20}$$

with initial condition $u(x,y,0)=\delta(x)\delta(y)$. The above fractional derivative is assumed to be Caputo derivative. So, Caputo derivative and its corresponding

Laplace transform [22] can be written as

$$\left({}^{C}D_{0^+}^{\alpha} f\right)(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t (t-s)^{\alpha} f'(s)\,\mathrm{d}s, \tag{21}$$

$$L\left\{\left({}^{C}D_{0^+}^{\alpha} u\right)(t); s\right\} = s^{\alpha} U(x,y,s) - s^{\alpha-1} U(x,y,0), (0<\alpha\le 1). \tag{22}$$

The two-dimensional Fourier transform [23]

$$F_2\left\{h(x,y); p,q\right\} = H(p,q) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x,y)\,\mathrm{e}^{-ixp}\mathrm{e}^{-iyq}\,\mathrm{d}x\mathrm{d}y. \tag{23}$$

Let's take the Laplace transform of both sides of this Equation (20), thus, we get an algebraic relation

$$s^{\alpha} U(x,y,s) - s^{\alpha-1} U(x,y,0) = \frac{\partial^2 U(x,y,s)}{\partial x^2} + \frac{\partial^2 U(x,y,s)}{\partial y^2} + U(x,y,s). \tag{24}$$

And likewise, taking the Fourier transform of both sides of this expression (24), we get an algebraic relation as follows

$$U(p,q,s) = \frac{s^{\alpha-1}}{s^{\alpha} + p^2 + q^2 - 1}. \tag{25}$$

In order to find the inversion of (25) and corresponding Green's function. We take the inverse Laplace transform of one parameter of the relation (25), it can be expressed in terms of the Mittag-Leffler function

$$U(p,q,t) = E_{\alpha}\left(-\left(p^2+q^2-1\right)t^{\alpha}\right). \tag{26}$$

We use the addition formula (19) of the Mittag-Leffler function to obtain the inverse Fourier transform as (by setting $c=\alpha$ and $v=-(1/2\alpha)$)

$$\begin{aligned} E_{\alpha}\left(-\left(p^2+q^2-1\right)t^{\alpha}\right) = \int_0^1 \mu^{-\frac{1}{2}}(1-\mu)^{-\frac{1}{2}} L\Big\{ & W_{\alpha,\frac{1}{2}}\left(\left(-p^2+\gamma\right)t^{\alpha}\tau\mu^{\alpha}\right) \\ & \times W_{\alpha,\frac{1}{2}}\left(\left(-q^2+\beta\right)t^{\alpha}\tau(1-\mu)^{\alpha}\right); s\to 1\Big\}\mathrm{d}x, \end{aligned} \tag{27}$$

so

$$u(x,y,t) = F_2^{-1}\left\{E_{\alpha}\left(-\left(p^2+q^2-1\right)t^{\alpha}\right); x,y\right\}, \tag{28}$$

or

$$\begin{aligned} u(x,y,t) = \int_0^1 \mu^{-\frac{1}{2}}(1-\mu)^{-\frac{1}{2}} L\Big\{ & F^{-1}\left\{W_{\alpha,\frac{1}{2}}\left(\left(-p^2+\gamma\right)t^{\alpha}\tau\mu^{\alpha}\right); x\right\} \\ & \times F^{-1}\left\{W_{\alpha,\frac{1}{2}}\left(\left(-q^2+\beta\right)t^{\alpha}\tau(1-\mu)^{\alpha}\right); y\right\}; s\to 1\Big\}\mathrm{d}\mu, \end{aligned} \tag{29}$$

where

$$\gamma + \beta = 1.$$

Now, we apply the Taylor expansion of exponential function

$$\mathrm{e}^{z} = \sum_{n=0}^{\infty}\frac{z^n}{n!} = \sum_{n=0}^{\infty}\frac{z^n}{\Gamma(n+1)} = E_{1,1}(z), \tag{30}$$

and employ the relation (16) to compute the inverse Fourier transform of Wright function as follows

$$
\begin{aligned}
&F^{-1}\left\{W_{\alpha,\frac{1}{2}}\left(\left(-p^2+\gamma\right)t^\alpha\tau\mu^\alpha\right);x\right\}\\
&=\frac{1}{2\pi}\int_{-\infty}^{\infty}W_{\alpha,\frac{1}{2}}\left(\left(-p^2+\gamma\right)t^\alpha\tau\mu^\alpha\right)\mathrm{e}^{ixp}\mathrm{d}p\\
&=\frac{1}{\pi}\int_0^{\infty}W_{\alpha,\frac{1}{2}}\left(\left(-p^2+\gamma\right)t^\alpha\tau\mu^\alpha\right)\cos(xp)\mathrm{d}p\\
&=\frac{1}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{\Gamma(2n+1)}\int_0^{\infty}p^{2n}W_{\alpha,\frac{1}{2}}\left(\left(-p^2+\gamma\right)t^\alpha\tau\mu^\alpha\right)\mathrm{d}p\\
&=\frac{1}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{\Gamma(2n+1)}\lim_{s\to0}\frac{\mathrm{d}^{2n}}{\mathrm{d}s^{2n}}L\left\{W_{\alpha,\frac{1}{2}}\left(\left(-p^2+\gamma\right)t^\alpha\tau\mu^\alpha\right);s\right\}\\
&=\frac{1}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{\Gamma(2n+1)}\lim_{s\to0}\frac{\mathrm{d}^{2n}}{\mathrm{d}s^{2n}}\int_0^{\infty}\mathrm{e}^{-sp}\frac{1}{2\pi i}\int_{Ha}\zeta^{-\frac{1}{2}}\mathrm{e}^{\zeta+\left(-p^2+\gamma\right)t^\alpha\tau\mu^\alpha\zeta^{-\alpha}}\mathrm{d}\zeta\mathrm{d}p\\
&=\frac{1}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{\Gamma(2n+1)}\lim_{s\to0}\frac{\mathrm{d}^{2n}}{\mathrm{d}s^{2n}}\frac{1}{2\pi i}\int_{Ha}\zeta^{-\frac{1}{2}}\mathrm{e}^{\zeta+\gamma t^\alpha\tau\mu^\alpha\zeta^{-\alpha}}\int_0^{\infty}\mathrm{e}^{-sp-p^2t^\alpha\tau\mu^\alpha\zeta^{-\alpha}}\mathrm{d}p\mathrm{d}\zeta\\
&=\frac{1}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{\Gamma(2n+1)}\lim_{s\to0}\frac{\mathrm{d}^{2n}}{\mathrm{d}s^{2n}}\frac{1}{2\pi i}\int_{Ha}\zeta^{-\frac{1}{2}}\mathrm{e}^{\zeta+\gamma t^\alpha\tau\mu^\alpha\zeta^{-\alpha}}\mathrm{e}^{\frac{s^2}{4t^\alpha\tau\mu^\alpha\zeta^{-\alpha}}}\\
&\quad\times\int_0^{\infty}\mathrm{e}^{-\left(pt^{\frac{\alpha}{2}}\tau^{\frac{1}{2}}\mu^{\frac{\alpha}{2}}\zeta^{-\frac{\alpha}{2}}+\frac{s}{t^{\frac{\alpha}{2}}\tau^{\frac{1}{2}}\mu^{\frac{\alpha}{2}}\zeta^{-\frac{\alpha}{2}}}\right)^2}\mathrm{d}p\mathrm{d}\zeta\\
&=\frac{1}{\sqrt{\pi}}\frac{1}{t^{\frac{\alpha}{2}}\tau^{\frac{1}{2}}\mu^{\frac{\alpha}{2}}}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{\Gamma(2n+1)}\lim_{s\to0}\frac{\mathrm{d}^{2n}}{\mathrm{d}s^{2n}}\frac{1}{2\pi i}\int_{Ha}\zeta^{-\frac{1}{2}+\frac{\alpha}{2}}\mathrm{e}^{\zeta+\gamma t^\alpha\tau\mu^\alpha\zeta^{-\alpha}}\mathrm{e}^{\frac{s^2}{4t^\alpha\tau\mu^\alpha\zeta^{-\alpha}}}\mathrm{d}\zeta\\
&=\frac{1}{\sqrt{\pi}}\frac{1}{t^{\frac{\alpha}{2}}\tau^{\frac{1}{2}}\mu^{\frac{\alpha}{2}}}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{\Gamma(2n+1)}\lim_{s\to0}\frac{\mathrm{d}^{2n}}{\mathrm{d}s^{2n}}\frac{1}{2\pi i}\\
&\quad\times\int_{Ha}\zeta^{-\frac{1}{2}+\frac{\alpha}{2}}\mathrm{e}^{\zeta+\gamma t^\alpha\tau\mu^\alpha\zeta^{-\alpha}}\sum_{k=0}^{\infty}\frac{s^{2k}}{k!\left(4t^\alpha\tau\mu^\alpha\zeta^{-\alpha}\right)^k}\mathrm{d}\zeta\\
&=\frac{1}{\sqrt{\pi}}\frac{1}{t^{\frac{\alpha}{2}}\tau^{\frac{1}{2}}\mu^{\frac{\alpha}{2}}}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{n!\left(4t^\alpha\tau\mu^\alpha\right)^n}\frac{1}{2\pi i}\int_{Ha}\zeta^{-\frac{1}{2}+\frac{\alpha}{2}+\alpha n}\mathrm{e}^{\zeta+\gamma t^\alpha\tau\mu^\alpha\zeta^{-\alpha}}\mathrm{d}\zeta\\
&=\frac{1}{\sqrt{\pi}}\frac{1}{t^{\frac{\alpha}{2}}\tau^{\frac{1}{2}}\mu^{\frac{\alpha}{2}}}E_{1,1}\left(\frac{-x^2}{4t^\alpha\tau\mu^\alpha}\right)W_{\alpha,\frac{1}{2}-\frac{\alpha}{2}-\alpha n}\left(\gamma t^\alpha\tau\mu^\alpha\right),
\end{aligned}
\tag{31}
$$

where the n in the subscript of the Wright function is the same as the n in the series expression of the Mittag-Leffler function.

Similarly,

$$
\begin{aligned}
&F^{-1}\left\{W_{\alpha,\frac{1}{2}}\left(\left(-q^2+\beta\right)t^\alpha\tau(1-\mu)^\alpha\right);y\right\}\\
&=\frac{1}{\sqrt{\pi}}\frac{1}{t^{\frac{\alpha}{2}}\tau^{\frac{1}{2}}(1-\mu)^{\frac{\alpha}{2}}}E_{1,1}\left(\frac{-y^2}{4t^\alpha\tau(1-\mu)^\alpha}\right)W_{\alpha,\frac{1}{2}-\frac{\alpha}{2}-\alpha m}\left(\beta t^\alpha\tau(1-\mu)^\alpha\right),
\end{aligned}
\tag{32}
$$

where the m in the subscript of the Wright function is the same as the m in the series expression of the Mittag-Leffler function.

Finally, we obtain the Green's formula of the generalized time fractional diffusion equation

$$u(x,y,t)=\frac{1}{\pi t^{\alpha}}\int_0^1 \mu^{-\frac{1+\alpha}{2}}(1-\mu)^{-\frac{1+\alpha}{2}} L\left\{\tau^{-1}E_{1,1}\left(\frac{-x^2}{4t^{\alpha}\tau\mu^{\alpha}}\right)W_{\alpha,\frac{1}{2}-\frac{\alpha}{2}-\alpha n}\left(\gamma t^{\alpha}\tau\mu^{\alpha}\right)\right.$$
$$\left.\times E_{1,1}\left(\frac{-y^2}{4t^{\alpha}\tau(1-\mu)^{\alpha}}\right)W_{\alpha,\frac{1}{2}-\frac{\alpha}{2}-\alpha m}\left(\beta t^{\alpha}\tau(1-\mu)^{\alpha}\right);s\to 1\right\}\mathrm{d}\mu. \tag{33}$$

Next we will prove the convergence of Green's function of generalized time fractional diffusion equation.

4. Convergence of the Green Function

Lemma 4.1. ([24], p. 38]) Let $F(x)$ be unbounded as $x\to 0$. Then

$$L\{F(x);s\}$$
$$=\int_0^{\infty}\mathrm{e}^{-sx}F(x)\mathrm{d}x$$

exists if the following conditions hold

(i) For some constant a, such that $0<a<1$, $\lim_{x\to 0}x^{a}F(x)=0$.

(ii) The function $F(x)$ be piecewise continuous in every finite interval $N\le x\le M$ ($N>0$).

(iii) For $x>M$, the function $F(x)$ be of exponential order ρ.

Theorem 4.2. For $0<\alpha\le 1$, the Green function (fundamental solution) of generalized time fractional diffusion equation is expressed as follows,

$$u(x,y,t)=\frac{1}{\pi t^{\alpha}}\int_0^1 \mu^{-\frac{1+\alpha}{2}}(1-\mu)^{-\frac{1+\alpha}{2}} L\left\{\tau^{-1}E_{1,1}\left(\frac{-x^2}{4t^{\alpha}\tau\mu^{\alpha}}\right)W_{\alpha,\frac{1}{2}-\frac{\alpha}{2}-\alpha n}\left(\gamma t^{\alpha}\tau\mu^{\alpha}\right)\right.$$
$$\left.\times E_{1,1}\left(\frac{-y^2}{4t^{\alpha}\tau(1-\mu)^{\alpha}}\right)W_{\alpha,\frac{1}{2}-\frac{\alpha}{2}-\alpha m}\left(\beta t^{\alpha}\tau(1-\mu)^{\alpha}\right);s\to 1\right\}\mathrm{d}\mu, \tag{34}$$

which converges.

Proof. Because the convergence condition of Mittag-Leffler function is $\Re(\rho>0)$, $\Re(\mu>0)$ (see (5)), in other words, the Mittag-Leffler function in expression (34) converges. It suffices to show that the Laplace transform of the products of Wright functions is exists.

$$L\left\{W_{\alpha,\frac{1}{2}-\frac{\alpha}{2}-\alpha n}\left(\gamma t^{\alpha}\tau\mu^{\alpha}\right)W_{\alpha,\frac{1}{2}-\frac{\alpha}{2}-\alpha m}\left(\beta t^{\alpha}\tau(1-\mu)^{\alpha}\right);s\to 1\right\}\mathrm{d}\mu$$
$$=\frac{1}{\pi}\int_0^{\infty}\mathrm{e}^{-r}F(r;s)\mathrm{d}r, \tag{35}$$

where

$$F(r;s)=r^{-\frac{1}{2}+\frac{\alpha}{2}+\alpha n}\sum_{k=0}^{\infty}\frac{\left(\beta t^{\alpha}(1-\mu)^{\alpha}\right)^{k}}{\Gamma\left(\alpha k+\frac{1}{2}-\frac{\alpha}{2}-\alpha m\right)}$$
$$\times\frac{1}{\left[\left(s+\gamma t^{\alpha}\mu^{\alpha}r^{-\alpha}\cos(\alpha\pi)\right)^{2}+\left(\gamma t^{\alpha}\mu^{\alpha}r^{-\alpha}\sin(\alpha\pi)\right)^{2}\right]^{\frac{k+1}{2}}} \tag{36}$$
$$\times\sin\left[\left(\frac{1}{2}-\frac{1}{\alpha}-\alpha n\right)\pi-(k+1)\arctan\left(\frac{\gamma t^{\alpha}\mu^{\alpha}r^{-\alpha}\sin(\alpha\pi)}{s+\gamma t^{\alpha}\mu^{\alpha}r^{-\alpha}\cos(\alpha\pi)}\right)\right],$$

We apply the integral representation of Wright function on the Hankel path, the Hankel path which consists of the upper edge of branch cut $\left(\xi=re^{i\pi},\varepsilon\le r<\infty\right)$ the circle $C_{\varepsilon}:=\left(\xi=\varepsilon e^{i\theta},-\pi\le\theta<\pi\right)$ the lower edge of branch cut $\left(\xi=re^{-i\pi},\varepsilon\le r<\infty\right)$ (see Figure 1).

Thus, we write the right hand side of (34) as

$$\begin{aligned}&L\left\{W_{\alpha,\frac{1}{2}-\frac{\alpha}{2}-\alpha n}\left(\gamma t^{\alpha}\tau\mu^{\alpha}\right)W_{\alpha,\frac{1}{2}-\frac{\alpha}{2}-\alpha m}\left(\beta t^{\alpha}\tau(1-\mu)^{\alpha}\right);s\to 1\right\}d\mu\\&=\int_{0}^{\infty}e^{-s\tau}\left(\frac{1}{2\pi i}\int_{Ha}\zeta^{-\frac{1}{2}+\frac{\alpha}{2}+\alpha n}e^{\zeta+\gamma\tau t^{\alpha}\mu^{\alpha}\zeta^{-\alpha}}d\zeta\right)W_{\alpha,\frac{1}{2}-\frac{\alpha}{2}-\alpha m}\left(\beta t^{\alpha}\tau(1-\mu)^{\alpha}\right)d\tau\\&=\frac{1}{2\pi i}\int_{Ha}\zeta^{-\frac{1}{2}+\frac{\alpha}{2}+\alpha n}e^{\zeta}\int_{0}^{\infty}e^{-s\tau}e^{\gamma\tau t^{\alpha}\mu^{\alpha}\zeta^{-\alpha}}W_{\alpha,\frac{1}{2}-\frac{\alpha}{2}-\alpha m}\left(\beta t^{\alpha}\tau(1-\mu)^{\alpha}\right)d\tau d\zeta\\&=\frac{1}{2\pi i}\int_{Ha}\zeta^{-\frac{1}{2}+\frac{\alpha}{2}+\alpha n}e^{\zeta}\left(\frac{1}{s-\gamma t^{\alpha}\mu^{\alpha}\zeta^{-\alpha}}E_{\alpha,\frac{1}{2}-\frac{\alpha}{2}-\alpha m}\left(\frac{\beta t^{\alpha}(1-\mu)^{\alpha}}{s-\gamma t^{\alpha}\mu^{\alpha}\zeta^{-\alpha}}\right)\right)d\zeta.\end{aligned} \tag{37}$$

According to the segmentation of Hankel path, the integral (37) is simplified as

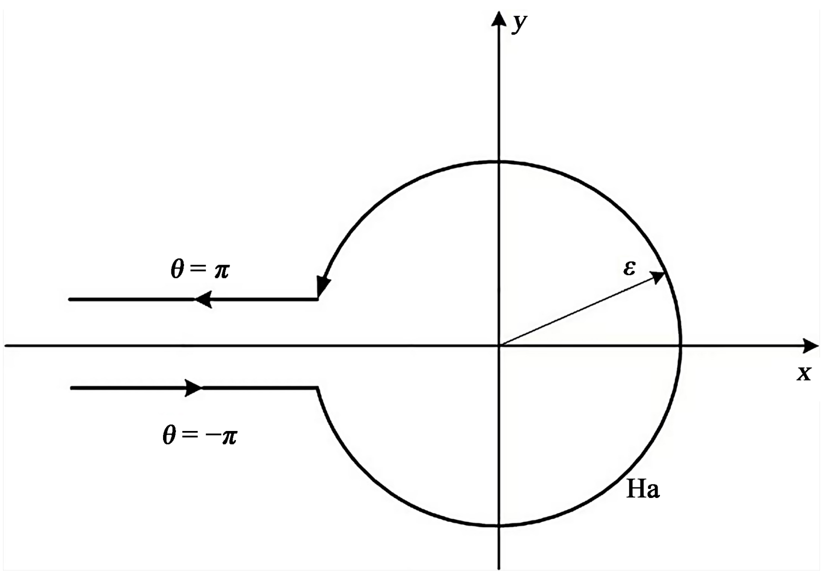

Figure 1. Hankel path.

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{Ha}\zeta^{-\frac{1}{2}+\frac{\alpha}{2}+\alpha n}e^{\zeta}\left(\frac{1}{s-\gamma t^{\alpha}\mu^{\alpha}\zeta^{-\alpha}}E_{\alpha,\frac{1}{2}-\frac{\alpha}{2}-\alpha m}\left(\frac{\beta t^{\alpha}(1-\mu)^{\alpha}}{s-\gamma t^{\alpha}\mu^{\alpha}\zeta^{-\alpha}}\right)\right)d\zeta \\
&=-\frac{1}{2\pi i}\int_{\varepsilon}^{\infty}r^{-\frac{1}{2}+\frac{\alpha}{2}+\alpha n}e^{-\left(\frac{1}{2}-\frac{\alpha}{2}-\alpha n\right)i\pi}e^{-r}\left(\frac{1}{s-\gamma t^{\alpha}\mu^{\alpha}r^{-\alpha}e^{-i\pi\alpha}}E_{\alpha,\frac{1}{2}-\frac{\alpha}{2}-\alpha m}\left(\frac{\beta t^{\alpha}(1-\mu)^{\alpha}}{s-\gamma t^{\alpha}\mu^{\alpha}r^{-\alpha}e^{-i\pi\alpha}}\right)\right)dr \\
&\quad+\frac{1}{2\pi}\int_{-\pi}^{\pi}\varepsilon^{\frac{1}{2}+\frac{\alpha}{2}+\alpha n}e^{\left(\frac{1}{2}+\frac{\alpha}{2}+\alpha n\right)i\theta}e^{\varepsilon e^{i\theta}}\left(\frac{1}{s-\gamma t^{\alpha}\mu^{\alpha}\varepsilon^{-\alpha}e^{-i\theta\alpha}}E_{\alpha,\frac{1}{2}-\frac{\alpha}{2}-\alpha m}\left(\frac{\beta t^{\alpha}(1-\mu)^{\alpha}}{s-\gamma t^{\alpha}\mu^{\alpha}\varepsilon^{-\alpha}e^{-i\theta\alpha}}\right)\right)d\theta \\
&\quad+\frac{1}{2\pi i}\int_{\varepsilon}^{\infty}r^{-\frac{1}{2}+\frac{\alpha}{2}+\alpha n}e^{\left(\frac{1}{2}-\frac{\alpha}{2}-\alpha n\right)i\pi}e^{-r}\left(\frac{1}{s-\gamma t^{\alpha}\mu^{\alpha}r^{-\alpha}e^{i\pi\alpha}}E_{\alpha,\frac{1}{2}-\frac{\alpha}{2}-\alpha m}\left(\frac{\beta t^{\alpha}(1-\mu)^{\alpha}}{s-\gamma t^{\alpha}\mu^{\alpha}r^{-\alpha}e^{i\pi\alpha}}\right)\right)dr.
\end{aligned} \tag{38}$$

Now, let's take the sum of the first and third terms

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{\varepsilon}^{\infty}r^{-\frac{1}{2}+\frac{\alpha}{2}+\alpha n}e^{-r}\left\{\left(\frac{e^{\left(\frac{1}{2}-\frac{\alpha}{2}-\alpha n\right)i\pi}}{s-\gamma t^{\alpha}\mu^{\alpha}r^{-\alpha}e^{i\pi\alpha}}E_{\alpha,\frac{1}{2}-\frac{\alpha}{2}-\alpha m}\left(\frac{\beta t^{\alpha}(1-\mu)^{\alpha}}{s-\gamma t^{\alpha}\mu^{\alpha}r^{-\alpha}e^{i\pi\alpha}}\right)\right)\right. \\
&\left.-\left(\frac{e^{-\left(\frac{1}{2}-\frac{\alpha}{2}-\alpha n\right)i\pi}}{s-\gamma t^{\alpha}\mu^{\alpha}r^{-\alpha}e^{-i\pi\alpha}}E_{\alpha,\frac{1}{2}-\frac{\alpha}{2}-\alpha m}\left(\frac{\beta t^{\alpha}(1-\mu)^{\alpha}}{s-\gamma t^{\alpha}\mu^{\alpha}r^{-\alpha}e^{-i\pi\alpha}}\right)\right)\right\}dr \\
&=\frac{1}{2\pi i}\int_{\varepsilon}^{\infty}r^{-\frac{1}{2}+\frac{\alpha}{2}+\alpha n}e^{-r}\left\{\sum_{k=0}^{\infty}\frac{\left(\beta t^{\alpha}(1-\mu)^{\alpha}\right)^{k}}{\Gamma\left(\alpha k+\frac{1}{2}-\frac{\alpha}{2}-\alpha m\right)}\frac{\cos\left(\left(\frac{1}{2}-\frac{\alpha}{2}-\alpha n\right)\pi+i\sin\left(\left(\frac{1}{2}-\frac{\alpha}{2}-\alpha n\right)\pi\right)\right)}{\left(s-\gamma t^{\alpha}\mu^{\alpha}r^{-\alpha}\cos(\pi\alpha)-i\gamma t^{\alpha}\mu^{\alpha}r^{-\alpha}\sin(\pi\alpha)\right)^{k+1}}\right. \\
&\left.-\sum_{k=0}^{\infty}\frac{\left(\beta t^{\alpha}(1-\mu)^{\alpha}\right)^{k}}{\Gamma\left(\alpha k+\frac{1}{2}-\frac{\alpha}{2}-\alpha m\right)}\frac{\cos\left(\left(\frac{1}{2}-\frac{\alpha}{2}-\alpha n\right)\pi-i\sin\left(\left(\frac{1}{2}-\frac{\alpha}{2}-\alpha n\right)\pi\right)\right)}{\left(s-\gamma t^{\alpha}\mu^{\alpha}r^{-\alpha}\cos(\pi\alpha)+i\gamma t^{\alpha}\mu^{\alpha}r^{-\alpha}\sin(\pi\alpha)\right)^{k+1}}\right\}dr.
\end{aligned} \tag{39}$$

The first series of (39) is rewritten as

$$\begin{aligned}
&\sum_{k=0}^{\infty}\frac{\left(\beta t^{\alpha}(1-\mu)^{\alpha}\right)^{k}}{\Gamma\left(\alpha k+\frac{1}{2}-\frac{\alpha}{2}-\alpha m\right)}\frac{\cos\left(\left(\frac{1}{2}-\frac{\alpha}{2}-\alpha n\right)\pi+i\sin\left(\left(\frac{1}{2}-\frac{\alpha}{2}-\alpha n\right)\pi\right)\right)}{\left(s-\gamma t^{\alpha}\mu^{\alpha}r^{-\alpha}\cos(\pi\alpha)-i\gamma t^{\alpha}\mu^{\alpha}r^{-\alpha}\sin(\pi\alpha)\right)^{k+1}} \\
&=\sum_{k=0}^{\infty}\frac{\left(\beta t^{\alpha}(1-\mu)^{\alpha}\right)^{k}\cos\left(\left(\frac{1}{2}-\frac{\alpha}{2}-\alpha n\right)\pi+i\sin\left(\left(\frac{1}{2}-\frac{\alpha}{2}-\alpha n\right)\pi\right)\right)}{\Gamma\left(\alpha k+\frac{1}{2}-\frac{\alpha}{2}-\alpha m\right)} \\
&\quad\times\frac{e^{-i(k+1)\arctan\left(\frac{\gamma t^{\alpha}\mu^{\alpha}r^{-\alpha}\sin(\pi\alpha)}{s-\gamma t^{\alpha}\mu^{\alpha}r^{-\alpha}\cos(\pi\alpha)}\right)}}{\left(\left(s-\gamma t^{\alpha}\mu^{\alpha}r^{-\alpha}\cos(\pi\alpha)\right)^{2}+\left(\gamma t^{\alpha}\mu^{\alpha}r^{-\alpha}\sin(\pi\alpha)\right)^{2}\right)^{\frac{k+1}{2}}}
\end{aligned}$$

$$=\sum_{k=0}^{\infty}\frac{\left(\beta t^{\alpha}(1-\mu)^{\alpha}\right)^{k}}{\Gamma\left(\alpha k+\frac{1}{2}-\frac{\alpha}{2}-\alpha m\right)\left(\left(s-\gamma t^{\alpha}\mu^{\alpha}r^{-\alpha}\cos(\pi\alpha)\right)^{2}+\left(\gamma t^{\alpha}\mu^{\alpha}r^{-\alpha}\sin(\pi\alpha)\right)^{2}\right)^{\frac{k+1}{2}}}$$
$$\times\left[\cos\left(\left(\frac{1}{2}-\frac{\alpha}{2}-\alpha n\right)\pi-(k+1)\arctan\left(\frac{\gamma t^{\alpha}\mu^{\alpha}r^{-\alpha}\sin(\pi\alpha)}{s-\gamma t^{\alpha}\mu^{\alpha}r^{-\alpha}\cos(\pi\alpha)}\right)\right)\right.$$
$$\left.+i\sin\left(\left(\frac{1}{2}-\frac{\alpha}{2}-\alpha n\right)\pi-(k+1)\arctan\left(\frac{\gamma t^{\alpha}\mu^{\alpha}r^{-\alpha}\sin(\pi\alpha)}{s-\gamma t^{\alpha}\mu^{\alpha}r^{-\alpha}\cos(\pi\alpha)}\right)\right)\right],\tag{40}$$

and the second series of (39) is rewritten as

$$\sum_{k=0}^{\infty}\frac{\left(\beta t^{\alpha}(1-\mu)^{\alpha}\right)^{k}}{\Gamma\left(\alpha k+\frac{1}{2}-\frac{\alpha}{2}-\alpha m\right)}\frac{\cos\left(\left(\frac{1}{2}-\frac{\alpha}{2}-\alpha n\right)\pi-i\sin\left(\left(\frac{1}{2}-\frac{\alpha}{2}-\alpha n\right)\pi\right)\right)}{\left(s-\gamma t^{\alpha}\mu^{\alpha}r^{-\alpha}\cos(\pi\alpha)-i\gamma t^{\alpha}\mu^{\alpha}r^{-\alpha}\sin(\pi\alpha)\right)^{k+1}}$$
$$=\sum_{k=0}^{\infty}\frac{\left(\beta t^{\alpha}(1-\mu)^{\alpha}\right)^{k}}{\Gamma\left(\alpha k+\frac{1}{2}-\frac{\alpha}{2}-\alpha m\right)\left(\left(s-\gamma t^{\alpha}\mu^{\alpha}r^{-\alpha}\cos(\pi\alpha)\right)^{2}+\left(\gamma t^{\alpha}\mu^{\alpha}r^{-\alpha}\sin(\pi\alpha)\right)^{2}\right)^{\frac{k+1}{2}}}$$
$$\times\left[\cos\left(\left(\frac{1}{2}-\frac{\alpha}{2}-\alpha n\right)\pi-(k+1)\arctan\left(\frac{\gamma t^{\alpha}\mu^{\alpha}r^{-\alpha}\sin(\pi\alpha)}{s-\gamma t^{\alpha}\mu^{\alpha}r^{-\alpha}\cos(\pi\alpha)}\right)\right)\right.$$
$$\left.-i\sin\left(\left(\frac{1}{2}-\frac{\alpha}{2}-\alpha n\right)\pi-(k+1)\arctan\left(\frac{\gamma t^{\alpha}\mu^{\alpha}r^{-\alpha}\sin(\pi\alpha)}{s-\gamma t^{\alpha}\mu^{\alpha}r^{-\alpha}\cos(\pi\alpha)}\right)\right)\right].\tag{41}$$

Finally, the integral (39) can be written as

$$I_1=\frac{1}{\pi}\int_{\varepsilon}^{\infty}r^{-\frac{1}{2}+\frac{\alpha}{2}+\alpha n}\mathrm{e}^{-r}\left[\sum_{k=0}^{\infty}\frac{\left(\beta t^{\alpha}(1-\mu)^{\alpha}\right)^{k}}{\Gamma\left(\alpha k+\frac{1}{2}-\frac{\alpha}{2}-\alpha m\right)\left(\left(s-\gamma t^{\alpha}\mu^{\alpha}r^{-\alpha}\cos(\pi\alpha)\right)^{2}+\left(\gamma t^{\alpha}\mu^{\alpha}r^{-\alpha}\sin(\pi\alpha)\right)^{2}\right)^{\frac{k+1}{2}}}\right.$$
$$\left.\times\sin\left(\left(\frac{1}{2}-\frac{\alpha}{2}-\alpha n\right)\pi-(k+1)\arctan\left(\frac{\gamma t^{\alpha}\mu^{\alpha}r^{-\alpha}\sin(\pi\alpha)}{s-\gamma t^{\alpha}\mu^{\alpha}r^{-\alpha}\cos(\pi\alpha)}\right)\right)\right]\mathrm{d}r.\tag{42}$$

For the second integral expression in the right hand of (38), we have

$$I_2=\frac{1}{2\pi}\int_{-\pi}^{\pi}\varepsilon^{\frac{1}{2}+\frac{\alpha}{2}+\alpha n}\mathrm{e}^{\left(\frac{1}{2}+\frac{\alpha}{2}+\alpha n\right)i\theta}\mathrm{e}^{\varepsilon\mathrm{e}^{i\theta}}\left(\frac{1}{s-\gamma t^{\alpha}\mu^{\alpha}\varepsilon^{-\alpha}\mathrm{e}^{-i\theta\alpha}}E_{\alpha,\frac{1}{2}-\frac{\alpha}{2}-\alpha m}\left(\frac{\beta t^{\alpha}(1-\mu)^{\alpha}}{s-\gamma t^{\alpha}\mu^{\alpha}\varepsilon^{-\alpha}\pi^{-i\theta\alpha}}\right)\right)\mathrm{d}\theta$$
$$=\frac{\varepsilon^{\frac{1}{2}+\frac{\alpha}{2}+\alpha n}}{2\pi}\int_{-\pi}^{\pi}\mathrm{e}^{\varepsilon\cos\theta}\left[\sum_{k=0}^{\infty}\frac{\left(\beta t^{\alpha}(1-\mu)^{\alpha}\right)^{k}}{\Gamma\left(\alpha k+\frac{1}{2}-\frac{\alpha}{2}-\alpha m\right)\left(\left(s-\gamma t^{\alpha}\mu^{\alpha}\varepsilon^{-\alpha}\cos(\theta\alpha)\right)^{2}+\left(\gamma t^{\alpha}\mu^{\alpha}\varepsilon^{-\alpha}\sin(\theta\alpha)\right)^{2}\right)^{\frac{k+1}{2}}}\right.$$
$$\left.\times\cos\left(\left(\frac{1}{2}+\frac{\alpha}{2}+\alpha n\right)\theta+\varepsilon\sin\theta+(k+1)\arctan\left(\frac{\gamma t^{\alpha}\mu^{\alpha}\varepsilon^{-\alpha}\sin(\theta\alpha)}{s-\gamma t^{\alpha}\mu^{\alpha}\varepsilon^{-\alpha}\cos(\theta\alpha)}\right)\right)\right]\mathrm{d}r.\tag{43}$$

To prove convergence of Green's function, we have to determine the behaviours of integrals I_1 and I_2 when $\varepsilon \to 0$. It is obvious that the integral I_1 converges to the following improper integral

$$\frac{1}{\pi}\int_0^\infty \mathrm{e}^{-r} F(r;s)\,\mathrm{d}r,$$

where

$$\begin{aligned} F(r;s) = {} & r^{-\frac{1}{2}+\frac{\alpha}{2}+\alpha n}\sum_{k=0}^{\infty}\frac{\left(\beta t^\alpha (1-\mu)^\alpha\right)^k}{\Gamma\left(\alpha k+\frac{1}{2}-\frac{\alpha}{2}-\alpha m\right)} \\ & \times \frac{1}{\left[\left(s-\gamma t^\alpha \mu^\alpha r^{-\alpha}\cos(\alpha\pi)\right)^2+\left(\gamma t^\alpha \mu^\alpha r^{-\alpha}\sin(\alpha\pi)\right)^2\right]^{\frac{k+1}{2}}} \\ & \times \sin\left[\left(\frac{1}{2}-\frac{1}{\alpha}-\alpha n\right)\pi-(k+1)\arctan\left(\frac{\gamma t^\alpha \mu^\alpha r^{-\alpha}\sin(\alpha\pi)}{s-\gamma t^\alpha \mu^\alpha r^{-\alpha}\cos(\alpha\pi)}\right)\right]. \end{aligned} \tag{44}$$

Obviously, $F(r)$ satisfies conditions (ii) and (iii) of Lemma 4.1, so now let's prove that $F(r)$ also satisfies the condition (i) of Lemma 4.1. Because $0<\alpha\le 1$, $n\in N$, $s\to 1$,

$$\begin{aligned} \lim_{r\to 0} r^a F(r;s) = {} & \lim_{r\to 0} r^{a-\frac{1}{2}+\frac{\alpha}{2}+\alpha n}\lim_{r\to 0}\sum_{k=0}^{\infty}\frac{\left(\beta t^\alpha (1-\mu)^\alpha\right)^k}{\Gamma\left(\alpha k+\frac{1}{2}-\frac{\alpha}{2}-\alpha m\right)} \\ & \times \frac{1}{\left[\left(s-\gamma t^\alpha \mu^\alpha r^{-\alpha}\cos(\alpha\pi)\right)^2+\left(\gamma t^\alpha \mu^\alpha r^{-\alpha}\sin(\alpha\pi)\right)^2\right]^{\frac{k+1}{2}}} \\ & \times \sin\left[\left(\frac{1}{2}-\frac{1}{\alpha}-\alpha n\right)\pi-(k+1)\arctan\left(\frac{\gamma t^\alpha \mu^\alpha r^{-\alpha}\sin(\alpha\pi)}{s-\gamma t^\alpha \mu^\alpha r^{-\alpha}\cos(\alpha\pi)}\right)\right] \\ = {} & 0. \end{aligned} \tag{45}$$

For $\left(\frac{1}{2}-\frac{1}{\alpha}-\alpha n\right)<a<1$, the function $r^a F(r)$ converges to zero when $r\to 0$. At the same time, when $\varepsilon\to 0$, the value of the integral I_2 is obtained as follows

$$\begin{aligned} \lim_{\varepsilon\to 0} I_2 = {} & \lim_{\varepsilon\to 0}\frac{1}{2\pi}\int_{-\pi}^{\pi}\varepsilon^{\frac{1}{2}+\frac{\alpha}{2}+\alpha n}\mathrm{e}^{\left(\frac{1}{2}+\frac{\alpha}{2}+\alpha n\right)i\theta}\mathrm{e}^{\varepsilon \mathrm{e}^{i\theta}}\left(\frac{1}{s-\gamma t^\alpha \mu^\alpha \varepsilon^{-\alpha}\mathrm{e}^{-i\theta\alpha}}E_{\alpha,\frac{1}{2}-\frac{\alpha}{2}-\alpha m}\left(\frac{\beta t^\alpha (1-\mu)^\alpha}{s-\gamma t^\alpha \mu^\alpha \varepsilon^{-\alpha}\mathrm{e}^{-i\theta\alpha}}\right)\right)\mathrm{d}\theta \\ = {} & \lim_{\varepsilon\to 0}\frac{\varepsilon^{\frac{1}{2}+\frac{\alpha}{2}+\alpha n}}{2\pi}\int_{-\pi}^{\pi}\mathrm{e}^{\varepsilon\cos\theta}\left[\sum_{k=0}^{\infty}\frac{\left(\beta t^\alpha (1-\mu)^\alpha\right)^k}{\Gamma\left(\alpha k+\frac{1}{2}-\frac{\alpha}{2}-\alpha m\right)\left(\left(s-\gamma t^\alpha \mu^\alpha \varepsilon^{-\alpha}\cos(\theta\alpha)\right)^2+\left(\gamma t^\alpha \mu^\alpha \varepsilon^{-\alpha}\sin(\theta\alpha)\right)^2\right)^{\frac{k+1}{2}}}\right. \\ & \left.\times\cos\left(\left(\frac{1}{2}+\frac{\alpha}{2}+\alpha n\right)\theta+\varepsilon\sin\theta+(k+1)\arctan\left(\frac{\gamma t^\alpha \mu^\alpha \varepsilon^{-\alpha}\sin(\theta\alpha)}{s-\gamma t^\alpha \mu^\alpha \varepsilon^{-\alpha}\cos(\theta\alpha)}\right)\right)\right]\mathrm{d}r. \end{aligned} \tag{46}$$

Because $\frac{1}{2}+\frac{\alpha}{2}+\alpha n>0$,

$$\lim_{\varepsilon\to 0} I_2 \to 0. \tag{47}$$

So theorem is proved.

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Notes on Contributors

All the authors contributed to each part of this study equally and agreed to the final version of the manuscript.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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