Journal of Applied Mathematics and Physics, 2022, 10, 2175-2180
https://www.scirp.org/journal/jamp
ISSN Online: 2327-4379
ISSN Print: 2327-4352

Some Vector Ky Fan Minimax Inequalities with Nonconvex Domain

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How to cite this paper: Chen, T. (2022) Some Vector Ky Fan Minimax Inequalities with Nonconvex Domain. *Journal of Applied Mathematics and Physics*, **10**, 2175-2180.
https://doi.org/10.4236/jamp.2022.107149

Received: June 8, 2022
Accepted: July 18, 2022
Published: July 21, 2022

Abstract

In this paper, by virtue of separation theorems of convex sets and scalarization functions, some minimax inequalities are first considered. As applications, some existence theorems of vector equilibrium problems with different order structures were also obtained.

Keywords

Vector-Valued Mapping, Minimax Theorem, Separation Theorem

1. Introduction

It is known to all that the Ky Fan minimax theorem acts a significant role in many fields ([1]). There are massive articles to study Ky Fan minimax inequality problems for vector-valued mappings and set-valued mappings. Chen [2] proved some Ky Fan minimax inequalities under some different assumptions. Zhang and Li [3] obtained two types of vector set-valued various minimax theorems by applying a fixed point theorem. Zhang and Li [4] investigated three types of Ky Fan minimax inequalities by using Ky Fan section theorem and KFG fixed point theorem. However, the domain assumptions of objection function of these results obtained were convex. Gao [5] investigated some matrix inequalities for the Fan product and the Hadamard Product of Matrices.

The number of papers about Ky Fan minimax inequalities for vector (set)-valued mappings with nonconvex domain assumption is very small. Motivated by these works, we establish some new vector various Ky Fan minimax inequalities with nonconvex domain structure. At the same time, we obtain some existence results.

2. Preliminaries

Let V be a topological sapces with Hausdorff structure, and P be a cone with pointed closed convex structure. We give the signs:

1) if $b \in \Lambda$, $b' \notin b - \operatorname{int} P$, $\forall b' \in \Delta$, then b is weakly minimal element in Λ;

2) if $b \in \Lambda$, $b' \notin b + \operatorname{int} P$, $\forall b' \in \Delta$, then b is weakly maximal element in Λ. The marginal set-valued functions $Min_w K(X_0, y)$ and $Max_w K(x, X_0)$ are u.s.c. and closed-valued in the setting of continuity of K and compactness of X_0.

Definition 2.1 Ref. [6] Let $K : X \to V$.

K is said to P-u.s.c. if $\forall v \in X$, $p \in \operatorname{int} P$, $\exists U_v$ of v s.t.

$$K(d) \in K(x) + p - \operatorname{int} P, \forall d \in U_v.$$

K is P-l.s.c. if $-K$ is P-u.s.c.

Clearly, if K is P-u.s.c., then $p^* K$ is u.s.c., $p^* \in P^* / \{\theta\}$.

Lemma 2.1 Let X_0 be compact and $K : X_0 \to V$, $p^* \in P^* / \{\theta\}$.

(i) If K is P-l.s.c., then the weakly minimal element of $K(X_0)$ is nonempty.

(ii) If K is P-u.s.c., then the weakly maximal element of $K(X_0)$ is nonempty.

Proof. (i) Let $p^* \in P^* / \{\theta\} = \{p^* \in V^* / \{\theta\} : p^*(p) \geq 0, \forall p \in P\}$.

There exists $v \in X_0$ such that

$$p^*(K(v)) = \min \cup_{b \in X_0} p^*(K(b)).$$

Thus, by the assumption of p^*, we have

$$K(v) \in Min_w \cup_{b \in X_0} K(b).$$

(ii) Similar way of (i).

3. Vector Various Ky Fan Minimax Inequalities

Theorem 3.1 Let X_0 be compact.

(i) If $\forall t$, $K(\cdot, t)$ is P-l.s.c.; $\forall s$, $K(s, s)$ is P-l.s.c.; K is P-convexlike in f.v. and K is P-concavelike in its s.v., then $\exists t_0 \in X_0$ s.t.

$$Min_w K(X_0, t_0) \subseteq co\left(Min_w \cup_{s \in X_0} K(s, s)\right) + P.$$

(ii) If $\forall s$, $K(s, \cdot)$ is P-u.s.c.; $\forall s$, $K(s, s)$ is P-u.s.c.; K is P-concavelike in f.v. and K is P-convexlike in s.v., then $\exists s_0 \in X_0$ s.t.

$$Max_w K(s_0, X_0) \subseteq co\left(Max_w \cup_{s \in X_0} K(s, s)\right) - P.$$

Proof. (i) Let $\alpha < \min \cup_{s \in X_0} \left(p^* K(s, s)\right)$. Define the multifunction G by the formula

$$G(y) := \{s \in X_0 : p^*(K(s, t)) \leq \alpha\}, y \in X_0.$$

Since $K(\cdot, t)$ is P-l.s.c. and Lemma 2.1, G is closed-valued, for each $s \in X_0$. We claim that

$$\bigcap_{t \in X_0} G(t) = \varnothing. \tag{1}$$

Indeed, if not, then there exists $s_0 \in X_0$ such that $s_0 \in \bigcap_{t \in X_0} G(t)$. Namely, $s_0 \in G(t)$, $\forall t \in X_0$. Particularly, taking $t = s_0$, we have that

$$p^*\left(K\left(s_0, s_0\right)\right) \le \alpha < \min \cup_{s \in X_0} p^*\left(K\left(s, s\right)\right).$$

Hence (1) holds. Thus, $\forall s$,

$$s \in X_0 \Big/ \bigcap_{t \in X_0} G(t) = \cup_{t \in X_0}\left(X_0 / G(t)\right).$$

Namely, $X_0 = \cup_{t \in X_0}\left(X_0 / G(t)\right)$. Since X_0 is compact and G is closed-valued, there is a finite subset $\{t_1, t_2, \cdots, t_n\} \subseteq X_0$ such that

$$X_0 = \cup_{1 \le i \le n}\left(X_0 / G(t_i)\right).$$

By virtue of G, $\forall s \in X_0$, $\exists i \in \{1, 2, \cdots, n\}$ s.t.

$$p^*\left(K\left(s, t_i\right)\right) > \alpha.$$

Then, we let

$$M = \left\{\left(u_1, u_2, \cdots, u_n, r\right) \in R^{n+1} : \exists s \in X_0,\, p^*\left(K\left(s, t_i\right)\right) \le r + u_i,\, i = 1, 2, \cdots, n\right\}.$$

Clearly, M is a convex set in R^{n+1}. In fact, let

$$\left(u_1', u_2', \cdots, u_n', r'\right), \left(u_1'', u_2'', \cdots, u_n'', r''\right) \in M$$

and $l \in [0,1]$. Thus, $\exists s', s'' \in X_0$ s.t.

$$p^*\left(K\left(s', t_i\right)\right) \le r' + u_i',\, p^*\left(K\left(s'', t_i\right)\right) \le r'' + u_i'',\, \forall i = 1, 2, \cdots, n.$$

By assumptions, $\exists s_0 \in X_0$ s.t.

$$\begin{aligned} p^*\left(K\left(s_0, t_i\right)\right) &\le l p^*\left(K\left(s', t_i\right)\right) + (1-l) p^*\left(K\left(s'', t_i\right)\right) \\ &\le l r' + (1-l) r'' + l u_i' + (1-l) u_i'',\, i = 1, 2, \cdots, n. \end{aligned}$$

Namely,

$$l\left(u_1', u_2', \cdots, u_n', r'\right) + (1-l)\left(u_1'', u_2'', \cdots, u_n'', r''\right) \in M.$$

By the assumption of α, we have $(\theta, \alpha) \notin M$. Next, by using separation theorem of convex sets, there exists $\left(e_1, e_2, \cdots, e_n, q\right) \in R^{n+1} / \{\theta\}$ such that

$$\sum_{i=1}^{n} e_i z_i + qr \ge qa,\ \forall\left(z_1, z_2, \cdots, z_n, r\right) \in M. \tag{2}$$

Letting $z_i \to \infty$ and $r \to \infty$, by (2), we have $e_i \ge 0$ and $q \ge 0$, $\forall i = 1, 2, \cdots, n$. By the assumption of α and the definition of M,

$$\alpha < \max_{1 \le i \le n} p^*\left(K\left(s, t_i\right)\right), \forall s \in X_0$$

and

$$\left(\theta, 1 + \max_{1 \le i \le n} p^*\left(K\left(s, t_i\right)\right)\right) \in \operatorname{int} M.$$

Thus, $q \ge 0$. Since

$$\left(p^*\left(K\left(s, t_1\right)\right) - t,\, p^*\left(K\left(s, t_2\right)\right) - t, \cdots, p^*\left(K\left(s, t_n\right)\right) - t, t\right) \in M, \forall t \in R,$$

by (2),

$$\sum_{i=1}^{n} e_i p^* K(s,t_i) + \left(q - \sum_{i=1}^{n} e_i\right) t \ge q\alpha$$

Namely,

$$\sum_{i=1}^{n} \frac{e_i}{q} p^*\left(K(s,t_i)\right) + t\left(1 - \sum_{i=1}^{n} \frac{e_i}{q}\right) \ge \alpha.$$

By the arbitrariness of *t*, we have that $\sum_{i=1}^{n} \frac{e_i}{q} = 1$. Because K is *P*-concave like in its second variable, $\exists s_0 \in X_0$ s.t.

$$p^*\left(K(s,t_0)\right) \ge \sum_{i=1}^{n} \frac{e_i}{q} p^*\left(K(s,t_i)\right) \ge \alpha, \forall s \in X_0.$$

Then, we have that $\exists s_0 \in X_0$ such that

$$\min p^*\left(K(X_0,t)\right) \ge \min \cup_{s\in X_0} p^*\left(K(s,s)\right). \tag{3}$$

Since $K(s,s)$ is *P*-l.s.c., the weakly minimal element of $\cup_{s\in X_0} K(s,s)$ is non-empty.

Suppose that $v \in V$ and $v \notin co\left(Max_w \cup_{s\in X_0} K(s,s)\right) + P$. Namely,

$$(v - P) \cap co\left(Min_w \cup_{s\in X_0} K(s,s)\right) = \varnothing.$$

Then, by the strong separation theorem of convex sets, there exists a linear-continuous function $p^* \ne \theta$ such that

$$p^*(v-p) < p^*(c),\ \forall c \in co\left(Min_w \cup_{s\in X_0} K(s,s)\right),\ p \in P. \tag{4}$$

By (4), letting $p = \theta$,

$$p^* \in P^* / \{\theta\}$$

and

$$p^*(v) < p^*(c),\ c \in co\left(Min_w \cup_{s\in X_0} K(s,s)\right).$$

By assumptions, there is $s_1 \in X_0$ s.t.

$$\min \cup_{s\in X_0} p^*\left(K(s,s)\right) = p^*\left(K(s_1,s_1)\right).$$

Then,

$$K(s_1,s_1) \in Min_w \cup_{s\in X_0} K(s,s) \subseteq co\left(Min_w \cup_{s\in X_0} K(s,s)\right).$$

By (3), $\exists s_0 \in X_0$ such that

$$p^*(v) < \min \cup_{s\in X_0} p^*\left(K(s,s)\right) \le \min p^*\left(K(X_0,s_0)\right).$$

Thus,

$$v \notin K(X_0,t_0) + P.$$

Then,

$$v \notin Min_w K(X_0,s_0).$$

By the assumption of *v*, we have that

$$Min_w K(X_0, t_0) \subseteq co\left(Min_w \cup_{s\in X_0} K(s,s)\right) + P.$$

Remark 3.2 In Theorm 3.1, X_0 can be nonconvex set. Hence, the result is differents from ones in [2] [3] [4].

4. Applications

In the following, the vector equilibrium problem and lexicographic vector equilibrium problem are considered: Let $K: X_0 \times X_0 \to V$.

(VEP) find $t \in X_0$ such that

$$K(s,t) \notin -P/\{\theta\}, \quad \forall s \in X_0.$$

Let V be $R^n\,(n \geq 2)$; $I = \{1, 2, \cdots, n\}$. The lexicographic cone of R^n is defined:

$$P_{lex} = \{\theta\} \cup \left\{p \in R^n : \exists i \in I_n \text{ s.t. } p_i > 0; \text{ no } \exists j \in I_n \text{ s.t. } p_j \neq 0\right\}.$$

(LVEP) find $t \in X_0$ such that

$$K(s,t) \in P_{lex}, \quad \forall s \in X_0.$$

Theorem 4.1 Assume that X_0 is compact and:

(i) $\forall t$, $K(\cdot, t)$ is P-l.s.c.;

(ii) K is P-convexlike in f.v. and K is P-concavelike in s.v.;

(iii) $co\left(\cup_{s\in X_0} K(s,s)\right) \subseteq V/\{-P/\{\theta\}\}$.

Then, $\exists t \in X_0$ which is a solution of VEP.

Proof. By applying vector various minimax inequality, $\exists t \in X_0$ s.t.

$$Min_w K(X_0, t_0) \subseteq co\left(Min_w \cup_{s\in X_0} K(s,s)\right) + P.$$

Then,

$$K(X_0, t) \subseteq Min_w K(X_0, t_0) + \operatorname{int} P \cup \{\theta\}.$$

Thus,

$$K(s,t) \in co\left(\cup_{s\in X_0} K(s,s)\right) + P, \ \forall s \in X_0.$$

By assumption (iii) and $P + V/-P/\{\theta\} = V/-P/\{\theta\}$,

$$K(s,t) \notin -P/\{\theta\}, \quad \forall s \in X_0.$$

By virtue of the above vector various Ky Fan minimax theorem, we can obtain the existence result in the general conditions, which is to verify easily than ones in the literatures.

Theorem 4.2 Assume that X_0 is compact and:

(i) $\forall t$, $K(\cdot, t)$ is P-l.s.c.;

(ii) K is P-convexlike in f.v. and K is P-concavelike in s.v.;

(iii) $co\left(\cup_{s\in X_0} K(s,s)\right) \subseteq P_{lex}$.

Then, there is $t \in X_0$ which is a solution of LVEP.

Proof. Similar to the proof of Theorem 4.1 and $P_{lex} + R^n_+ = P_{lex}$, one can show that the result holds as well.

5. Concluding Remark

We obtain some new vector various Ky Fan minimax inequalities in the setting of nonconvex domain. As applications, we obtained some existence results for VEP and LVEP with nonconvex domain assumptions, respectively. These results improve and generalize the relevant ones in the papers.

Acknowledgements

The author is grateful to the referees for suggestions. This paper was funded by the project of Yunnan education department (No: 2022J0601).

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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