Journal of Applied Mathematics and Physics, 2022, 10, 2181-2199
https://www.scirp.org/journal/jamp
ISSN Online: 2327-4379
ISSN Print: 2327-4352

The Long-Term Dynamic Behavior of Solutions to a Class of Generalized Higher-Order Kirchhoff-Type Coupled Wave Equations

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How to cite this paper: Lin, G.G. and Shao, M. (2022) The Long-Term Dynamic Behavior of Solutions to a Class of Generalized Higher-Order Kirchhoff-Type Coupled Wave Equations. *Journal of Applied Mathematics and Physics*, **10**, 2181-2199.
https://doi.org/10.4236/jamp.2022.107150

Received: May 7, 2022
Accepted: July 18, 2022
Published: July 21, 2022

Abstract

In this paper, we study the long-term dynamic behavior of a class of generalized high-order Kirchhoff-type coupled wave equations. Firstly, the existence of uniqueness global solution of this kind of equations in E_k space is proved by prior estimation and Galerkin method; Then, through using Rellich-Kondrachov compact embedding theorem, it is proved that the solution semigroup $S(t)$ has the family of the global attractors A_k in space E_k; Finally, through linearization method, proves that the operator semigroup $S(t)$ Frechet differentiable and the attenuation of linearization problem volume element. Furthermore, we can obtain the finite Hausdorff dimension and Fractal dimension of the family of the global attractors A_k.

Keywords

Kirchhoff Equation, Existence and Uniqueness of Solutions, Global Attractor Family, Dimension Estimation

1. Introduction

In this paper, we study the long-term dynamic behavior of a class of generalized high-order Kirchhoff-type coupled wave equations:

$$u_{tt}+M\left(\left\|\nabla^m u\right\|^2+\left\|\nabla^m v\right\|^2\right)(-\Delta)^m u+\beta(-\Delta)^m u_t+g(u_t,v)=f_1(x), \tag{1}$$

$$v_{tt}+M\left(\left\|\nabla^m u\right\|^2+\left\|\nabla^m v\right\|^2\right)(-\Delta)^{2m} v+\beta(-\Delta)^{2m} v_t+g(u,v_t)=f_2(x), \tag{2}$$

the boundary conditions:

$$\frac{\partial^i u}{\partial n^i}=0, i=0,1,2,\cdots,m-1, x\in\partial\Omega, t>0, \tag{3}$$

$$\frac{\partial^j v}{\partial v^j}=0,\ j=0,1,2,\cdots,2m-1,x\in\partial\Omega,t>0, \tag{4}$$

the initial conditions:

$$u(x,0)=u_0(x),u_t(x,0)=u_1(x),v(x,0)=v_0(x),v_t(x,0)=v_1(x),x\in\Omega, \tag{5}$$

where Ω is a bounded domain in R^n with smooth boundary $\partial\Omega$, $u_0(x),u_1(x)$ is a known function, $g(u,v)$, $f_i(x),i=1,2$ are nonlinear source term and the external force interference terms, $m>1,\beta$ is real number.

Recently, the global attractor and its dimension estimation for Kirchhoff type equations have been favored by many scholars. Many scholars have done a lot of research on this kind of problems and obtained good results [1] [2] [3].

Lin Guoguang, Gao Yunlong [1] studied the longtime behavior of solution to initial boundary value problem for a class of strongly damped higher-order Kirchhoff type equation:

$$u_{tt}+(-\Delta)^m u_t+\left(\alpha+\beta\left\|\nabla^m u\right\|^2\right)^q(-\Delta)^m u+g(u)=f(x),\ (x,t)\in\Omega\times[0,+\infty),$$

they got the existence and uniqueness of the solution by the Galerkin method and obtained the existence of the global attractor in $H_0^m(\Omega)\times L^2(\Omega)$ according to the attractor theorem, besides, the estimation of the upper bound of Hausdorff dimension for the attractor was established.

Guoguang Lin, Ming Zhang [2] studied the initial boundary value problem for a class of Kirchhoff-type coupled equations:

$$u_{tt}-M\left(\left\|\nabla u\right\|^2+\left\|\nabla v\right\|^2\right)\Delta u-\beta\Delta u_t+g_1(u,v)=f_1(x),$$

$$v_{tt}-M\left(\left\|\nabla u\right\|^2+\left\|\nabla v\right\|^2\right)\Delta v-\beta\Delta v_t+g_2(u,v)=f_2(x),$$

they obtained the existence of the global attractor and a precise estimate of upper bound of Hausdorff dimension.

Lin Guoguang, Yang Lujiao [3] studied the long-time properties of solutions of generalized Kirchhoff-type equation with strongly damped terms:

$$u_{tt}+M\left(\left\|\nabla^m u\right\|_p^p\right)(-\Delta)^{2m}u+\beta(-\Delta)^{2m}u_t+g(u)=f(x),$$

by assuming the nonlinear source terms $g(u)$ and Kirchhoff stress term $M(s)$, the author verified the appropriateness of the solution and proved the existence of the global attractor, obtained the upper boundary estimation of the Hausdorff dimension and Fractal dimension of a family of the global attractor.

For more significant research results about the global attractor and its dimension estimation of Kirchhoff equation, please refer to the literature [4]-[18].

2. Existence and Uniqueness of Solutions

The following symbols and assumptions are introduced for the convenience of statement:

$$V_m = H^m(\Omega)\cap H_0^1(\Omega), V_{2m} = H^m(\Omega)\cap H_0^1(\Omega), V_{4m} = H^{4m}(\Omega)\cap H_0^1(\Omega),$$
$$\|\cdot\| = \|\cdot\|_{L^2(\Omega)}, E_0 = V_m \times V_0 \times V_{2m} \times V_0, E_1 = V_{2m} \times V_0 \times V_{4m} \times V_0,$$
$$E_k = V_{m+k} \times V_k \times V_{2m+2k} \times V_{2k}, V_0 = L^2(\Omega).$$

In order to obtain our results, we consider system (1)-(5) under some assumptions on $M(s)$ and $g(u,v)$. Precisely, we state the general assumptions:

(A1) $M(s)\in C^2([0,+\infty),R)$ is not decreasing function and for positive constants δ_0,δ_1,

(1) $\varepsilon+1\le\delta_0\le M(s)\le\delta_1$,

(2) $M(s)$ is a non-negative Lipschitz function, L is associated with the Lipschitz constant $M(s)$, $M(s)=M\left(\|\nabla^m u\|^2+\|\nabla^m v\|^2\right)$.

(A2) For any $u,v,p,q\in V$, $g(u_t,v), g(u,v_t)\in C^2(\mathbb{R})$, there exist $\alpha\ge 0$, $\varepsilon\ge 0$, $C(\alpha,\varepsilon)\ge 0$, such that

$$\begin{aligned}&\left(g(u_t,v),p\right)+\left(g(u,v_t),q\right)\\ &\ge\alpha\left(\|p\|^2+\|q\|^2\right)-\varepsilon\left(\|u\|^2+\|v\|^2\right)-C(\alpha,\varepsilon)\left(\|u_t\|^2+\|v_t\|^2\right).\end{aligned}$$

Lemma 1 Assuming (A1)-(A2) are true, letting $(u_0,p_0,v_0,q_0)\in E_0$, $f_1(x), f_2(x)\in L^2(\Omega)$, then there is a solution (u,p,v,q) for problem (1)-(5) which has the following properties:

(i) $(u,p,v,q)\in L^\infty((0,+\infty);E_0)$;

(ii)

$$y(t)\le y(0)\mathrm{e}^{-\frac{2\varepsilon_0}{k_1}t}+\frac{k_1}{2\varepsilon\varepsilon_0}\left(\|f_1(x)\|^2+\|f_2(x)\|^2\right),\tag{6}$$

where $y(t)=\|\nabla^m u\|^2+\|p\|^2+\|\nabla^{2m}v\|^2+\|q\|^2$.

(iii) There are normal numbers $C(R_0)$ and $t_0=t_0(\Omega)>0$, such that

$$\|(u,p,v,q)\|_{E_0}^2=\|\nabla^m u\|^2+\|p\|^2+\|\nabla^{2m}v\|^2+\|q\|^2\le C(R_0).\tag{7}$$

Proof: Let $p=u_t+\varepsilon u$ inner product with Equation (1),

$$\left(u_{tt}+M\left(\|\nabla^m u\|^2+\|\nabla^m v\|^2\right)(-\Delta)^m u+\beta(-\Delta)^m u_t+g(u_t,v),p\right)=\left(f_1(x),p\right),\tag{8}$$

according to the hypothesis (A1), using the Young inequality, Holder inequation, Poincare inequality, etc., there are

$$(u_{tt},p)=\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|p\|^2-\varepsilon\|p\|^2+\varepsilon^2(u,p),\tag{9}$$

$$\left(M\left(\|\nabla^m u\|^2+\|\nabla^m v\|^2\right)(-\Delta)^m u,p\right)\ge\frac{\delta}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla^m u\|^2+\varepsilon\delta_0\|\nabla^m u\|^2,\tag{10}$$

$$\left(\beta(-\Delta)^m u_t,p\right)=\beta\|\nabla^m u_t\|^2+\frac{\beta\varepsilon}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla^m u\|^2,\tag{11}$$

$$\left(f_1(x),p\right)\le\frac{\varepsilon}{2}\|p\|^2+\frac{1}{2\varepsilon}\|f_1(x)\|^2,\tag{12}$$

where $\delta=\delta_0$ or δ_1.

Similarly, letting $q=v_t+\varepsilon v$ inner product with Equation (2), the treatment of each item is similar to (9)-(11), and the above results are sorted out,

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left((\delta+\beta\varepsilon)\left(\left\|\nabla^m u\right\|^2+\left\|\nabla^{2m}v\right\|^2\right)+\|p\|^2+\|q\|^2\right)-\frac{3\varepsilon}{2}\left(\|p\|^2+\|q\|^2\right)\\&+\varepsilon^2\left[(u,p)+(v,q)\right]+\varepsilon\delta_0\left(\left\|\nabla^m u\right\|^2+\left\|\nabla^{2m}v\right\|^2\right)+\beta\left(\left\|\nabla^m u_t\right\|^2+\left\|\nabla^{2m}v_t\right\|^2\right)\\&+\left(g(u_t,v),p\right)+\left(g(u,v_t),q\right)=\frac{1}{2\varepsilon}\left(\left\|f_1(x)\right\|^2+\left\|f_2(x)\right\|^2\right),\end{aligned}\tag{13}$$

using the Young inequality, the Poincare inequality, and the assumptions (A2), The individual items in Equation (13) are treated as follows:

$$\varepsilon^2\left[(u,p)+(v,q)\right]\geq-\varepsilon\left(\|u\|^2+\|v\|^2\right)-\frac{\varepsilon^3}{4}\left(\|p\|^2+\|q\|^2\right),\tag{14}$$

by the Poincare inequality has

$$-\varepsilon\left(\|u\|^2+\|v\|^2\right)\geq-\varepsilon\left(\lambda_1^{-m}+\lambda_1^{-2m}\right)\left(\left\|\nabla^m u\right\|^2+\left\|\nabla^{2m}v\right\|^2\right),\tag{15}$$

where λ_1 is the first eigenvalue with homogeneous Dirichlet boundary conditions of $-\Delta$, in the same way

$$-C(\alpha,\varepsilon)\left(\|u_t\|^2+\|v_t\|^2\right)\geq-C(\alpha,\varepsilon)\left(\lambda_1^{-m}+\lambda_1^{-2m}\right)\left(\left\|\nabla^m u_t\right\|^2+\left\|\nabla^{2m}v_t\right\|^2\right),\tag{16}$$

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left((\delta+\beta\varepsilon)\left(\left\|\nabla^m u\right\|^2+\left\|\nabla^{2m}v\right\|^2\right)+\|p\|^2+\|q\|^2\right)\\&+\left(\alpha-\frac{3\varepsilon}{2}-\frac{\varepsilon^3}{4}\right)\left(\|p\|^2+\|q\|^2\right)+\varepsilon\left(\delta_0-2\left(\lambda_1^{-m}+\lambda_1^{-2m}\right)\right)\left(\left\|\nabla^m u\right\|^2+\left\|\nabla^{2m}v\right\|^2\right)\\&+\left(\beta-C(\alpha,\varepsilon)\left(\lambda_1^{-m}+\lambda_1^{-2m}\right)\right)\left(\left\|\nabla^m u_t\right\|^2+\left\|\nabla^{2m}v_t\right\|^2\right),\\&\leq\frac{1}{2\varepsilon}\left(\left\|f_1(x)\right\|^2+\left\|f_2(x)\right\|^2\right),\end{aligned}\tag{17}$$

where $\alpha>\frac{3\varepsilon}{2}+\frac{\varepsilon^3}{4}$, $\delta_0>2\left(\lambda_1^{-m}+\lambda_1^{-2m}\right)$, $\beta>C(\alpha,\varepsilon)\left(\lambda_1^{-m}+\lambda_1^{-2m}\right)$.

Let $\bar{y}(t)=(\delta+\beta\varepsilon)\left(\left\|\nabla^m u\right\|^2+\left\|\nabla^{2m}v\right\|^2\right)+\|p\|^2+\|q\|^2$,

$y(t)=\left\|\nabla^m u\right\|^2+\left\|\nabla^{2m}v\right\|^2+\|p\|^2+\|q\|^2$, there are normal numbers $k_1=\min\{1,\delta+\beta\varepsilon\}$, such that

$$\bar{y}(t)\geq k_1 y(t)\geq 0,\tag{18}$$

let $\varepsilon_0=\min\left\{\alpha-\frac{3\varepsilon}{2}-\frac{\varepsilon^3}{4},\varepsilon\left(\delta_0-2\left(\lambda_1^{-m}+\lambda_1^{-2m}\right)\right)\right\}$, there are

$$\frac{\mathrm{d}}{\mathrm{d}t}y(t)+\frac{2\varepsilon_0}{k_1}y(t)\leq\frac{1}{\varepsilon}\left(\left\|f_1(x)\right\|^2+\left\|f_2(x)\right\|^2\right),\tag{19}$$

by Gronwall inequality,

$$y(t)\leq y(0)\mathrm{e}^{-\frac{2\varepsilon_0}{k_1}t}+\frac{k_1}{2\varepsilon\varepsilon_0}\left(\left\|f_1(x)\right\|^2+\left\|f_2(x)\right\|^2\right),\tag{20}$$

so there are normal numbers $C(R_0)$ and $t_0=t_0(\Omega)>0$, such that

$$\|(u,p,v,q)\|_{E_0}^2=\|\nabla^m u\|^2+\|p\|^2+\|\nabla^{2m}v\|^2+\|q\|^2\le C(R_0). \tag{21}$$

Lemma 1 is proved.

Lemma 2 Assuming (A1)-(A2) are true, $(u_0,p_0,v_0,q_0)\in E_k$, $f_1(x)\in H^m(\Omega)$, $f_2(x)\in H^{2m}(\Omega)$, then there is a solution (u,p,v,q) for problem (1)-(5), which has the following properties:

(i) $(u,p,v,q)\in L^\infty((0,+\infty);E_k)$;

(ii)

$$y_1(t)\le y_1(0)\mathrm{e}^{-\frac{2\varepsilon_1}{k_2}t}+\frac{k_2}{2\varepsilon\varepsilon_1}\left(\|\nabla^k f_1(x)\|^2+\|\nabla^{2k}f_2(x)\|^2\right), \tag{22}$$

where $y(t)=\|\nabla^m u\|^2+\|p\|^2+\|\nabla^{2m}v\|^2+\|q\|^2$.

(iii) There are normal numbers $C(R_k)$ and t_{0k}, such that

$$\|(u,p,v,q)\|_{E_k}^2=\|\nabla^{m+k}u\|^2+\|\nabla^{2m+2k}v\|^2+\|\nabla^k p\|^2+\|\nabla^{2k}q\|^2\le C(R_k). \tag{23}$$

Proof: Let $(-\Delta)^k p$ inner product with Equation (1),

$$\begin{aligned}&\left(u_{tt}+M\left(\|\nabla^m u\|^2+\|\nabla^m v\|^2\right)(-\Delta)^m u+\beta(-\Delta)^m u_t+g(u_t,v),(-\Delta)^k p\right)\\&=\left(f_1(x),(-\Delta)^k p\right),\end{aligned} \tag{24}$$

according to the hypothesis (A1), using the Young inequality, Holder inequation, Poincare inequality, etc., there are

$$\left(u_{tt},(-\Delta)^k p\right)\ge\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla^k p\|^2-\left(\varepsilon+\frac{\varepsilon^2}{2}\right)\|\nabla^k p\|^2-\frac{\varepsilon^2\lambda_1^{-m}}{2}\|\nabla^{m+k}u\|^2, \tag{25}$$

$$\left(M\left(\|\nabla^m u\|^2+\|\nabla^m v\|^2\right)(-\Delta)^m u,(-\Delta)^k p\right)\ge\frac{\delta}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla^{m+k}u\|^2+\varepsilon\delta_0\|\nabla^{m+k}u\|^2, \tag{26}$$

$$\begin{aligned}&\left(\beta(-\Delta)^m u_t,(-\Delta)^k p\right)\\&=\beta\|\nabla^{m+k}u_t\|^2+\frac{\beta\varepsilon}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla^{m+k}u\|^2,\end{aligned} \tag{27}$$

where $\delta=\delta_0$ or δ_1.

Similarly, letting $(-\Delta)^{2m}q$ inner product with Equation (2), the treatment of each item is similar to (25)-(27), using Young inequality, Poincare inequality, and assumption (A2), the individual terms are treated as follows:

$$\begin{aligned}&\left(g(u_t,v),(-\Delta)^k p\right)+\left(g(u,v_t),(-\Delta)^{2k}q\right)\\&\le\frac{C_1}{2\varepsilon}\left(\lambda_1^{-m}+\lambda_1^{-2m}\right)\left(\|\nabla^{m+k}u_t\|^2+\|\nabla^{2m+2k}v_t\|^2\right)+\frac{\varepsilon C_1}{2}\left(\|\nabla^k p\|^2+\|\nabla^{2k}q\|^2\right),\end{aligned} \tag{28}$$

$$\begin{aligned}&\left(f_1(x),(-\Delta)^k p\right)+\left(f_2(x),(-\Delta)^{2k}2q\right)\\&\le\frac{\varepsilon}{2}\left(\|\nabla^k p\|^2+\|\nabla^{2k}q\|^2\right)+\frac{1}{2\varepsilon}\left(\|\nabla^k f_1(x)\|^2+\|\nabla^{2k}f_2(x)\|^2\right).\end{aligned} \tag{29}$$

Then sort out the result of appeal and other inner product items, get

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left((\delta+\beta\varepsilon)\left(\left\|\nabla^{m+k}u\right\|^2+\left\|\nabla^{2m+2k}v\right\|^2\right)+\left\|\nabla^k p\right\|^2+\left\|\nabla^{2k}q\right\|^2\right)\\&+\frac{\varepsilon^2-\varepsilon C_1-3\varepsilon}{2}\left(\left\|\nabla^k p\right\|^2+\left\|\nabla^{2k}q\right\|^2\right)\\&+\left(\varepsilon\delta_0-\frac{\varepsilon^2}{2}\left(\lambda_1^{-m}+\lambda_1^{-2m}\right)\right)\left(\left\|\nabla^{m+k}u\right\|^2+\left\|\nabla^{2m+2k}v\right\|^2\right)\\&+\left(\beta-\frac{C_1}{2\varepsilon}\left(\lambda_1^{-m}+\lambda_1^{-2m}\right)\right)\left(\left\|\nabla^{m+k}u_t\right\|^2+\left\|\nabla^{2m+2k}v_t\right\|^2\right)\\&\le\frac{1}{2\varepsilon}\left(\left\|\nabla^k f_1(x)\right\|^2+\left\|\nabla^{2k}f_2(x)\right\|^2\right),\end{aligned}\tag{30}$$

using the Young inequality, Poincare inequality, and assumption (A2), the individual terms in Equation (30) are treated as follows:

$$\begin{aligned}&\left(g(u_t,v),(-\Delta)^k p\right)+\left(g(u,v_t),(-\Delta)^{2k}q\right)\\&\le\frac{C_1}{2\varepsilon}\left(\lambda_1^{-m}+\lambda_1^{-2m}\right)\left(\left\|\nabla^{m+k}u_t\right\|^2+\left\|\nabla^{2m+2k}v_t\right\|^2\right)+\frac{\varepsilon C_1}{2}\left(\left\|\nabla^k p\right\|^2+\left\|\nabla^{2k}q\right\|^2\right),\end{aligned}\tag{31}$$

$$\begin{aligned}&\left(f_1(x),(-\Delta)^k p\right)+\left(f_2(x),(-\Delta)^{2k}2q\right)\\&\le\frac{\varepsilon}{2}\left(\left\|\nabla^k p\right\|^2+\left\|\nabla^{2k}q\right\|^2\right)+\frac{1}{2\varepsilon}\left(\left\|\nabla^k f_1(x)\right\|^2+\left\|\nabla^{2k}f_2(x)\right\|^2\right),\end{aligned}\tag{32}$$

where $\varepsilon-C_1-3>0$, $\delta_0>\frac{\varepsilon}{2}\left(\lambda_1^{-m}+\lambda_1^{-2m}\right)$, $\beta>\frac{C_1}{2\varepsilon}\left(\lambda_1^{-m}+\lambda_1^{-2m}\right)$.

Let $\bar{y}_1(t)=(\delta+\beta\varepsilon)\left(\left\|\nabla^{m+k}u\right\|^2+\left\|\nabla^{2m+2k}v\right\|^2\right)+\left\|\nabla^k p\right\|^2+\left\|\nabla^{2k}q\right\|^2$, $y_1(t)=\left\|\nabla^{m+k}u\right\|^2+\left\|\nabla^{2m+2k}v\right\|^2+\left\|\nabla^k p\right\|^2+\left\|\nabla^{2k}q\right\|^2$, there are normal numbers $k_2=\min\{1,\delta+\beta\varepsilon\}$, such that

$$\bar{y}_1(t)\ge k_2 y_1(t)\ge 0,\tag{33}$$

let $\varepsilon_1=\min\left\{\frac{\varepsilon^2-\varepsilon C_1-3\varepsilon}{2}-\frac{\varepsilon^3}{4},\varepsilon\delta_0-\frac{\varepsilon^2}{2}\left(\lambda_1^{-m}+\lambda_1^{-2m}\right)\right\}$, get

$$\frac{\mathrm{d}}{\mathrm{d}t}y_1(t)+\frac{2\varepsilon_1}{k_2}y_1(t)\le\frac{1}{\varepsilon}\left(\left\|\nabla^k f_1(x)\right\|^2+\left\|\nabla^{2k}f_2(x)\right\|^2\right),\tag{34}$$

by Gronwall inequality,

$$y_1(t)\le y_1(0)\mathrm{e}^{-\frac{2\varepsilon_1}{k_2}t}+\frac{k_2}{2\varepsilon\varepsilon_1}\left(\left\|\nabla^k f_1(x)\right\|^2+\left\|\nabla^{2k}f_2(x)\right\|^2\right),\tag{35}$$

so there are normal numbers $C(R_k)$ and $t_{0k}>0$, such that

$$\left\|(u,p,v,q)\right\|_{E_k}^2=\left\|\nabla^{m+k}u\right\|^2+\left\|\nabla^{2m+2k}v\right\|^2+\left\|\nabla^k p\right\|^2+\left\|\nabla^{2k}q\right\|^2\le C(R_k)\tag{36}$$

Lemma 1 is proved.

Theorem 1 Assuming (A1)-(A2) is true, $(u_0,p_0,v_0,q_0)\in E_k$, $f_1(x)\in V_k$, $f_2(x)\in V_{2k}$, then the initial boundary value problem (1)-(5) has a unique solution

$$\left(u(x,t),p(x,t),v(x,t),q(x,t)\right)\in L^\infty\left((0,+\infty);E_k\right).\tag{37}$$

Proof: According to literature [9] and Galerkin method, combining with lemma 1 and lemma 2, we can easily obtain the existence of solutions.

Next, prove the uniqueness of the solution:

Assuming $(u_1, p_1, v_1, q_1), (u_2, p_2, v_2, q_2) \in E_k$ are the two solutions of the problem (1)-(5), letting $\bar{u} = u_1 - u_2, \bar{v} = v_1 - v_2$, obtain that

$$\begin{cases} \bar{u}_{tt} + M(s_1)(-\Delta)^m u_1 - M(s_2)(-\Delta)^m u_2 + \beta(-\Delta)^m \bar{u}_t + g(u_{1t}, v_1) - g(u_{2t}, v_2) = 0, \\ \bar{v}_{tt} + M(s_1)(-\Delta)^{2m} v_1 - M(s_2)(-\Delta)^{2m} v_2 + \beta(-\Delta)^{2m} \bar{v}_t + g(u_1, v_{1t}) - g(u_2, v_{2t}) = 0, \\ \bar{u}(x,0) = 0, \bar{u}_t(x,0) = 0, \bar{v}(x,0) = 0, \bar{v}_t(x,0) = 0, x \in \Omega, \\ \dfrac{\partial^i \bar{u}}{\partial n^i} = 0, i = 0,1,2,\cdots,m-1, x \in \partial\Omega, t > 0, \\ \dfrac{\partial^j \bar{v}}{\partial v^j} = 0, j = 0,1,2,\cdots,2m-1, x \in \partial\Omega, t > 0, \end{cases} \tag{38}$$

where $s_1 = \|\nabla^m u_1\|^2 - \|\nabla^m v_1\|^2, s_2 = \|\nabla^m u_2\|^2 - \|\nabla^m v_2\|^2$.

Let $\bar{u}_t, \bar{v}_t$ inner product with the first two equations in (38) and obtain,

$$\begin{cases} \dfrac{1}{2}\dfrac{\mathrm{d}}{\mathrm{d}t}\|\bar{u}_t\|^2 + \left(M(s_1)(-\Delta)^m u_1 - M(s_2)(-\Delta)^m u_2, \bar{u}_t\right) + \beta\|\nabla^m \bar{u}_t\|^2 = \left(g(u_{2t}, v_2) - g(u_{1t}, v_1), \bar{u}_t\right), \\ \dfrac{1}{2}\dfrac{\mathrm{d}}{\mathrm{d}t}\|\bar{v}_t\|^2 + \left(M(s_1)(-\Delta)^{2m} v_1 - M(s_2)(-\Delta)^{2m} v_2, \bar{v}_t\right) + \beta\|\nabla^{2m} \bar{v}_t\|^2 = \left(g(u_2, v_{2t}) - g(u_1, v_{1t}), \bar{v}_t\right), \end{cases} \tag{39}$$

using the Young inequality, Poincare inequality, as well as the assumptions (A2), for processing on the type of individual items as follows:

$$\begin{aligned} &\left(M(s_1)(-\Delta)^m \bar{u} + M(s_1)(-\Delta)^m u_2 - M(s_2)(-\Delta)^m u_2, \bar{u}_t\right) \\ &\geq \frac{M(s)}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla^m u\|^2 - \left(M(s_1)(-\Delta)^m u_2 - M(s_2)(-\Delta)^m u_2, \bar{u}_t\right) \\ &\geq \frac{M(s)}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla^m u\|^2 - L\left(\left(\|\nabla^m u_1\| + \|\nabla^m u_2\|\right)\|\nabla^m \bar{u}\| \right. \\ &\quad \left. + L\left(\|\nabla^m v_1\| + \|\nabla^m v_2\|\right)\|\nabla^m \bar{v}\|\right)\left\|(-\Delta)^m \bar{u}_2\right\|\|\bar{u}_t\| \\ &\geq C_2\left(\|\nabla^m \bar{u}\| + \|\nabla^{2m} \bar{v}\|\right)\|\bar{u}_t\| \\ &\geq C_2\left(\|\nabla^m \bar{u}\|^2 + \|\nabla^{2m} \bar{v}\|^2 + \frac{\|\bar{u}_t\|^2}{2}\right), \end{aligned} \tag{40}$$

similarly, we obtain

$$\left(M(s_1)(-\Delta)^{2m} v_1 - M(s_2)(-\Delta)^{2m} v_2, \bar{v}_t\right) \geq C_3\left(\|\nabla^m \bar{u}\|^2 + \|\nabla^{2m} \bar{v}\|^2 + \frac{\|\bar{v}_t\|^2}{2}\right), \tag{41}$$

$$\begin{aligned} &\left|\left(g(u_{1t}, v_1) - g(u_{2t}, v_1) + g(u_{2t}, v_1) - g(u_{2t}, v_2), \bar{u}_t\right)\right| \\ &\leq \left|\left(g'(\xi_t, v_1)|\bar{u}_t| + g'(u_{2t}, \eta)|\bar{v}|, \bar{u}_t\right)\right| \\ &\leq \|g'(\xi_t, v_1)\|_\infty \|\bar{u}_t\|^2 + \|g'(u_{2t}, \eta)_\infty\|\|\bar{v}\|\|\bar{u}_t\| \leq C_4\left(\frac{\|\bar{v}\|^2}{2} + \frac{3\|\bar{u}_t\|^2}{2}\right), \end{aligned} \tag{42}$$

$$\left|\left(g\left(u_1,v_{1t}\right)-g\left(u_2,v_{2t}\right),\overline{v}_t\right)\right|\le C_5\left(\frac{3\left\|\overline{v}_t\right\|^2}{2}+\frac{\left\|\overline{u}\right\|^2}{2}\right), \tag{43}$$

Through the (40)-(43), finally will become

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\left\|\overline{u}_t\right\|^2+\left\|\overline{v}_t\right\|^2+M\left(s_1\right)\left(\left\|\nabla^m\overline{u}\right\|^2+\left\|\nabla^{2m}\overline{v}\right\|^2\right)\right)+\beta\left(\left\|\nabla^m\overline{u}_t\right\|^2+\left\|\nabla^{2m}\overline{v}_t\right\|^2\right)\\&\le\left(C_2+C_3\right)\left(\left\|\nabla^m\overline{u}\right\|^2+\left\|\nabla^{2m}\overline{v}\right\|^2\right)+\frac{C_2+3C_4}{2}\left\|\overline{u}_t\right\|^2\\&\quad+\frac{C_3+3C_5}{2}\left\|\overline{v}_t\right\|^2+\frac{C_4}{2}\left\|v\right\|^2+\frac{C_5}{2}\left\|u\right\|^2\\&\le C_6\left(\left\|\nabla^m\overline{u}\right\|^2+\left\|\nabla^{2m}\overline{v}\right\|^2\right)+C_7\left(\left\|\overline{u}_t\right\|^2+\left\|\overline{v}_t\right\|^2\right).\end{aligned} \tag{44}$$

Let $y_2\left(t\right)=\left\|\overline{u}_t\right\|^2+\left\|\overline{v}_t\right\|^2+M\left(s_1\right)\left(\left\|\nabla^m\overline{u}\right\|^2+\left\|\nabla^{2m}\overline{v}\right\|^2\right)$, there are normal numbers $k_3=\min\left\{C_7,C_8\right\}$, where $C_6\le C_8M\left(s_1\right)$, such that

$$\frac{\mathrm{d}}{\mathrm{d}t}y_2\left(t\right)\le k_3y_2\left(t\right), \tag{45}$$

using the Gronwall inequality, we have

$$y_2\left(t\right)\le y_2\left(0\right)\mathrm{e}^{k_4t}=0, \tag{46}$$

$$y_2\left(t\right)=\left\|\overline{u}_t\right\|^2+\left\|\overline{v}_t\right\|^2+M\left(s_1\right)\left(\left\|\nabla^m\overline{u}\right\|^2+\left\|\nabla^{2m}\overline{v}\right\|^2\right)=0, \tag{47}$$

so

$$\overline{u}=\overline{v}=0. \tag{48}$$

Theorem 1 is proved.

3. The Family of Global Attractors and Dimension Estimation

Theorem 2 [9] Assume E_0 is a Banach space, $\left\{S\left(t\right)\right\}_{t\ge0}$ is the operator semigroup, $S\left(t\right):E\to E$, $S\left(t+\tau\right)=S\left(t\right)\cdot S\left(\tau\right)\left(\forall t,\tau\ge0\right)$, $S\left(0\right)=I$, where I is the identity operator. If $S\left(t\right)$ satisfies

1) Semigroup $S\left(t\right)$ is uniformly bounded in *E*, *i.e.* $\forall R>0$, exists a constant $C\left(R\right)$ such that when $\left\|u\right\|_E\le R$, there is $\left\|S\left(t\right)u\right\|_E\le C\left(R\right)\left(\forall t\in\left[0,+\infty\right)\right)$;

2) There exists a bounded absorbing set B in *E*, that is, for any bounded set $B\subset E$, there exists a constant $t_0>0$, such that

$$S\left(t\right)B\subset B\left(t>t_0\right), \tag{49}$$

3) $\left\{S\left(t\right)\right\}_{t\ge0}$ is completely continuous operator.

Then operator semigroup $S\left(t\right)$ has compact global attractor *A*.

Theorem 3 Let $S\left(t\right)$ is a solution semigroup generated by the initial boundary value problems (1)-(5) under the hypothesis of lemma 1 and lemma 2, then the initial boundary value problems (1)-(5) have the family of global attractors. There are compact sets satisfying:

$$A_k\subset E_k\subset E_0\quad\text{and}\quad A_k=\omega\left(B_{0k}\right)=\bigcap_{\tau\ge0}\overline{\bigcup_{t\ge\tau}S\left(t\right)B_{0k}},k=1,2,\cdots,m.$$

where $B_{0k}=\left\{(u,p,v,q)\in E_k:\left\|\nabla^{m+k}u\right\|^2+\left\|\nabla^k p\right\|^2+\left\|\nabla^{2m+2k}v\right\|^2+\left\|\nabla^{2k}q\right\|^2\le R_{0k}\right\}$,

1) $S(t)A_k=A_k, t>0$,

2) A_k attracts all bounded sets of E_k, that is, any bounded set $B_{0k}\subset E_k$, having

$\lim_{t\to\infty} dist\left(S(t)B_k,A_k\right)=0$, where $dist\left(S(t)B_k,A_k\right)=\sup_{x\in B_{0k}}\inf_{y\in A_k}\left\|S(t)-y\right\|_{E_k}$.

then compact set A_k are called the family of global attractors of semigroup $S(t)$.

Proof: From lemma 1, lemma 2, for any bounded set $B_{0k}\subset E_k$ and $B_{0k}\subset\left\{\left\|(u,p,v,q)\right\|_{E_k}\le R_k\right\}$, the equation has solution semigroups $S(t):E_k\to E_k$, and

$$\left\|S(t)(u_0,p_0,v_0,q_0)\right\|_{E_k}^2=\|u\|_{V_{m+k}}^2+\|p\|_{V_k}^2+\|v\|_{V_{2m+2k}}^2+\|q\|_{V_{2k}}^2\le C(R_k), \tag{50}$$

where $t_0\ge 0$, $(u_0,v_0)\in B_{0k}$, shows that $S(t)_{t\ge 0}$ is uniformly bounded in E_k;

Further,

$B_{0k}=\left\{(u,p,v,q)\in E_k:\left\|\nabla^{m+k}u\right\|^2+\left\|\nabla^k p\right\|^2+\left\|\nabla^{2m+2k}v\right\|^2+\left\|\nabla^{2k}q\right\|^2\le R_{0k}\right\}$ is a bounded absorption set of semigroup $S(t)$; E_k is compactly embedded in E_0, *i.e.*, the bounded set in E_k is a compact set in E_0, so the operator semigroup $S(t)$ is completely continuous operator. Then there exists a global attractor family of equations

$$A_k=\omega(B_{0k})=\bigcap_{\tau\ge 0}\overline{\bigcup_{t\ge\tau}S(t)B_{0k}}, k=1,2,\cdots,m.$$

Theorem 3 is proved.

After the family of global attractors are obtained, in order to estimate the Hausdroff dimension and Fractal dimension of the family of global attractors, the initial boundary value problem (1)-(5) is linearized and obtain that

$$\begin{cases} U_{tt}+M'\left(\left\|\nabla^m u\right\|^2+\left\|\nabla^m v\right\|^2\right)\left[\left(\nabla^m u,\nabla^m U\right)+\left(\nabla^m v,\nabla^m V\right)\right](-\Delta)^m u \\ +M\left(\left\|\nabla^m u\right\|^2+\left\|\nabla^m v\right\|^2\right)(-\Delta)^m U+\beta(-\Delta)^m U_t+\dfrac{\partial g(u_t,v)}{\partial u_t}U_t+\dfrac{\partial g(u_t,v)}{\partial v}V=0, \\ U_{tt}+M'\left(\left\|\nabla^m u\right\|^2+\left\|\nabla^m v\right\|^2\right)\left[\left(\nabla^m u,\nabla^m U\right)+\left(\nabla^m v,\nabla^m V\right)\right](-\Delta)^m u \\ +M\left(\left\|\nabla^m u\right\|^2+\left\|\nabla^m v\right\|^2\right)(-\Delta)^m U+\beta(-\Delta)^m U_t+\dfrac{\partial g(u_t,v)}{\partial u_t}U_t+\dfrac{\partial g(u_t,v)}{\partial v}V=0, \\ U(x,0)=\xi_1,U_t(x,0)=\xi_2,V(x,0)=\eta_1,V_t(x,0)=\eta_2, \\ U(x,0)\big|_{x\in\Omega}=V(x,0)\big|_{x\in\Omega}=0,t>0. \end{cases} \tag{51}$$

where $(\xi_1,\xi_2,\eta_1,\eta_2)\in E_0$, $(u,p,v,q)=S(t)(u_0,p_0,v_0,q_0)$ is the solution of the initial boundary value problem (51).

Given $(u_0,p_0,v_0,q_0)\in A_k$, $S(t):E_k\to E_k$, for any $(\xi_1,\xi_2,\eta_1,\eta_2)\in E_k$, there exists a unique solution $\left(U(t),P(t),V(t),Q(t)\right)\in L^\infty(0,T;E_k)$ to the linear initial boundary value problem (51).

Lemma 3 For any $t>0$, $R>0$, the mapping $S(t):E_k\to E_k$ is Frechet differentiable. The derivative on $\rho_0=(u_0,p_0,v_0,q_0)^{\mathrm{T}}$ is a linear operator on E_k,

$$DS(t)\rho_0:(\xi,\zeta,\eta,\sigma)\to(U,P,V,Q),$$

where $(U(t),P(t),V(t),Q(t))$ is the solution of the problem (51).

Proof: suppose $\rho_0=(u_0,p_0,v_0,q_0)\in E_k$, $\tilde{\rho}_0=(u_0+\xi,p_0+\zeta,v_0+\eta,q_0+\sigma)\in E_k$ with $\|\rho_0\|_{E_k}\le R_k$, $\|\tilde{\rho}_0\|_{E_k}\le R_k$, we denote

$$S(t)\rho_0=\rho=(u_1,p_1,v_1,q_1),S(t)\tilde{\rho}_0=(u_2,p_2,v_2,q_2).$$

First, we can prove a Lipschitz property of $S(t)$ on the bounded sets on E_k, that is

$$\left\|S(t)\rho_0-S(t)\tilde{\rho}_0\right\|_{E_k}^2\le \mathrm{e}^{ct}\left\|(\xi,\zeta,\eta,\sigma)\right\|_{E_k}^2. \tag{52}$$

We now consider the difference $\theta=u_2-u_1-U,\omega=v_2-v_1-V$ is the solution to problem (53),

$$\begin{cases}\theta_{tt}+M\left(\left\|\nabla^m u_1\right\|^2+\left\|\nabla^m v_1\right\|^2\right)(-\Delta)^m\theta+\beta(-\Delta)^m\theta_t=h_1,\\ \omega_{tt}+M\left(\left\|\nabla^m u_1\right\|^2+\left\|\nabla^m v_1\right\|^2\right)(-\Delta)^{2m}\omega+\beta(-\Delta)^{2m}\omega_t=h_2,\\ \omega(0)=\omega_t(0)=0,\theta(0)=\theta_t(0)=0.\end{cases} \tag{53}$$

where

$$\begin{aligned}h_1=&\left[M(s)-M(\tilde{s})\right](-\Delta)^m u_2\\&+2M'(s)\left[\left(\nabla^m u_1,\nabla^m U\right)+\left(\nabla^m v_1,\nabla^m V\right)\right](-\Delta)^m u_1\\&+\frac{\partial g(u_{1t},v_1)}{\partial u_{1t}}U_t+\frac{\partial g(u_{1t},v_1)}{\partial v_1}V+g(u_{1t},v_1)-g(u_{2t},v_2),\end{aligned} \tag{54}$$

$$\begin{aligned}h_2=&\left[M(s)-M(\tilde{s})\right](-\Delta)^{2m} v_2\\&+2M'(s)\left[\left(\nabla^m u_1,\nabla^m U\right)+\left(\nabla^m v_1,\nabla^m V\right)\right](-\Delta)^{2m} v_1\\&+\frac{\partial g(u_1,v_{1t})}{\partial u_1}U+\frac{\partial g(u_1,v_{1t})}{\partial v_{1t}}V_t+g(u_1,v_{1t})-g(u_2,v_{2t}),\end{aligned} \tag{55}$$

where $s=\left\|\nabla^m u_1\right\|^2+\left\|\nabla^m v_1\right\|^2,\tilde{s}=\left\|\nabla^m u_2\right\|^2+\left\|\nabla^m v_2\right\|^2$.

Some items are treated as follows

$$\begin{aligned}&\left[M(s)-M(\tilde{s})\right](-\Delta)^m u_2+2M'(s)\left[\left(\nabla^m U,\nabla^m u_1\right)+\left(\nabla^m V,\nabla^m v_1\right)\right](-\Delta)^m u_1\\&=\left(M(s)-M(\tilde{s})\right)(-\Delta)^m u_2+2M'(s)\left[\left(\nabla^m u_1,\nabla^m U\right)+\left(\nabla^m v_1,\nabla^m V\right)\right](-\Delta)^m u_1\\&=-2M'(s)\left[\left(-\nabla^m U,\nabla^m u_1\right)+\left(-\nabla^m V,\nabla^m v_1\right)\right](-\Delta)^m u_1+M'\left(\alpha s+(1-\alpha)\tilde{s}\right)\\&\quad\times\left[\left(\nabla^m(u_1-u_2),\nabla^m(u_2+u_1)\right)+\left(\nabla^m(v_1-v_2),\nabla^m(v_2+v_1)\right)\right](-\Delta)^m u_2\end{aligned}$$

$$
\begin{aligned}
&\le M''(\xi)\left[\left(\nabla^m(u_1-u_2),\nabla^m(u_2+u_1)\right)+\left(\nabla^m(v_1-v_2),\nabla^m(v_2+v_1)\right)\right]^2(-\Delta)^m u_2\\
&\quad+M'(s)\left[\left(\nabla^m(u_1-u_2),\nabla^m(u_2-u_1)\right)+\left(\nabla^m(v_1-v_2),\nabla^m(v_2-v_1)\right)\right](-\Delta)^m u_2\\
&\quad+2M'(s)\left[\left(\nabla^m(u_1-u_2),\nabla^m u_1\right)+\left(\nabla^m(v_1-v_2),\nabla^m v_1\right)\right](-\Delta)^m(u_2-u_1)\\
&\quad-2M'(s)\left[\left(\nabla^m\theta,\nabla^m u_1\right)+\left(\nabla^m\omega,\nabla^m v_1\right)\right](-\Delta)^m u_1\\
&=I_1+I_2+I_3+I_4.
\end{aligned}
\tag{56}
$$

Let $(-\Delta)^k\theta_t$ inner product with the first equation in (53), $(-\Delta)^{2k}\omega_t$ and inner product with the second equation in (53), obtain

$$
\begin{aligned}
\left(I_1,(-\Delta)^k\theta_t\right)&\le\left(2M''(\xi)\left[\left(\nabla^m(u_1-u_2),\nabla^m(u_2+u_1)\right)^2+\left(\nabla^m(v_1-v_2),\nabla^m(v_2+v_1)\right)^2\right]\nabla^{m+k}u_2,\nabla^{m+k}\theta_t\right)\\
&\le 2c_1\left(\frac{4}{\varepsilon^2}\left\|\nabla^m(u_2-u_1)\right\|^4+\frac{4}{\varepsilon^2}\left\|\nabla^m(v_2-v_1)\right\|^4+\frac{\varepsilon^2\left\|\nabla^{m+k}\theta_t\right\|^2}{8}\right),\\
&\le 2c_1\left(\frac{4}{\varepsilon^2}\left(\lambda_1^{-2k}+\lambda_1^{-2m-4k}\right)\left(\left\|\nabla^{m+k}(u_2-u_1)\right\|^4+\left\|\nabla^{2m+2k}(v_2-v_1)\right\|^4\right)+\frac{\varepsilon^2\left\|\nabla^{m+k}\theta_t\right\|^2}{8}\right),
\end{aligned}
$$

$$
\begin{aligned}
\left(I_2,(-\Delta)^k\theta_t\right)&=\left(M'(s)\left[\left(\nabla^m(u_1-u_2),\nabla^m(u_2-u_1)\right)+\left(\nabla^m(v_1-v_2),\nabla^m(v_2-v_1)\right)\right](-\Delta)^{m+k}u_2,\nabla^{m+k}\theta_t\right)\\
&\le c_2\left(\frac{4}{\varepsilon^2}\left\|\nabla^m(u_2-u_1)\right\|^4+\frac{4}{\varepsilon^2}\left\|\nabla^m(v_2-v_1)\right\|^4+\frac{\varepsilon^2\left\|\nabla^{m+k}\theta_t\right\|^2}{8}\right)\\
&\le c_2\left(\frac{4}{\varepsilon^2}\left(\lambda_1^{-2k}+\lambda_1^{-2m-4k}\right)\left(\left\|\nabla^{m+k}(u_2-u_1)\right\|^4+\left\|\nabla^{2m+2k}(v_2-v_1)\right\|^4+\right)\frac{\varepsilon^2\left\|\nabla^{m+k}\theta_t\right\|^2}{8}\right),
\end{aligned}
$$

$$
\begin{aligned}
\left(I_3,(-\Delta)^k\theta_t\right)&\le\left(2M'(s)\left[\left(\nabla^m(u_1-u_2),\nabla^m u_1\right)+\left(\nabla^m(v_1-v_2),\nabla^m v_1\right)\right]\nabla^{m+k}(u_2-u_1),\nabla^{m+k}\theta_t\right)\\
&\le 2c_3\left(\frac{4}{\varepsilon^2}\left\|\nabla^m(u_2-u_1)\right\|^4+\frac{4}{\varepsilon^2}\left\|\nabla^m(v_2-v_1)\right\|^4+\frac{4}{\varepsilon^2}\left\|\nabla^{m+k}(u_2-u_1)\right\|^4+\frac{3\varepsilon^2\left\|\nabla^{m+k}\theta_t\right\|^2}{16}\right)\\
&\le 2c_3\left(\frac{4}{\varepsilon^2}\left(\lambda_1^{-2k}+\lambda_1^{-2m-4k}+1\right)\left(\left\|\nabla^{m+k}(u_2-u_1)\right\|^4+\left\|\nabla^{2m+2k}(v_2-v_1)\right\|^4\right)+\frac{3\varepsilon^2\left\|\nabla^{m+k}\theta_t\right\|^2}{16}\right),
\end{aligned}
$$

$$
\begin{aligned}
\left(I_4,(-\Delta)^k\theta_t\right)&=\left(-2M'(s)\left[\left(\nabla^m\theta,\nabla^m u_1\right)+\left(\nabla^m\omega,\nabla^m v_1\right)\right]\nabla^{m+k}u_1,\nabla^{m+k}\theta_t\right)\\
&\le 2c_4\left(\frac{4\left\|\nabla^{m+k}\theta\right\|^2}{\varepsilon^2}+\frac{4\left\|\nabla^{m+k}\omega\right\|^2}{\varepsilon^2}+\frac{\varepsilon^2\left\|\nabla^{m+k}\theta_t\right\|^2}{8}\right)\\
&\le 2c_4\left(\frac{4}{\varepsilon^2}\left(\lambda_1^{-2k}+\lambda_1^{-2m-4k}\right)\left(\left\|\nabla^{m+k}\theta\right\|^2+\left\|\nabla^{2m+2k}\omega\right\|^2\right)+\frac{\varepsilon^2\left\|\nabla^{m+k}\theta_t\right\|^2}{8}\right),
\end{aligned}
$$

which implies that

$$\left|\left(\left[M(s)-M(\tilde{s})\right](-\Delta)^m u_2+2M'(s)\left[\left(\nabla^m U,\nabla^m u_1\right)+\left(\nabla^m V,\nabla^m v_1\right)\right](-\Delta)^m u_1,(-\Delta)^k\theta_t\right)\right|$$
$$\le\frac{c_5}{\varepsilon^2}\left(\lambda_1^{-2k}+\lambda_1^{-2m-4k}+1\right)\left(\left\|\nabla^{m+k}(u_2-u_1)\right\|^4+\left\|\nabla^{2m+2k}(v_2-v_1)\right\|^4\right.$$
$$\left.+\left\|\nabla^{m+k}\theta\right\|^2+\left\|\nabla^{2m+2k}\omega\right\|^2\right)+c_6\varepsilon^2\left\|\nabla^{m+k}\theta_t\right\|^2. \tag{57}$$

Analogously,

$$\left|\left(\left[M(s)-M(\tilde{s})\right](-\Delta)^{2m} v_2+2M'(s)\left[\left(\nabla^m u_1,\nabla^m U\right)+\left(\nabla^m v_1,\nabla^m V\right)\right](-\Delta)^{2m} v_1,(-\Delta)^{2k}\omega_t\right)\right|$$
$$\le\frac{c_7}{\varepsilon^2}\left(\lambda_1^{-2k}+\lambda_1^{-2m-4k}+1\right)\left(\left\|\nabla^{m+k}(u_2-u_1)\right\|^4+\left\|\nabla^{2m+2k}(v_2-v_1)\right\|^4\right.$$
$$\left.+\left\|\nabla^{m+k}\theta\right\|^2+\left\|\nabla^{2m+2k}\omega\right\|^2\right)+c_8\varepsilon^2\left\|\nabla^{2m+2k}\omega_t\right\|^2. \tag{58}$$

Further,

$$\left(\frac{\partial g(u_{1t},v_1)}{\partial u_{1t}}U_t+\frac{\partial g(u_{1t},v_1)}{\partial v_1}V+g(u_{1t},v_1)-g(u_{2t},v_2),(-\Delta)^k\theta_t\right)$$
$$=\left(\frac{\partial g(\xi_1,v_1)}{\partial \xi_1}(u_{1t}-u_{2t})+\frac{\partial g(u_{2t},\eta_1)}{\partial \eta_1}(v_1-v_2)+\frac{\partial g(u_{1t},v_1)}{\partial u_{1t}}U_t+\frac{\partial g(u_{1t},v_1)}{\partial v_1}V,(-\Delta)^k\theta_t\right)$$
$$\le\left(\frac{\partial^2 g(\xi_2,v_1)}{\partial \xi_2}(u_{1t}-u_{2t})(\xi_1-u_{1t})+\frac{\partial g(u_{1t},v_1)}{\partial u_{1t}}(-\theta_t)+\frac{\partial^2 g(u_{2t},\eta_2)}{\partial,\eta_2^2}(v_1-v_2)(\eta_1-v_1)\right.$$
$$\left.+\frac{\partial^2 g(\xi_3,v_1)}{\partial\xi_3\partial v}(v_1-v_2)(u_{2t}-u_{1t})+\frac{\partial g(u_{1t},v_1)}{\partial v_1}(-\omega),(-\Delta)^k\theta_t\right)$$
$$\le c_9\lambda_1^{-\frac{k}{2}-m}\left\|\nabla^{m+k}(u_{1t}-u_{2t})\right\|^2\left\|\nabla^k\theta_t\right\|+c_{10}\left\|\nabla^k\theta_t\right\|^2+c_{11}\lambda_1^{-\frac{3k}{2}-2m}\left\|\nabla^{2m+2k}(v_1-v_2)\right\|^2\left\|\nabla^k\theta_t\right\|$$
$$+c_{12}\lambda_1^{-k-\frac{3m}{2}}\left\|\nabla^{m+k}(u_{1t}-u_{2t})\right\|\left\|\nabla^{2k+2m}(v_1-v_2)\right\|\left\|\nabla^k\theta_t\right\|+c_{13}\lambda_1^{-\frac{k}{2}-m}\left\|\nabla^{2m+2k}\omega\right\|\left\|\nabla^k\theta_t\right\|$$
$$\le\frac{c_{14}}{2}\left(\frac{3}{2\varepsilon^2}\left(\left\|\nabla^{m+k}(u_{1t}-u_{2t})\right\|^4+\left\|\nabla^{2m+2k}(v_1-v_2)\right\|^4\right)+\varepsilon^2\left(\left\|\nabla^k\theta_t\right\|^2+5\left\|\nabla^{2m+2k}\omega\right\|^2\right)\right), \tag{59}$$

where,

$$\xi_1=r_1u_{1t}-(1-r_1)u_{2t},\eta_1=r_2v_1-(1-r_2)v_2,\xi_2=r_3\xi_1-(1-r_3)u_{1t},$$
$$\eta_2=r_4\eta_1-(1-r_4)v_1,\xi_3=r_5u_{2t}-(1-r_5)u_{1t},$$
$$c_{14}=\max\left\{c_9\lambda_1^{-\frac{k}{2}-m},\frac{2c_{10}}{\varepsilon^2},c_{11}\lambda_1^{-\frac{3k}{2}-2m},c_{12}\lambda_1^{-k-\frac{3m}{2}},c_{13}\lambda_1^{-\frac{k}{2}-m}\right\}.$$

Similarly

$$\left(\frac{\partial g(u,v_t)}{\partial u}U+\frac{\partial g(u,v_t)}{\partial v_t}V_t+g(u,v_t)-g(\tilde{u},\tilde{v}_t),\omega_t\right)$$
$$\le\frac{c_{15}}{2}\left(\frac{3}{2\varepsilon^2}\left(\left\|\nabla^{m+k}(u_1-u_2)\right\|^4+\left\|\nabla^{2m+2k}(v_{1t}-v_{2t})\right\|^4\right)\right. \tag{60}$$
$$\left.+\varepsilon^2\left(\left\|\nabla^{m+k}\theta\right\|^2+5\left\|\nabla^{2k}\omega_t\right\|^2\right)\right).$$

Based on the above Equations (57)-(60), it is sorted out that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\{\left\|\nabla^{k}\theta_{t}\right\|^{2}+\left\|\nabla^{2k}\omega_{t}\right\|^{2}+\mu\left(\left\|\nabla^{m+k}\theta\right\|^{2}+\left\|\nabla^{2m+2k}\omega\right\|^{2}\right)\right\}\\&+\left(\beta-\left(c_{6}+c_{8}\right)\varepsilon^{2}\right)\left(\left\|\nabla^{m+k}\theta_{t}\right\|^{2}+\left\|\nabla^{2m+2k}\omega_{t}\right\|^{2}\right)\\&\le c_{16}\left(\left\|\nabla^{m+k}\left(u_{1}-u_{2}\right)\right\|^{4}+\left\|\nabla^{m+k}\left(u_{1t}-u_{2t}\right)\right\|^{4}+\left\|\nabla^{2m+2k}\left(v_{1}-v_{2}\right)\right\|^{4}\right.\\&\quad\left.+\left\|\nabla^{2m+2k}\left(v_{1t}-v_{2t}\right)\right\|^{4}\right)+c_{17}\left(\left\|\nabla^{k}\theta_{t}\right\|^{2}+\left\|\nabla^{2k}\omega_{t}\right\|^{2}+\left\|\nabla^{m+k}\theta\right\|^{2}+\left\|\nabla^{2m+2k}\omega\right\|^{2}\right),\end{aligned}\tag{61}$$

where $\beta\ge\left(c_{6}+c_{8}\right)\varepsilon^{2}$,
by Gronwall inequality

$$\begin{aligned}&\left\|\theta_{t}\right\|^{2}+\left\|\omega_{t}\right\|^{2}+\mu\left(\left\|\nabla^{m}\theta\right\|^{2}+\left\|\nabla^{2m}\omega\right\|^{2}\right)\\&\le c_{18}\mathrm{e}^{C_{19}t}\int_{0}^{t}\left(\left\|\nabla^{m+k}\left(u_{1}-u_{2}\right)\right\|^{4}+\left\|\nabla^{m+k}\left(u_{1t}-u_{2t}\right)\right\|^{4}\right.\\&\quad\left.+\left\|\nabla^{2m+2k}\left(v_{1}-v_{2}\right)\right\|^{4}+\left\|\nabla^{2m+2k}\left(v_{1t}-v_{2t}\right)\right\|^{4}\right)\mathrm{d}\tau\\&\le c_{20}\mathrm{e}^{c_{21}t}\left\|\left(\xi,\zeta,\eta,\sigma\right)\right\|_{E_{k}}^{4},\end{aligned}\tag{62}$$

so as $\left\|\left(\xi,\zeta,\eta,\sigma\right)\right\|_{E_{k}}^{2}\to 0$ in E_{k}, there are

$$\frac{\left\|S(t)\tilde{\rho}_{0}-S(t)\rho_{0}-\left(DS(t)\rho_{0}\right)\left(\xi_{1},\xi_{2},\eta_{1},\eta_{2}\right)\right\|_{E_{k}}^{2}}{\left\|\left(\xi,\zeta,\eta,\sigma\right)\right\|_{E_{k}}^{2}}\le c_{20}\mathrm{e}^{c_{21}t}\left\|\left(\xi,\zeta,\eta,\sigma\right)\right\|_{E_{k}}^{2}\to 0.$$

The differentiability of $S(t)$ is proved.

The next step will be used in demonstrating the process of dimension estimation. it seems obvious that the Equations (1)-(5) also can be written as

$$\varphi_{t}+H(\varphi)=F(\varphi),\tag{63}$$

where $\varphi=\left(u,p,v,q\right)^{\mathrm{T}}$, $p=u_{t}+\varepsilon u$, $q=v_{t}+\varepsilon v$,

$$H(\varphi)=\begin{pmatrix}\varepsilon u-p\\(-\Delta)^{m}u-\varepsilon\left(\beta(-\Delta)^{m}-\varepsilon\right)u+\left(\beta(-\Delta)^{m}-\varepsilon\right)p\\\varepsilon v-q\\(-\Delta)^{2m}v-\varepsilon\left(\beta(-\Delta)^{2m}-\varepsilon\right)v+\left(\beta(-\Delta)^{2m}-\varepsilon\right)q\end{pmatrix},\tag{64}$$

$$F(\varphi)=\begin{pmatrix}0\\f_{1}(x)-g\left(u_{t},v\right)+\left(1-M\left(\left\|\nabla^{m}u\right\|^{2}+\left\|\nabla^{m}v\right\|^{2}\right)\right)(-\Delta)^{m}u\\0\\f_{2}(x)-g\left(u,v_{t}\right)+\left(1-M\left(\left\|\nabla^{m}u\right\|^{2}+\left\|\nabla^{m}v\right\|\right)\right)(-\Delta)^{2m}v\end{pmatrix}.\tag{65}$$

Consider the first variation equation of (63)

$$\Psi'+P(\varphi)\Psi=\Gamma_{1}(\varphi)\Psi+\Gamma_{2}(\varphi)\Psi,\tag{66}$$

where $\Psi=\left(U,P,V,Q\right)^{\mathrm{T}}$, $P=U_{t}+\varepsilon U$, $Q=V_{t}+\varepsilon V$, and $\varphi=\left(u,p,v,q\right)^{\mathrm{T}}$ is the solution to problem (63), and

$$P(\varphi)=\begin{pmatrix} \varepsilon I & -I & 0 & 0 \\ (1-\beta\varepsilon)(-\Delta)^m+\varepsilon^2 I & \beta(-\Delta)^m-\varepsilon I & 0 & 0 \\ 0 & 0 & \varepsilon I & -I \\ 0 & 0 & (1-\beta\varepsilon)(-\Delta)^{2m}+\varepsilon^2 I & \beta(-\Delta)^{2m}-\varepsilon I \end{pmatrix}, \tag{67}$$

$$\Gamma_1(\varphi)=\begin{pmatrix} 0 & 0 & 0 & 0 \\ \varepsilon\frac{\partial g(u_t,v)}{\partial u_t} & -\frac{\partial g(u_t,v)}{\partial u_t} & -\frac{\partial g(u_t,v)}{\partial v} & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{\partial g(u,v_t)}{\partial u} & 0 & \varepsilon\frac{\partial g(u,v_t)}{\partial v_t} & -\frac{\partial g(u,v_t)}{\partial v_t} \end{pmatrix}, \tag{68}$$

$$\Gamma_2(\varphi)\Psi=\begin{pmatrix} 0 \\ (1-M(s))(-\Delta)^m U-2M'(s)\left[(\nabla^m u,\nabla^m U)+(\nabla^m v,\nabla^m V)\right](-\Delta)^m u \\ 0 \\ (1-M(s))(-\Delta)^{2m} V-2M'(s)\left[(\nabla^m u,\nabla^m U)+(\nabla^m v,\nabla^m V)\right](-\Delta)^{2m} v \end{pmatrix}. \tag{69}$$

Theorem 4 Under the condition of Theorem 3, the global attractors of initial boundary value problems (1)-(5) have finite dimensional Hausdroff dimension and fractal dimension, and then $d_H(A_k)<\frac{2}{3}n_0$, $d_F(A_k)<\frac{4}{3}n_0$.

Proof: For any fixed $(u_0,p_0,v_0,q_0)\in E_k$, assume that $\chi_1,\chi_2,\cdots,\chi_n$ are n_0 elements in E_k, and $\psi_1(t),\psi_2(t),\cdots,\psi_{n_0}(t)$ are n_0 solutions of the linearized Equation (66) with an initial value $\psi_1(0)=\chi_1,\psi_2(0)=\chi_2,\cdots,\psi_{n_0}(0)=\chi_{n_0}$, where n_0 is a natural number. It can be obtained by calculation

$$\begin{aligned}&\left\|\psi_1(t)\wedge\cdots\wedge\psi_{n_0}(t)\right\|_{\wedge^{n_0}E_k}\\ &\le\left\|\chi_1\wedge\cdots\wedge\chi_n\right\|_{\wedge^{n_0}E_k}\exp\int_0^t trF'(\varphi(\tau))\circ Q_{n_0}(\tau)\mathrm{d}\tau.\end{aligned} \tag{70}$$

where $\wedge$ represents the outer product, *tr* represents the trace of the operator, $Q_{n_0}(\tau)=Q_{n_0}(\tau,\varphi_0;\chi_1,\chi_2,\cdots,\chi_{n_0})$ represents the orthogonal projection from E_k to $span\{\psi_1(t),\psi_2(t),\cdots,\psi_{n_0}(t)\}$.

At a given time τ, let $h_j(\tau)=(\xi_j(\tau),\zeta_j(\tau),\eta_j(\tau),\sigma_j(\tau))^{\mathrm{T}}$, $j=1,2,\cdots,n_0$ are the standard orthogonal basis of space $span\{\psi_1(t),\psi_2(t),\cdots,\psi_{n_0}(t)\}$, then define the inner product on E_k

$$\begin{aligned}(h_j,\overline{h_j})=&(\nabla^{m+k}\xi_j,\nabla^{m+k}\overline{\xi_j})+(\nabla^k\zeta_j,\nabla^k\overline{\zeta_j})\\ &+(\nabla^{2m+2k}\eta_j,\nabla^{2m+2k}\overline{\eta_j})+(\nabla^{2k}\sigma_j,\nabla^{2k}\overline{\sigma_j}),\end{aligned}$$

$$\|h_j\|_{E_k}^2=(h_j,h_j)_{E_k}=\|\nabla^{m+k}\xi_j\|^2+\|\nabla^k\zeta_j\|^2+\|\nabla^{2m+2k}\eta_j\|^2+\|\nabla^{2k}\sigma_j\|^2=1.$$

Through the above conditions, can get

$$trF'(\varphi(\tau))\circ Q_{n_0}(\tau)=\sum_{j=1}^{+\infty}\left(F'(\varphi(\tau))\circ Q_{n_0}(\tau)h_j(\tau),h_j(\tau)\right)_{E_k}$$
$$=\sum_{j=1}^{n_0}\left(F'(\varphi(\tau))h_j(\tau),h_j(\tau)\right)_{E_k},$$

by the Holder inequality, Young and Poincare inequality

$$\begin{aligned}
&\left(P(\varphi)h_j,h_j\right)\\
&=\varepsilon\left(\nabla^{m+k}\xi_j,\nabla^{m+k}\xi_j\right)-\left(\nabla^{m+k}\zeta_j,\nabla^{m+k}\xi_j\right)+(1-\beta\varepsilon)\left(\nabla^{m+k}\xi_j,\nabla^{m+k}\zeta_j\right)\\
&\quad+\varepsilon^2\left(\nabla^k\xi_j,\nabla^k\zeta_j\right)+\beta\left(\nabla^{m+k}\zeta_j,\nabla^{m+k}\zeta_j\right)-\varepsilon\left(\nabla^k\zeta_j,\nabla^k\zeta_j\right)\\
&\quad+\varepsilon\left(\nabla^{2m+2k}\eta_j,\nabla^{2m+2k}\eta_j\right)-\left(\nabla^{2m+2k}\sigma_j,\nabla^{2m+2k}\eta_j\right)+\varepsilon^2\left(\nabla^{2k}\eta_j,\nabla^{2k}\sigma_j\right)\\
&\quad+\beta\left(\nabla^{2m+2k}\sigma_j,\nabla^{2m+2k}\sigma_j\right)-\varepsilon\left(\nabla^{2k}\sigma_j,\nabla^{2k}\sigma_j\right)\\
&\quad+(1-\beta\varepsilon)\left(\nabla^{2m+2k}\eta_j,\nabla^{2m+2k}\sigma_j\right)\\
&\geq\varepsilon\left\|\nabla^{m+k}\xi_j\right\|^2-(2-\beta\varepsilon)\left\|\nabla^{m+k}\zeta_j\right\|\left\|\nabla^{m+k}\xi_j\right\|-\varepsilon^2\left\|\nabla^k\xi_j\right\|\left\|\nabla^k\zeta_j\right\|\\
&\quad+\beta\left\|\nabla^{m+k}\zeta_j\right\|^2-\varepsilon\left\|\nabla^k\zeta_j\right\|^2-(2-\beta\varepsilon)\left\|\nabla^{2m+2k}\sigma_j\right\|\left\|\nabla^{2m+2k}\eta_j\right\|\\
&\quad-\varepsilon^2\left\|\nabla^{2k}\eta_j\right\|\left\|\nabla^{2k}\sigma_j\right\|+\beta\left\|\nabla^{2m+2k}\sigma_j\right\|^2-\varepsilon\left\|\nabla^{2k}\sigma_j\right\|^2+\varepsilon\left\|\nabla^{2m+2k}\eta_j\right\|^2\\
&\geq\left(\beta\lambda_1^m-\frac{\varepsilon^2+2\varepsilon}{2}-\frac{c_{21}(2-\beta\varepsilon)\lambda_1^m}{2}\right)\left\|\nabla^k\zeta_j\right\|^2-\frac{\varepsilon^2}{2}\left\|\nabla^k\xi_j\right\|^2-\frac{\varepsilon^2}{2}\left\|\nabla^{2k}\eta_j\right\|^2\\
&\quad+\left(\varepsilon-\frac{c_{21}(2-\beta\varepsilon)}{2}\right)\left\|\nabla^{m+k}\xi_j\right\|^2+\left(\varepsilon-\frac{c_{22}(2-\beta\varepsilon)}{2}\right)\left\|\nabla^{2m+2k}\eta_j\right\|^2\\
&\quad+\left(\beta\lambda_1^{2m}-\frac{\varepsilon^2+2\varepsilon}{2}-\frac{c_{22}(2-\beta\varepsilon)\lambda_1^{2m}}{2}\right)\left\|\nabla^{2k}\sigma_j\right\|^2,
\end{aligned}\tag{71}$$

$$\begin{aligned}
&\left(\Gamma_1(\varphi)h_j,h_j\right)\\
&=\varepsilon\left(\frac{\partial g(u_t,v)}{\partial u_t}\nabla^k\xi_j,\nabla^k\zeta_j\right)-\left(\frac{\partial g(u_t,v)}{\partial u_t}\nabla^k\zeta_j,\nabla^k\zeta_j\right)\\
&\quad-\left(\frac{\partial g(u_t,v)}{\partial v}\nabla^k\eta_j,\nabla^k\zeta_j\right)-\left(\frac{\partial g(u,v_t)}{\partial u}\nabla^{2k}\xi_j,\nabla^{2k}\sigma_j\right)\\
&\quad+\varepsilon\left(\frac{\partial g(u,v_t)}{\partial v_t}\nabla^{2k}\eta_j,\nabla^{2k}\sigma_j\right)-\left(\frac{\partial g(u,v_t)}{\partial v_t}\nabla^{2k}\sigma_j,\nabla^{2k}\sigma_j\right)\\
&\leq\varepsilon\left\|\frac{\partial g(u_t,v)}{\partial u_t}\right\|_\infty\left\|\nabla^k\xi_j\right\|\left\|\nabla^k\zeta_j\right\|-\left\|\frac{\partial g(u_t,v)}{\partial u_t}\right\|_\infty\left\|\nabla^k\zeta_j\right\|^2\\
&\quad-\left\|\frac{\partial g(u_t,v)}{\partial v}\right\|_\infty\left\|\nabla^k\eta_j\right\|\left\|\nabla^k\zeta_j\right\|-\left\|\frac{\partial g(u,v_t)}{\partial u}\right\|_\infty\left\|\nabla^{2k}\xi_j\right\|\left\|\nabla^{2k}\sigma_j\right\|\\
&\quad+\varepsilon\left\|\frac{\partial g(u,v_t)}{\partial v_t}\right\|_\infty\left\|\nabla^{2k}\eta_j\right\|\left\|\nabla^{2k}\sigma_j\right\|-\left\|\frac{\partial g(u,v_t)}{\partial v_t}\right\|_\infty\left\|\nabla^{2k}\sigma_j\right\|^2\\
&\leq\frac{\varepsilon^2c_{23}}{2}\left\|\nabla^k\xi_j\right\|^2+\frac{c_{23}-c_{24}}{2}\left\|\nabla^k\zeta_j\right\|^2+\frac{\varepsilon^2c_{25}}{2}\left\|\nabla^{2k}\eta_j\right\|^2+\frac{c_{23}-c_{26}}{2}\left\|\nabla^{2k}\sigma_j\right\|^2,
\end{aligned}\tag{72}$$

$$
\begin{aligned}
&\left(\Gamma_2(\varphi)h_j,h_j\right)\\
&=\left(1-M(s)\right)\left(\nabla^{m+k}\xi_j,\nabla^{m+k}\zeta_j\right)-2M'(s)\left(\nabla^m u,\nabla^m\xi_j\right)\left(\nabla^{2m+k}u,\nabla^k\zeta_j\right)\\
&\quad-2M'(s)\left(\nabla^m v,\nabla^m\eta_j\right)\left(\nabla^{2m+k}u,\nabla^k\zeta_j\right)+\left(1-M(s)\right)\left(\nabla^{2m+2k}\eta_j,\nabla^{2m+2k}\sigma_j\right)\\
&\quad-2M'(s)\left(\nabla^m u,\nabla^m\xi_j\right)\left(\nabla^{4m+2k}v,\nabla^{2k}\sigma_j\right)\\
&\quad-2M'(s)\left(\nabla^m v,\nabla^m\eta_j\right)\left(\nabla^{4m+2k}v,\nabla^{2k}\sigma_j\right)\\
&\le\left(1-\delta_0\right)\lambda_1^{\frac{m}{2}}\left(\left\|\nabla^{m+k}\xi_j\right\|\left\|\nabla^k\zeta_j\right\|+\left\|\nabla^{2m+2k}\eta_j\right\|\left\|\nabla^{2k}\sigma_j\right\|\right)\\
&\quad+2c_{26}\left(\lambda_1^{-\frac{k}{2}}+\lambda_1^{-\frac{m}{2}-k}\right)\left(\left\|\nabla^{m+k}\xi_j\right\|\left\|\nabla^k\zeta_j\right\|+\left\|\nabla^{2m+2k}\eta_j\right\|\left\|\nabla^k\zeta_j\right\|\right)\\
&\quad+2c_{27}\left(\lambda_1^{-\frac{k}{2}}+\lambda_1^{-\frac{m}{2}-k}\right)\left(\left\|\nabla^{m+k}\xi_j\right\|\left\|\nabla^{2k}\sigma_j\right\|+\left\|\nabla^{2m+2k}\eta_j\right\|\left\|\nabla^{2k}\sigma_j\right\|\right)\\
&\le\frac{1-\delta_0}{2}\lambda_1^{\frac{m}{2}}\left(\left\|\nabla^{m+k}\xi_j\right\|^2+\left\|\nabla^k\zeta_j\right\|^2+\left\|\nabla^{2m+2k}\eta_j\right\|^2+\left\|\nabla^{2k}\sigma_j\right\|^2\right)\\
&\quad+2c_{28}\left(\lambda_1^{-\frac{k}{2}}+\lambda_1^{-\frac{m}{2}-k}\right)\left(\left\|\nabla^{m+k}\xi_j\right\|^2+\left\|\nabla^k\zeta_j\right\|^2+\left\|\nabla^{2m+2k}\eta_j\right\|^2+\left\|\nabla^{2k}\sigma_j\right\|^2\right)\\
&\le\frac{1-\delta_0}{2}\lambda_1^{\frac{m}{2}}+2c_{28}\left(\lambda_1^{-\frac{k}{2}}+\lambda_1^{-\frac{m}{2}-k}\right).
\end{aligned}
\tag{73}
$$

Based on the above Equations (71)-(73), it is sorted out that

$$
\begin{aligned}
&\left(F'(\varphi(\tau))h_j(\tau),h_j(\tau)\right)_{E_k}=\left(\left(-P(\varphi)+\Gamma_1(\varphi)+\Gamma_2(\varphi)\right)h_j,h_j\right)\\
&\le-\left(\beta\lambda_1^m-\frac{\varepsilon^2+2\varepsilon}{2}-\frac{c_{21}(2-\beta\varepsilon)\lambda_1^m}{2}-\frac{c_{23}-c_{24}}{2}\right)\left\|\nabla^k\zeta_j\right\|^2\\
&\quad-\left(\beta\lambda_1^{2m}-\frac{\varepsilon^2+2\varepsilon}{2}-\frac{c_{22}(2-\beta\varepsilon)\lambda_1^{2m}}{2}-\frac{c_{25}-c_{26}}{2}\right)\left\|\nabla^{2k}\sigma_j\right\|^2\\
&\quad-\left(\varepsilon-\frac{c_{21}(2-\beta\varepsilon)}{2}\right)\left\|\nabla^{m+k}\xi_j\right\|^2-\left(\varepsilon-\frac{c_{22}(2-\beta\varepsilon)}{2}\right)\left\|\nabla^{2m+2k}\eta_j\right\|^2\\
&\quad+\frac{\varepsilon^2(c_{23}-1)}{2}\left\|\nabla^k\xi_j\right\|^2+\frac{\varepsilon^2(c_{25}-1)}{2}\left\|\nabla^{2k}\eta_j\right\|^2+\frac{1-\delta_0}{2}\lambda_1^{\frac{m}{2}}\\
&\quad+2c_{28}\left(\lambda_1^{-\frac{k}{2}}+\lambda_1^{-\frac{m}{2}-k}\right),
\end{aligned}
\tag{74}
$$

let

$$
\begin{aligned}
b=\min\Bigg\{&\beta\lambda_1^m-\frac{\varepsilon^2+2\varepsilon}{2}-\frac{c_{21}(2-\beta\varepsilon)\lambda_1^m}{2}-\frac{c_{23}-c_{24}}{2},\varepsilon-\frac{c_{21}(2-\beta\varepsilon)}{2},\\
&\varepsilon-\frac{c_{22}(2-\beta\varepsilon)}{2},\beta\lambda_1^{2m}-\frac{\varepsilon^2+2\varepsilon}{2}-\frac{c_{22}(2-\beta\varepsilon)\lambda_1^{2m}}{2}-\frac{c_{25}-c_{26}}{2},\\
&-\frac{1-\delta_0}{2}\lambda_1^{\frac{m}{2}}-2c_{28}\left(\lambda_1^{-\frac{k}{2}}+\lambda_1^{-\frac{m}{2}-k}\right)\Bigg\},
\end{aligned}
$$

$$a = \max\left\{\frac{\varepsilon^2(c_{23}-1)}{2}, \frac{\varepsilon^2(c_{25}-1)}{2}\right\},$$

we can obtain

$$\sum_{j=1}^{n_0}\left(F'(\varphi(\tau))h_j(\tau), h_j(\tau)\right)_{E_k} \le -n_0 b + a\sum_{j=1}^{n_0}\left(\left\|\nabla^k \xi_j\right\|^2 + \left\|\nabla^{2k}\eta_j\right\|^2\right), \quad (75)$$

for almost all times *t*, there is $\sum_{i=1}^{n_0}\left\|\nabla^k \xi_i\right\|^2 \le \sum_{i=1}^{n_0}\lambda_i^{s'-1}$, $\sum_{j=1}^{n_0}\left\|\nabla^{2k}\eta_j\right\|^2 \le \sum_{j=1}^{n_0}\lambda_j^{s''-1}$, so

$$trF'(\varphi(\tau))\circ Q_{n_0}(\tau) \le -n_0 b + a\left(\sum_{i=1}^{n_0}\lambda_i^{s'-1} + \sum_{j=1}^{n_0}\lambda_j^{s''-1}\right), \quad (76)$$

because of

$$q_{n_0}(t) = \sup_{\varphi_0\in B_{0k}}\sup_{\Psi_j(0)\in E_k}\left\{\frac{1}{t}\int_0^t trF'(\varphi(\tau))\circ Q_{n_0}(\tau)\,\mathrm{d}\tau\right\}, q_{n_0} = \lim_{t\to\infty} q_{n_0}(t), \quad (77)$$

$$q_{n_0}(t) \le -n_0 b + a\left(\sum_{i=1}^{n_0}\lambda_i^{s'-1} + \sum_{j=1}^{n_0}\lambda_j^{s''-1}\right), q_{n_0} \le -n_0 b + a\left(\sum_{i=1}^{n_0}\lambda_i^{s'-1} + \sum_{j=1}^{n_0}\lambda_j^{s''-1}\right), \quad (78)$$

Therefore, the Lyapunov exponent $K_1, K_2, \cdots, K_{n_0}$ on set B_{0k} is uniformly bounded, and

$$K_1 + K_2 + \cdots + K_{n_0} \le -n_0 b + a\left(\sum_{i=1}^{n_0}\lambda_i^{s'-1} + \sum_{j=1}^{n_0}\lambda_j^{s''-1}\right), \quad (79)$$

so

$$(q_{ij})_+ \le -n_0 b + a\left(\sum_{i=1}^{n_0}\lambda_i^{s'-1} + \sum_{j=1}^{n_0}\lambda_j^{s''-1}\right) \le a\left(\sum_{i=1}^{n_0}\lambda_i^{s'-1} + \sum_{j=1}^{n_0}\lambda_j^{s''-1}\right) \le \frac{2}{5}n_0 b, \quad (80)$$

$$q_{n_0} \le -n_0 b\left(1 - \frac{a}{n_0 b}\left(\sum_{i=1}^{n_0}\lambda_i^{s'-1} + \sum_{j=1}^{n_0}\lambda_j^{s''-1}\right)\right) \le -\frac{3}{5}n_0 b, \quad (81)$$

further

$$\max_{1\le i,j\le n_0}\frac{(q_{ij})_+}{\left|q_{n_0}\right|} \le \frac{2}{3}. \quad (82)$$

Thus, can obtain $d_H(A_k) < \frac{2}{3}n_0$, $d_F(A_k) < \frac{4}{3}n_0$.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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