Journal of Applied Mathematics and Physics, 2022, 10, 2690-2697
https://www.scirp.org/journal/jamp
ISSN Online: 2327-4379
ISSN Print: 2327-4352

Analytical Solutions to Definite Integrals for Combinations of Legendre, Bessel and Trigonometric Functions Encountered in Propagation and Scattering Problems in Spherical Coordinates

Farhad Azadi Namin

Department of Electrical Engineering, Amirkabir University of Technology (Tehran Polytechnic), Tehran, Iran
Email: namin_farhad@aut.ac.ir

How to cite this paper: Namin, F.A. (2022) Analytical Solutions to Definite Integrals for Combinations of Legendre, Bessel and Trigonometric Functions Encountered in Propagation and Scattering Problems in Spherical Coordinates. *Journal of Applied Mathematics and Physics*, **10**, 2690-2697.
https://doi.org/10.4236/jamp.2022.109179

Received: July 19, 2022
Accepted: September 17, 2022
Published: September 20, 2022

Abstract

Mie theory is a rigorous solution to scattering problems in spherical coordinate system. The first step in applying Mie theory is expansion of some arbitrary incident field in terms of spherical harmonics fields in terms of spherical which in turn requires evaluation of certain definite integrals whose integrands are products of Bessel functions, associated Legendre functions and periodic functions. Here we present analytical results for two specific integrals that are encountered in expansion of arbitrary fields in terms of summation of spherical waves. The analytical results are in terms of finite summations which include Lommel functions. A concise analytical expression is also derived for the special case of Lommel functions that arise, rendering expensive numerical integration or other iterative techniques unnecessary.

Keywords

Mie Theory, Vector Spherical Wave Function, Eigen-Function Expansion, Spherical Harmonics, Special Functions

1. Introduction

In recent years, generalized multiparticle Mie theory (GMT) [1] has been widely used to analyze plasmonic nanoparticle arrays [2] [3]. The first step in applying GMT requires the expansion of incident fields in terms of vector spherical wave functions (VSWF) [4]. VSWFs constitute a complete orthogonal eigen-basis by which every physically realizable wave can be expanded. Determining the expansion coefficients requires evaluating the inner product of the wave of interest with VSWFs. These inner products are definite double integrals where the integrands are products of special functions. With very few exceptions [4] [5] [6] these integrals are intractable and have to be evaluated using quadrature methods which itself presents a degree of difficulty due to the fact that evaluating special function can be a computationally expensive task.

In many practical and experimental applications an incident field with a finite beamwidth is obtained by placing circular aperture in front of a plane wave. The diffracted waves in term act as the incident field for an array of spheres. This scattering problem can be evaluated using generalized multiparticle Mie theory if we can expand the diffracted fields in terms of spherical harmonics. In doing so, we encountered two integrals to which we were not able to find analytical solutions in the literature or using several commercial mathematical softwares. As noted before numerical evaluations were computationally expensive and also precise enough. We were eventually able to obtain closed form expressions for these two integrals which greatly reduced the computational burden. These results were successfully used in two other publications [7] [8], however we did not explain in detail how they were obtained. But we felt a detailed derivation of these results and the methods used can be beneficial to the scientific community and in this paper we present the results along with complete details.

We assume a mono-chromatic plane wave with time-variation of the form $\mathrm{e}^{-i\omega t}$ and linearly polarized electric field of the form $\mathbf{E} = \mathbf{x}\mathrm{e}^{ikz}$ incident on a circular aperture of radius a in the xy-plane at $z = 0$ from the upper half-space. The fields diffracted from the aperture in the lower half-space ($z < 0$) generally act as the incident field for an array of spherical particles [7]. We can obtain an expression for the far-field approximation diffracted fields which we denote by $\mathbf{E}_{\mathrm{inc}}$ using the Kirchhoff integral [9]:

$$\mathbf{E}_{\mathrm{inc}} = E_\theta \hat{\boldsymbol{\theta}} + E_\phi \hat{\boldsymbol{\phi}} \tag{1}$$

where,

$$E_\theta = \frac{-iak\mathrm{e}^{i\rho} J_1(ka\sin\theta)\cos\phi}{\rho\sin\theta} \tag{2}$$

$$E_\phi = \frac{iak\mathrm{e}^{i\rho} J_1(ka\sin\theta)\sin\phi\cos\theta}{\rho\sin\theta} \tag{3}$$

where J_1 is the first order Bessel function of the first kind.

The incident electric field can be expanded in terms of vector spherical basis functions $\mathbf{N}_{mn}^{(1)}$, and $\mathbf{M}_{mn}^{(1)}$ as

$$\mathbf{E}_{\mathrm{inc}} = -\sum_{n=1}^{\infty}\sum_{m=-n}^{n} iE_{mn}\left[p_{mn}\mathbf{N}_{mn}^{(1)}(\rho,\theta,\phi) + q_{mn}\mathbf{M}_{mn}^{(1)}(\rho,\theta,\phi)\right] \tag{4}$$

where (r,θ,ϕ) are the spherical coordinates and $\rho = kr$. The terms E_{mn}, $\mathbf{N}_{mn}^{(1)}$, and $\mathbf{M}_{mn}^{(1)}$ are vector spherical wave functions (VSWF) defined as [1]

$$\mathbf{N}_{mn}^{(1)}(\rho,\theta,\phi)=\left\{\hat{\boldsymbol{r}}n(n+1)P_n^m(\cos\theta)\frac{j_n(\rho)}{\rho}\right.$$
$$\left.+\left[\hat{\boldsymbol{\theta}}\tau_{mn}(\cos\theta)+\hat{\boldsymbol{\phi}}i\pi_{mn}(\cos\theta)\right]\frac{\psi_n'(\rho)}{\rho}\right\}\mathrm{e}^{im\phi} \tag{5}$$

$$\mathbf{M}_{mn}^{(1)}(\rho,\theta,\phi)=\left[\hat{\boldsymbol{\theta}}i\pi_{mn}(\cos\theta)-\hat{\boldsymbol{\phi}}\tau_{mn}(\cos\theta)\right]j_n(\rho)\mathrm{e}^{im\phi} \tag{6}$$

$$E_{mn}=i^n\left[\frac{(2n+1)(n-m)!}{n(n+1)(n+m)!}\right]^{1/2} \tag{7}$$

where $P_n^m(\cos\theta)$ is the associated Legendre function of the first kind of degree n and order m, j_n is the spherical Bessel function of the first kind, and $\psi_n(\rho)=\rho j_n(\rho)$ is the Riccati-Bessel function. In addition, the functions $\pi_{mn}(\cos\theta)$ and $\tau_{mn}(\cos\theta)$ are defiend as

$$\pi_{mn}(\cos\theta)=\frac{m}{\sin\theta}P_n^m(\cos\theta) \tag{8}$$

$$\tau_{mn}(\cos\theta)=\frac{\mathrm{d}}{\mathrm{d}\theta}P_n^m(\cos\theta) \tag{9}$$

Vector spherical wave functions $\mathbf{N}_{mn}^{(1)}$, and $\mathbf{M}_{mn}^{(1)}$ form a complete eigen-basis. The expansion coefficients p_{mn} and q_{mn} are obtained using the orthogonality of VSWFs which leads to

$$q_{mn}=\frac{i\int_0^{2\pi}\int_0^{\pi}\mathbf{E}_{\text{inc}}\cdot\mathbf{M}_{mn}^{(1)*}\sin\theta\mathrm{d}\theta\mathrm{d}\phi}{E_{mn}\int_0^{2\pi}\int_0^{\pi}\left|\mathbf{M}_{mn}^{(1)}\right|^2\sin\theta\mathrm{d}\theta\mathrm{d}\phi} \tag{10}$$

$$p_{mn}=\frac{i\int_0^{2\pi}\int_0^{\pi}\mathbf{E}_{\text{inc}}\cdot\mathbf{N}_{mn}^{(1)*}\sin\theta\mathrm{d}\theta\mathrm{d}\phi}{E_{mn}\int_0^{2\pi}\int_0^{\pi}\left|\mathbf{N}_{mn}^{(1)}\right|^2\sin\theta\mathrm{d}\theta\mathrm{d}\phi} \tag{11}$$

In the process of evaluating the expansion coefficients, two integrals are encountered that involve Bessel, trigonometric, and associated Legendre functions which we denote as $\Pi_n^1(\alpha)$ and $\Pi_n^2(\alpha)$:

$$\Pi_n^1(\alpha)=\int_0^{\pi/2}J_1(\alpha\sin\theta)\big(\pi_{1n}(\cos\theta)+\cos\theta\tau_{1n}(\cos\theta)\big)\mathrm{d}\theta \tag{12}$$

$$\Pi_n^2(\alpha)=\int_0^{\pi/2}J_1(\alpha\sin\theta)\big(\tau_{1n}(\cos\theta)+\cos\theta\pi_{1n}(\cos\theta)\big)\mathrm{d}\theta \tag{13}$$

Integrals (12) and (13) are encountered are encountered when evaluating the integrals for p_{mn} and q_{mn} and closed form expressions for them are not available in the literature, including mathematical handbooks, and cannot be found using commercial software package such as Mathematica. Here we obtain analytical expressions for these integrals in terms of finite summations involving Lommel functions. In general, evaluation of Lommel functions is a numerically expensive task; however it is possible to derive a concise analytical expression for the particular order of Lommel functions that were encountered here. There have been several publications in recent years on related integrals [10] [11]. The availability of these analytical expressions can be valuable in different fields of physics such as electromagnetics or optics which require wave expansions in terms of spherical harmonics.

2. Evaluation of $\Pi_n^1(\alpha)$

We start by considering integral (12) and rewriting the integrand in terms of $P_n^1(\cos\theta)$ using (8) and (9)

$$\Pi_n^1(\alpha) = \int_0^{\pi/2} J_1(\alpha\sin\theta)\left[\frac{P_n^1(\cos\theta)}{\sin\theta} + \cos\theta\frac{\mathrm{d}P_n^1(\cos\theta)}{\mathrm{d}\theta}\right]\mathrm{d}\theta \tag{14}$$

Here we adopt the convention for the associated Legendre function which omits the $(-1)^m$ term [12]

$$P_n^m(x) = (1-x^2)^{m/2}\frac{\mathrm{d}^m}{\mathrm{d}x^m}P_n(x) \tag{15}$$

where P_n represents the Legendre polynomial of order *n*. In the remainder of the paper for the sake of brevity we forgo writing the argument for functions π_{mn}, τ_{mn}, P_n^m, and P_n. Unless otherwise stated, it is always assumed that these functions have $\cos\theta$ as their argument. As the next step, we consider Legendre's differential equation [13]

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}P_n}{\mathrm{d}\theta}\right) = -n(n+1)\sin\theta P_n \tag{16}$$

By performing the differentiation in (16) and using $\frac{\mathrm{d}P_n}{\mathrm{d}\theta} = -P_n^1$, it can be shown that

$$\frac{P_n^1}{\sin\theta} + \cos\theta\frac{\mathrm{d}P_n^1}{\mathrm{d}\theta} = n(n+1)\cos\theta P_n + \sin\theta P_n^1 \tag{17}$$

Thus $\Pi_n^1(\alpha)$ can be decomposed as

$$n(n+1)\int_0^{\pi/2} J_1(\alpha\sin\theta)\cos\theta P_n\mathrm{d}\theta + \int_0^{\pi/2} J_1(\alpha\sin\theta)\sin\theta P_n^1\mathrm{d}\theta \tag{18}$$

Here we define two auxiliary functions $\Omega_n(\alpha)$ and $\Xi_n(\alpha)$ based on the two integrals in (18)

$$\Omega_n(\alpha) \equiv \int_0^{\pi/2} J_1(\alpha\sin\theta)\cos\theta P_n\mathrm{d}\theta \tag{19}$$

$$\Xi_n(\alpha) \equiv \int_0^{\pi/2} J_1(\alpha\sin\theta)\sin\theta P_n^1\mathrm{d}\theta \tag{20}$$

From Gradshteyn and Ryzhik [14] we have the following expansion for $P_n(x)$

$$P_n(x) = \frac{1}{2^n}\sum_{k=0}^{\left\lfloor \frac{n}{2}\right\rfloor}\frac{(-1)^k(2n-2k)!}{k!(n-k)!(n-2k)!}x^{n-2k} \tag{21}$$

where $\lfloor\cdot\rfloor$ denotes the floor function.

We start by considering $\Omega_n(\alpha)$ where, by using (21), the integrand in (19) can be expressed as a finite summation. Furthermore, due to the linearity of the integral operation, the order for the integration and summation can be reversed. As a result, the problem is reduced to dealing with a summation of integrals of the form

$$\int_0^{\pi/2} (\cos\theta)^{n-2t+1} J_1(\alpha\sin\theta)\mathrm{d}\theta \tag{22}$$

The integral in (22) can be evaluated using the following result [14]

$$\int_0^{\pi/2} J_\mu(z\sin\theta)(\sin\theta)^{1-\mu}(\cos\theta)^{2\nu+1}\mathrm{d}\theta = \frac{s_{(\mu+\nu,\nu-\mu+1)}(z)}{2^{\mu-1}z^{\nu+1}\Gamma(\mu)} \tag{23}$$

where Γ is the gamma function and $s_{\mu,\nu}(z)$ is the Lommel function defined as

$$s_{\mu,\nu}(z) = \frac{\pi}{2}\left[Y_\nu(z)\int_0^z z^\mu J_\nu(z)\mathrm{d}z - J_\nu(z)\int_0^z z^\mu Y_\nu(z)\mathrm{d}z\right] \tag{24}$$

and Y_ν is the Bessel function of the second kind of order ν. Letting $\mu=1$, $2\nu=n-2t$, and $z=\alpha$ in (23) we arrive at

$$\int_0^{\pi/2} \cos\theta^{(n-2t+1)} J_1(\alpha\sin\theta)\mathrm{d}\theta = \frac{s_{\left(1-t+\frac{n}{2},\frac{n}{2}-t\right)}(\alpha)}{\alpha^{\left(1-t+\frac{n}{2}\right)}} \tag{25}$$

Numerical evaluation of Lommel functions can be in general a challenging undertaking. However, at this point a closer inspection of (25) reveals that it is only necessary to consider Lommel functions of the form $s_{\nu+1,\nu}$. Based on the integral definition given in (24) we have

$$s_{\nu+1,\nu}(z) = \frac{\pi}{2}\left[Y_\nu(z)\int_0^z z^{\nu+1} J_\nu(z)\mathrm{d}z - J_\nu(z)\int_0^z z^{\nu+1} Y_\nu(z)\mathrm{d}z\right] \tag{26}$$

Fortunately both integrals encountered in (26) have closed form solutions

$$\int x^{p+1} Z_p(x)\mathrm{d}x = x^{p+1} Z_{p+1}(x) \tag{27}$$

where Z_p represents an arbitrary Bessel function of order *p*. In evaluating (26) using (27), we encounter a $0\times\infty$ indeterminate form due to the term $z^{\nu+1}Y_{\nu+1}(z)$ when evaluated at $z=0$. In this case the small argument limit of $Y_\nu(z)$ [13] can be used to show that

$$\lim_{z\to 0} z^{\nu+1}Y_{\nu+1}(z) = \frac{-\Gamma(\nu+1)2^{\nu+1}}{\pi} \tag{28}$$

After some algebra, the Lommel functions may be written in the form

$$s_{\nu+1,\nu}(z) = z^\nu - 2^\nu\Gamma(\nu+1)J_\nu(z) \tag{29}$$

Combining all the results and after some further simplifications we arrive at the following expression for $\Omega_n(\alpha)$

$$\Omega_n(\alpha) = \frac{1}{\alpha} - \frac{1}{2^n}\sum_{t=0}^{\left\lfloor\frac{n}{2}\right\rfloor}\frac{(-1)^t(2n-2t)!}{t!(n-t)!(n-2t)!}\frac{\Gamma\left(1-t+\frac{n}{2}\right)2^{\left(\frac{n}{2}-t\right)}J_{\left(\frac{n}{2}-t\right)}(\alpha)}{\alpha^{\left(1-t+\frac{n}{2}\right)}} \tag{30}$$

Next we consider the function $\Xi_n(\alpha)$ defined in (20). By differentiating (21) with respect to θ and using the fact that $P_n^1 = -\frac{\mathrm{d}P_n}{\mathrm{d}\theta}$ the following expansion for P_n^1 can be obtained:

$$P_n^1(\cos\theta) = \frac{\sin\theta}{2^n} \sum_{t=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \frac{(-1)^t (2n-2t)!}{t!(n-t)!(n-2t-1)!} (\cos\theta)^{n-2t-1} \tag{31}$$

Using this expansion for P_n^1, the integrand in (20) can be cast in the form of the summation given below:

$$\frac{J_1(\alpha\sin\theta)\sin^2\theta}{2^n} \sum_{t=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \frac{(-1)^t (2n-2t)!(\cos\theta)^{(n-2t-1)}}{t!(n-t)!(n-2t-1)!} \tag{32}$$

As done previously, the order of integration and summation can be reversed. The integrals inside the summation can be evaluated using the following result [14]

$$\int_0^{\pi/2} J_\mu(\alpha\sin\theta)(\sin\theta)^{\mu+1}(\cos\theta)^{2\rho+1}\,\mathrm{d}\theta = 2^\rho\Gamma(\rho+1)\alpha^{(-\rho-1)}J_{(\rho+\mu+1)}(\alpha) \tag{33}$$

Setting $\mu = 1$ and $\rho = \frac{n}{2} - t - 1$ leads to

$$\int_0^{\pi/2} J_1(\alpha\sin\theta)\sin^2\theta(\cos\theta)^{(n-2t-1)}\,\mathrm{d}\theta = 2^{\left(\frac{n}{2}-t-1\right)}\Gamma\left(\frac{n}{2}-t\right)\alpha^{\left(t-\frac{n}{2}\right)}J_{\left(\frac{n}{2}-t+1\right)}(\alpha) \tag{34}$$

Finally by using this result we arrive at the following expression for $\Xi_n(\alpha)$

$$\Xi_n(\alpha) = \frac{1}{2^n} \sum_{t=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \frac{(-1)^t (2n-2t)!2^{\left(\frac{n}{2}-t-1\right)}\Gamma\left(\frac{n}{2}-t\right)\alpha^{\left(t-\frac{n}{2}\right)}J_{\left(\frac{n}{2}-t+1\right)}(\alpha)}{t!(n-t)!(n-2t-1)!} \tag{35}$$

Thus we have obtained the solution for $\Pi_n^1(\alpha)$ defined by integral (12) as

$$\Pi_n^1(\alpha) = n(n+1)\Omega_n(\alpha) + \Xi_n(\alpha) \tag{36}$$

where $\Omega_n(\alpha)$ and $\Xi_n(\alpha)$ were shown to have closed form representations given by (30) and (35) respectively.

3. Evaluation of $\Pi_n^2(\alpha)$

To evaluate $\Pi_n^2(\alpha)$, we start by writing (13) as

$$\Pi_n^2(\alpha) = \int_0^{\pi/2} \frac{J_1(\alpha\sin\theta)}{\sin\theta}\left(\frac{\mathrm{d}P_n^1}{\mathrm{d}\theta} + \cos\theta\frac{P_n^1}{\sin\theta}\right)\sin\theta\,\mathrm{d}\theta \tag{37}$$

It can easily be shown that

$$\left(\frac{\mathrm{d}P_n^1}{\mathrm{d}\theta} + \cos\theta\frac{P_n^1}{\sin\theta}\right)\sin\theta = \frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta P_n^1\right) \tag{38}$$

Using Legendre's differential Equation (16) and $\frac{\mathrm{d}P_n}{\mathrm{d}\theta} = -P_n^1$ it can be shown that

$$\frac{\Pi_n^2(\alpha)}{n(n+1)} = \int_0^{\pi/2} J_1(\alpha \sin\theta) P_n \mathrm{d}\theta \tag{39}$$

Evaluating the integral in (39) follows a very similar procedure to that already presented in Section 2 for $\Omega_n(\alpha)$. We start by using the expansion in (21) to rewrite the integrand as a summation. Reversing the order of summation and integration, the resulting integrals can be evaluated using (23). As before, Lommel functions of the form $s_{\nu+1,\nu}(z)$ are encountered in the results, which can be evaluated using (29). Combining all the results and upon further simplifications, we arrive at

$$\frac{\Pi_n^2(\alpha)}{n(n+1)} = \frac{1}{\alpha} - \frac{1}{2^n} \sum_{t=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{(-1)^t (2n-2t)!}{t!(n-t)!(n-2t)!} \frac{\Gamma\left(\frac{n+1}{2} - t\right) 2^{\left(\frac{n-1}{2} - t\right)} J_{\left(\frac{n-1}{2} - t\right)}(\alpha)}{\alpha^{\left(\frac{n+1}{2} - t\right)}} \tag{40}$$

4. Conclusions and Suggestions

In this paper, we obtained analytical results for two definite integrals which include products of Bessel, Legendre, and trigonometric functions. These integrals are encountered when expanding aperture-diffracted fields in terms of VSWF. These results are especially important for electromagnetic and optical scattering problems involving general Mie theory with arbitrary incident fields, but may also have applications in other branches of physics.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

References

[1] Xu, Y.L. (1995) Electromagnetic Scattering by an Aggregate of Spheres. *Applied Optics*, **34**, 4573-4588. https://doi.org/10.1364/AO.34.004573

[2] Gopinath, A., Boriskina, S.V., Feng, N., Reinhard, B.M. and Negro, L.D. (2008) Photonic-Plasmonic Scattering Resonances in Deterministic Aperiodic Structures. *Nano Letters*, **8**, 2423-2431. https://doi.org/10.1021/nl8013692

[3] Gopinath, A., Boriskina, S.V., Reinhard, B.M. and Negro, L.D. (2009) Deterministic Aperiodic Arrays of Metal Nanoparticles for Surface-Enhanced Raman Scattering (SERS). *Optics Express*, **17**, 3741-3753. https://doi.org/10.1364/OE.17.003741

[4] Bohren, C.F. and Huffman, D.R. (2004) Absorption and Scattering of Light by Small Particles. Wiley-VCH, Germany.

[5] Guillaume, T. (2020) On the Telegrapher's Equation with Three Space Variables in Non-Rectangular Coordinates. *Journal of Applied Mathematics and Physics*, **8**, 910-926. https://doi.org/10.4236/jamp.2020.85070

[6] Nijimbere, V. (2020) Analytical Evaluation of Non-Elementary Integrals Involving Some Exponential, Hyperbolic and Trigonometric Elementary Functions and Derivation of New Probability Measures Generalizing the Gamma-Type and Gaussian-Type Distributions. *Advances in Pure Mathematics*, **10**, 371-392. https://doi.org/10.4236/apm.2020.107023

[7] Namin, F.A., Yuwan, Y.A., Liu, L., Panaretos, A.H., Werner, D.H. and Mayer, T.S. (2016) Efficient Design, Accurate Fabrication and Effective Characterization of Plasmonic Quasicrystalline Arrays of Nanospherical Particles. *Scientific Reports*, **6**, 1-12. https://doi.org/10.1038/srep22009

[8] Namin, F.A., Wang, X. and Werner, D.H. (2013) Reflection and Transmission Coefficients for Finite-Sized Aperiodic Aggregates of Spheres. *Journal of Optical Society of America B*, **30**, 1008-1016. https://doi.org/10.1364/JOSAB.30.001008

[9] Jackson, J.D. (1975) Classical Electrodynamics. Wiley, New York.

[10] Neves, A.A.R., Fontes, A., Rodriguez, E. and Cruz, C.H.B. (2009) Analytical Results for a Bessel Function Times Legendre Polynomials Class Integrals. *Journal of Physics A*: *Mathematical and General*, **39**, L293. https://doi.org/10.1088/0305-4470/39/18/L06

[11] Neves, A.A.R., Fontes, A., Rodriguez, E., Cruz, C.H.B., Barbosa, L.C. and Cesar, C.L. (2006) Exact Partial Wave Expansion of Optical Beams with Respect to an Arbitrary Origin. *Optics Letters*, **31**, 2477-2479. https://doi.org/10.1364/OL.31.002477

[12] Arfken, G. (1985) Mathematical Methods for Physicists. Academic Press, Orlando.

[13] Abramowitz, M. and Stegun, I.A. (1965) Handbook of Mathematical Functions. Dover, New York.

[14] Gradshteyn, I.S. and Ryzhik, I.M. (2000) Table of Integrals, Series, and Products. Academic Press, San Diego.