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Study on the Dynamics of an SIR Epidemic Model with Saturated Growth Rate

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Abstract

In this paper, we investigate the dynamic properties of an SIR epidemic model with saturated growth rate. Under the conditions of an arbitrary initial value, we prove that the system exists unique positive solution, and give the sufficient conditions caused by random environmental factors leading to the extinction of infectious diseases. Moreover, we verify the conditions for the persistence of infectious diseases in the mean sense. Finally, we provide the biology interpretation and some strategies to control the infectious diseases.

Keywords

SIR Epidemic Model, Ito Formula, Extinction, Persistence in the Mean Sense

1. Introduction

Since Kennack and McKendrick proposed the SIR model in 1927 (see [1]), the epidemic model has been well developed. SIR epidemic models play an important role in revealing the laws of infectious disease spread and providing a theoretical basis for prevention and control of the diseases [1] [2] [3] [4].

In 2013, Gong and Yang studied the following SIR epidemic model with saturated growth rate in [5],

$$\begin{cases} S'(t) = rS(t)\left(1-\dfrac{S(t)}{K}\right) - \dfrac{\beta S(t) I(t)}{1+\alpha S(t)} + cI(t) - dS(t), \\ I'(t) = \dfrac{\beta S(t) I(t)}{1+\alpha S(t)} - dI(t) - cI(t) - \mu I(t), \\ R'(t) = \mu I(t) - dR(t) \end{cases} \tag{1}$$

where $S(t), I(t), R(t)$ represent the numbers of susceptible, infected and re-

covered individuals at time t respectively. r is the intrinsic natural growth rate, K is the environmental carrying capacity, β is the infection rate of the infectious disease, c is the recovery rate, d is the natural mortality rate, μ is the removal rate, α is the psychological effect coefficient, that is, when the susceptible people know that the infected person is infected, he will take corresponding measures to affect the incidence. All parameters in the system are positive.

Since the first two equations of System (1) are independent of the third equation, it is sufficient to consider the first two equations of (1). So, we study the following simplified model

$$\begin{cases} S'(t) = rS(t)\left(1-\dfrac{S(t)}{K}\right) - \dfrac{\beta S(t)I(t)}{1+\alpha S(t)} + cI(t) - dS(t), \\ I'(t) = \dfrac{\beta S(t)I(t)}{1+\alpha S(t)} - dI(t) - cI(t) - \mu I(t). \end{cases} \tag{2}$$

In System (2), the basic reproductive number

$$R_0 = \frac{\beta K\left(1-\dfrac{d}{r}\right)}{(d+\mu+c)\left(1+\alpha K\left(1-\dfrac{d}{r}\right)\right)}.$$

When $R_0 < 1$, the disease-free equilibrium is asymptotically stable, which means that the epidemic will disappear; when $R_0 > 1$, the disease-free equilibrium is unstable, and there is an endemic equilibrium that is globally asymptotically stable, which means that the epidemic will prevail and persist in the population.

In real life, environmental noise is ubiquitous. It is very important to study the impact of environmental noise on the spread of infectious diseases in the prevention and control of infectious diseases. In [6], Wang pointed out that every parameter in the epidemic model may be randomly perturbed by the environment, which behaves as a random fluctuation. For example, the contact rate and the disease mortality rate in the epidemic model are randomly disturbed by external factors such as age, gender, constitution, mood, climate and season of the individual. Compared with the deterministic model, that with environmental noise can provide additional realism because deterministic model does not take into account these random factors and can only roughly reflect the real situation of infectious diseases to some extent. The research on random model can also be referred to the article [7]-[13].

Inspired by the ideas in the work of [5] [8] [9] [10], we consider the System (2) with a random interference in this paper. We assume that the parameter β is affected by white noise, so, the random driving force of Brownian motion is introduced into the System (2) as a random factor, that is, β is replaced with $\beta+\sigma B(t)$ where $B(t)$ represents Brownian motion and σ^2 represents the intensity of Brownian motion. After added a random term, the system (2) is described by

$$\begin{cases} \mathrm{d}S(t) = \left[rS(t)\left(1-\dfrac{S(t)}{K}\right) - \dfrac{\beta S(t)I(t)}{1+\alpha S(t)} + cI(t) - dS(t) \right]\mathrm{d}t - \dfrac{\sigma S(t)I(t)}{1+\alpha S(t)}\mathrm{d}B(t), \\ \mathrm{d}I(t) = \left[\dfrac{\beta S(t)I(t)}{1+\alpha S(t)} - (d+\mu+c)I(t) \right]\mathrm{d}t + \dfrac{\sigma S(t)I(t)}{1+\alpha S(t)}\mathrm{d}B(t). \end{cases} \tag{3}$$

The arrangement of this paper is as follows. In Section 2, we study the existence of a unique positive solution of System (3) for any positive initial value and then prove that the positive solution stays in R_+^2 with probability 1. In Section 3, we establish the sufficient conditions for the extinction of the infectious diseases. In Section 4, we give the conditions for the persistence of the infectious diseases in the mean value. Finally, we give some biological explanations and prevention and control measures for the epidemic.

2. The Existence and Uniqueness of Positive Solution of System

We first give the following notations and definitions. (Ω, F, P) is a complete probability space, $\{F_t : t \in R_+\}$ is a σ algebraic current on (Ω, F, P) satisfying the usual conditions.

Definition 1 (Locally Lipschitz condition).Function $f : R^n \to R$ is said to be locally Lipschitz continuous if there is positive constants L and r such that for $\forall x'', x' \in O(\xi, r)$, there is $\left| f(x''-x') \right| \le L\left\| x''-x' \right\|$, where $\xi \notin R^n$.

Definition 2 (Blow-up time). If the solution of the equation exists in region $\Omega\times(0,t_0)$, but does not exist in region $\Omega\times(0,t_0+\varepsilon)$ for an arbitrarily small constant $\varepsilon > 0$, then t_0 is called the blow-up time of the solution of the equation.

Definition 3 (Stopping time). If a function $\tau : \Omega \to T \bigcup \{\infty\}$ satisfies condition $\{\tau \le t\} \in F_t$, $\forall t \in T$, then τ is called a stopping time.

Remark 1. It is necessary to allow τ to get ∞. For example, If $\tau = \inf\{t \in T : X_t \in B\}$, where B is any given Borel set, and τ can be regarded as the first arrival time of X_t into B or the first exit time of B^C, then $\tau = \infty \Leftrightarrow X_t \notin B, \forall t \in T$.

Remark 2 Obviously, $\tau \equiv t$ (constant time) is a stopping time, which is a generalization of time.

Definition 4 (Ito Formula). Let $V(t,x) \in C^{1,2}\left(R_+ \times R^d\right)$, $x(t) = x(t_0) + \int_0^t f(s)\mathrm{d}s + \int_o^t g(s)\mathrm{d}w(s)$, $t \in J \subset R_+$ where $f \in L^1\left(J, R^d\right)$, $g \in L^2\left(J, R^{d\times m}\right)$, then $V(t, x(t))$ is a Ito-process, and

$$\mathrm{d}V(t,x(t)) = \left[V_t(t,x(t)) + V_x(t,x(t))f(t) + \frac{1}{2}V_{xx}(t,x(t))g^2(t) \right]\mathrm{d}t + V_x(t,x(t))g(t)\mathrm{d}w(t).$$

Since $S(t)$ and $I(t)$ in System (3) represent the size of the susceptible and infected populations at time t respectively, they must be nonnegative. We first give the result for System (3) having a global positive solution.

Theorem 1 For any initial value $(S(0),I(0))\in \mathrm{R}_+^2$, System (3) has a unique positive solution for $t\in(0,+\infty)$ and the solution stays in R_+^2 with probability 1, *i.e.*, $(S(t),I(t))\in \mathrm{R}_+^2$ is almost sure for $t\in(0,+\infty)$.

Proof Since the coefficients of System (3) are locally Lipschitz continuous, for any initial value $(S(0),I(0))\in \mathrm{R}_+^2$, System (3) has a unique local solution $(S(t),I(t))\in \mathrm{R}_+^2$, $t\in(0,\tau_a)$, where τ_a is the blow-up time (see [10]). In order to prove that the above local solution is global, it is only necessary to prove $\tau_a=\infty$ *a.s.* Therefore, the stopping time τ^* is defined as $\tau^*=\inf\{t\in(0,\tau_a): S(t)\le 0, I(t)\le 0\}$. From the definition of stopping time, we can see that, if $\tau^*=\infty$ *a.s.* can be proved, then when $t>0$, $\tau_a=\infty$ *a.s.* and $(S(t),I(t))\in \mathrm{R}_+^2$ *a.s.* Assume $\tau^*<\infty$, then there exists a constant $T>0$ such that $P\{\tau^*<T\}>0$.

Defining the C² function $V:\mathrm{R}_+^2\to R$ satisfying $V(S(t),I(t))=\ln S(t)I(t)$ and using the Ito formula (see [10]), for $\beta\in\{\tau^*<T\}$ and $t\in(0,\tau^*)$, we have:

$$
\begin{aligned}
&\mathrm{d}V(S(t),I(t))\\
&=\left[r\left(1-\frac{S(t)}{K}\right)-\frac{\beta I(t)}{1+\alpha S(t)}+\frac{cI(t)}{S(t)}-d-\frac{1}{2}\left(\frac{\sigma I(t)}{1+\alpha S(t)}\right)^2\right]\mathrm{d}t-\frac{\sigma I(t)}{1+\alpha S(t)}\mathrm{d}B(t)\\
&\quad+\left[\frac{\beta S(t)}{1+\alpha S(t)}-(d+\mu+c)-\frac{1}{2}\left(\frac{\sigma S(t)}{1+\alpha S(t)}\right)^2\right]\mathrm{d}t+\frac{\sigma S(t)}{1+\alpha S(t)}\mathrm{d}B(t)\\
&=\left[r\left(1-\frac{S(t)}{K}\right)-\frac{\beta I(t)}{1+\alpha S(t)}+\frac{cI(t)}{S(t)}+\frac{\beta S(t)}{1+\alpha S(t)}-(2d+\mu+c)\right.\\
&\quad\left.-\frac{1}{2}\left(\frac{\sigma I(t)}{1+\alpha S(t)}\right)^2-\frac{1}{2}\left(\frac{\sigma S(t)}{1+\alpha S(t)}\right)^2\right]\mathrm{d}t-\frac{\sigma I(t)}{1+\alpha S(t)}\mathrm{d}B(t)+\frac{\sigma S(t)}{1+\alpha S(t)}\mathrm{d}B(t)\\
&\ge\left[-\beta I(t)-(2d+\mu+c)-\frac{1}{2}\sigma^2I^2(t)-\frac{1}{2}\sigma^2S^2(t)\right]\mathrm{d}t\\
&\quad-\frac{\sigma I(t)}{1+\alpha S(t)}\mathrm{d}B(t)+\frac{\sigma S(t)}{1+\alpha S(t)}\mathrm{d}B(t)
\end{aligned}
$$

Here $1+\alpha S\ge 1$, therefore, we obtain

$$
\begin{aligned}
\mathrm{d}V(S(t),I(t))\ge&\left[-\beta I(t)-(2d+\mu+c)-\frac{1}{2}\sigma^2I^2(t)-\frac{1}{2}\sigma^2S^2(t)\right]\mathrm{d}t\\
&-\frac{\sigma I(t)}{1+\alpha S(t)}\mathrm{d}B(t)+\frac{\sigma S(t)}{1+\alpha S(t)}\mathrm{d}B(t).
\end{aligned}
$$

Let $F(S,I)=-\beta I-(2d+\mu+c)-\frac{1}{2}\sigma^2I^2-\frac{1}{2}\sigma^2S^2$, then we can get

$$
\mathrm{d}V(S(t),I(t))\ge F(S(t),I(t))\mathrm{d}t-\frac{\sigma I(t)}{1+\alpha S(t)}\mathrm{d}B(t)+\frac{\sigma S(t)}{1+\alpha S(t)}\mathrm{d}B(t). \quad (4)
$$

Integrating both sides of inequality (4) from 0 to t, we have

$$
V(S(t),I(t))\ge V(S(0),I(0))+\int_0^t F(S(u),I(u))\mathrm{d}u+\sigma\int_0^t\frac{S(u)-I(u)}{1+\alpha S(u)}\mathrm{d}B(u). \quad (5)
$$

Assume that $\left(S\left(\tau^*\right), I\left(\tau^*\right)\right) = (0,0)$, because $V\left(S(t), I(t)\right) = \ln S(t) I(t)$, and from the definition of the stopping time, we know that $\lim_{t\to\tau^*} V\left(S(t), I(t)\right) = \infty$.

Letting $t \to \tau^*$ in inequality (5), we get:

$$\infty \geq V\left(S(0), I(0)\right) + \int_0^{\tau^*} F\left(S(u), I(u)\right) \mathrm{d}u + \sigma \int_0^{\tau^*} \frac{S(u) - I(u)}{1 + \alpha S(u)} \mathrm{d}B(u) > \infty$$

This is a contradiction with the assumption $\tau^* < \infty$. Therefore, $\tau^* \to +\infty$, which also proves that $S(t)$ and $I(t)$ will not blow up in finite time and with probability 1. We obtain the forward invariant set of System (3)

$$\Psi = \left\{(S, I) : S > 0, I > 0, S + I \leq K\left(1 - \frac{d}{r}\right)\right\}.$$

In the following, it is enough to consider the solution of System (3) in Ψ.

3. Sufficient Conditions for Extinction of Infectious Diseases

Before giving the extinction theorem, we give a lemma which can be found in [10].

Lemma 1 Let $M(t)$ $(t \geq 0)$ be a locally continuous martingale with initial value $M(0) = 0$ and $\langle M(t) \rangle$ be a quadratic variation of $M(t)$. Let $\delta > 1$ and ν_n, τ_n be two sequences of positive terms. Then, for almost all $w \in \Omega$, there exists a positive integer $n_0 = n_0(w)$ such that for any $n \geq n_0$, we have

$$M(t) \leq \frac{1}{2}\nu_n \langle M(t), M(t) \rangle + \frac{\delta \ln n}{\nu_n}, 0 \leq t \leq \tau_n. \tag{6}$$

Theorem 2 Let $\left(S(t), I(t)\right)$ be a solution of System (3) with initial value $\left(S(0), I(0)\right) \in \Psi$, we have

$$\limsup_{t\to+\infty} \frac{\ln I(t)}{t} < \frac{\beta^2}{2\sigma^2} - (d + \mu + c). \tag{7}$$

If $\frac{\beta^2}{2\sigma^2} < (d + \mu + c)$, then $I(t)$ tends to 0 exponentially with probability 1 and $\lim_{t\to+\infty} S(t) = K\left(1 - \frac{d}{r}\right)$, *a.s.*

Proof Applying I to formula to the System (3), we have

$$\mathrm{d}\left(\ln I(t)\right) = \left[\frac{\beta S(t)}{1 + \alpha S(t)} - (d + \mu + c) - \frac{\sigma^2 S^2(t)}{2\left(1 + \alpha S(t)\right)^2}\right] \mathrm{d}t + \frac{\sigma S(t)}{1 + \alpha S(t)} \mathrm{d}B(t). \tag{8}$$

Integrating both sides of (8) from 0 to t

$$\begin{aligned} \ln I(t) &= \ln I(0) + \int_0^t \left[\frac{\beta S(u)}{1 + \alpha S(u)} - (d + \mu + c) - \frac{\sigma^2 S^2(u)}{2\left(1 + \alpha S(u)\right)^2}\right] \mathrm{d}u \\ &\quad + \int_0^t \frac{\sigma S(u)}{1 + \alpha S(u)} \mathrm{d}B(u) \\ &= \ln I(0) + \int_0^t \left[\frac{\beta S(u)}{1 + \alpha S(u)} - (d + \mu + c) - \frac{\sigma^2 S^2(u)}{2\left(1 + \alpha S(u)\right)^2}\right] \mathrm{d}u + M(t) \end{aligned} \tag{9}$$

where $M(t)=\int_0^t \frac{\sigma S(u)}{1+\alpha S(u)}\mathrm{d}B(u)$ is a locally continuous martingale with quadratic variation:

$$\langle M(t), M(t)\rangle = \sigma^2 \int_0^t \frac{S^2(u)}{(1+\alpha S(u))^2}\mathrm{d}u. \tag{10}$$

Taking $\delta=2$, $\nu_n=\nu>0$ and $\tau_n=n$ in Lemma 1, for almost all $w\in\Omega$, there exists a positive integer $n_0=n_0(w)$ such that for any $n\geq n_0$, we have

$$M(t)\leq \frac{1}{2}\nu\sigma^2\int_0^t \frac{S^2(u)}{1+\alpha S(u)}\mathrm{d}u+\frac{2\ln n}{\nu}. \tag{11}$$

From (9) and (11):

$$\begin{aligned}\ln I(t) &< \ln I(0)+\int_0^t\left[\frac{\beta S(u)}{1+\alpha S(u)}-(d+\mu+c)-\frac{\sigma^2 S^2(u)}{2(1+\alpha S(u))}\right]\mathrm{d}u\\ &\quad+\frac{1}{2}\nu\sigma^2\int_0^t\frac{S^2(u)}{(1+\alpha S(u))^2}\mathrm{d}u+\frac{2\ln n}{\nu}\\ &=\ln I(0)+\int_0^t\left[-\frac{1}{2}(1-\nu)\sigma^2\frac{S^2(u)}{(1+\alpha S(u))^2}+\frac{\beta S(u)}{1+\alpha S(u)}-(d+\mu+c)\right]\mathrm{d}u\\ &\quad+\frac{2\ln n}{\nu}\end{aligned} \tag{12}$$

where

$$\begin{aligned}&-\frac{1}{2}(1-\nu)\sigma^2\left(\frac{S^2(t)}{(1+\alpha S(t))^2}-\frac{2\beta}{\sigma^2(1-\nu)}\frac{S(t)}{1+\alpha S(t)}\right)\\ &=-\frac{1}{2}(1-\nu)\sigma^2\left(\frac{S(t)}{1+\alpha S(t)}-\frac{\beta}{\sigma^2(1-\nu)}\right)^2+\frac{\beta^2}{2\sigma^2(1-\nu)}\\ &\leq\frac{\beta^2}{2\sigma^2(1-\nu)}.\end{aligned} \tag{13}$$

From (12) and (13), we get

$$\ln I(t)<\ln I(0)+\left(\frac{\beta^2}{2\sigma^2(1-\nu)}-(d+\mu+c)\right)t+\frac{2\ln n}{\nu}. \tag{14}$$

Therefore, for $n-1\leq t\leq n$, by dividing t on both sides of inequality (14), we obtain

$$\frac{\ln I(t)}{t}<\frac{\ln I(0)}{t}+\left(\frac{\beta^2}{2\sigma^2(1-\nu)}-(d+\mu+c)\right)+\frac{2}{\nu}\frac{\ln n}{n-1}. \tag{15}$$

Let $n\to+\infty$, thus $t\to+\infty$, we have

$$\limsup_{t\to+\infty}\frac{\ln I(t)}{t}<\frac{\beta^2}{2\sigma^2(1-\nu)}-(d+\mu+c). \tag{16}$$

Let $\nu\to 0$, then we have

$$\limsup_{t\to+\infty}\frac{\ln I(t)}{t}<\frac{\beta^2}{2\sigma^2}-(d+\mu+c). \tag{17}$$

If $\frac{\beta^2}{2\sigma^2}<(d+\mu+c)$ holds, we get $\lim_{t\to+\infty} I(t)=0$ *a.s.* From system (3),

$$\begin{aligned}&\frac{\mathrm{d}(S(t)+I(t))}{\mathrm{d}t}\\&=rS(t)\left(1-\frac{S(t)}{K}\right)-d(S(t)+I(t))-\mu I(t),\end{aligned} \tag{18}$$

then

$$S(t)+I(t)=\mathrm{e}^{-dt}\left\{C+\int_0^t\left[rS(u)\left(1-\frac{S(u)}{K}\right)-\mu I(u)\right]\mathrm{e}^{dt}\mathrm{d}u\right\} \tag{19}$$

where C is any arbitrary constant. By (19) there is

$$\lim_{t\to+\infty}(S(t)+I(t))=\lim_{t\to+\infty}\mathrm{e}^{-dt}\left\{C+\int_0^t\left[rS(u)\left(1-\frac{S(u)}{K}\right)-\mu I(u)\right]\mathrm{e}^{dt}\mathrm{d}u\right\}.$$

Applying L'Hopital's rule and $\lim_{t\to+\infty} I(t)=0$, we have $\lim_{t\to+\infty} S(t)=K\left(1-\frac{d}{r}\right)$ almost everywhere.

4. Persistence of Infectious Diseases in the Meansense

Definition 5. If $\liminf_{t\to+\infty}\frac{1}{t}\int_0^t I(u)\mathrm{d}u>0$ *a.s.*, then System (3) is persistent in the mean sense (see [8]).

Theorem 3. If

$$\bar{R}_0=\left(1-\frac{\sigma^2K\left(1-\frac{d}{r}\right)}{2\beta}\right)R_0>1, \tag{20}$$

then for any initial value $(S(0),I(0))\in\Psi$ the solution of System (3) has the following properties

$$\liminf_{t\to+\infty}\frac{1}{t}\int_0^t I(u)\mathrm{d}u>\frac{(d-r)(d+\mu+c)\left(1+\alpha K\left(1-\frac{d}{r}\right)\right)(\bar{R}_0-1)}{\beta(d+\mu)}\quad a.s. \tag{21}$$

Proof By using the Newton-Leibniz formula, from the System (3) we obtain

$$\begin{aligned}&S(t)-S(0)+I(t)-I(0)\\&=r\int_0^t S(u)\mathrm{d}u-\frac{r}{K}\int_0^t S^2(u)\mathrm{d}u+c\int_0^t I(u)\mathrm{d}u\\&\quad-d\int_0^t S(u)\mathrm{d}u-(d+\mu+c)\int_0^t I(u)\mathrm{d}u.\end{aligned} \tag{22}$$

Dividing both sides of inequality (22) by t, we get

$$\begin{aligned}&\frac{S(t)-S(0)}{t}+\frac{I(t)-I(0)}{t}\\&=\frac{r}{t}\int_0^t S(u)\mathrm{d}u-\frac{r}{tK}\int_0^t S^2(u)\mathrm{d}u+\frac{c}{t}\int_0^t I(u)\mathrm{d}u\\&\quad-\frac{d}{t}\int_0^t S(u)\mathrm{d}u-\frac{d+\mu+c}{t}\int_0^t I(u)\mathrm{d}u\\&=\frac{r-d}{t}\int_0^t S(u)\mathrm{d}u-\frac{r}{tK}\int_o^t S^2(u)\mathrm{d}u-\frac{d+\mu}{t}\int_0^t I(u)\mathrm{d}u.\end{aligned}\tag{23}$$

So, we have

$$\begin{aligned}\frac{1}{t}\int_0^t S(u)\mathrm{d}u&=\phi(t)+\frac{r}{tK(r-d)}\int_0^t S^2(u)\mathrm{d}u+\frac{d+\mu}{t(r-d)}\int_0^t I(u)\mathrm{d}u\\&\le K\left(1-\frac{d}{r}\right)+\frac{d+\mu}{r-d}\frac{1}{t}\int_0^t I(u)\mathrm{d}u+\phi(t)\end{aligned}\tag{24}$$

where $\phi(t)=\dfrac{1}{r-d}\left(\dfrac{S(t)-S(0)}{t}+\dfrac{I(t)-I(0)}{t}\right)$.

Applying the Itoformula, there is

$$\begin{aligned}&\mathrm{d}\left(\ln I(t)+\alpha S(t)\right)\\&=\left[\beta S(t)-(d+\mu+c)-\alpha(d+\mu+c)S(t)-\frac{\sigma^2S^2(t)}{2}\right]\mathrm{d}t+\sigma S(t)\mathrm{d}B(t)\\&\ge\left[\beta S(t)-(d+\mu+c)-\alpha K\left(1-\frac{d}{r}\right)(d+\mu+c)-\frac{\sigma^2K^2\left(1-\frac{d}{r}\right)^2}{2}\right]\mathrm{d}t\\&\quad+\sigma S(t)\mathrm{d}B(t).\end{aligned}\tag{25}$$

Integrating both sides of inequality (25) from 0 to t, we get

$$\begin{aligned}&\frac{\ln I(t)-\ln I(0)}{t}+\alpha\frac{S(t)-S(0)}{t}\\&\ge\frac{\beta}{t}\int_0^t S(u)\mathrm{d}u-(d+\mu+c)-\alpha K\left(1-\frac{d}{r}\right)(d+\mu+c)\\&\quad-\frac{\sigma^2K^2\left(1-\frac{d}{r}\right)^2}{2}+\frac{\sigma}{t}\int_0^t S(u)\mathrm{d}B(u)\end{aligned}\tag{26}$$

Substituting (24) into (26) yields

$$\begin{aligned}&\frac{\ln I(t)-\ln I(0)}{t}+\alpha\frac{S(t)-S(0)}{t}\\&\ge\beta K\left(1-\frac{d}{r}\right)-\frac{\beta(d+\mu)}{d-r}\frac{1}{t}\int_0^t I(u)\mathrm{d}u+\beta\phi(t)\\&\quad-\left(1+\alpha K\left(1-\frac{d}{r}\right)\right)(d+\mu+c)-\frac{\sigma^2K^2\left(1-\frac{d}{r}\right)^2}{2}+\frac{\sigma}{t}\int_0^t S(u)\mathrm{d}B(u)\end{aligned}$$

$$= \beta K\left(1-\frac{d}{r}\right)-\left(1+\alpha K\left(1-\frac{d}{r}\right)\right)(d+\mu+c)-\frac{\sigma^2K^2\left(1-\frac{d}{r}\right)^2}{2}$$
$$-\frac{\beta(d+\mu)}{d-r}\frac{1}{t}\int_0^t I(u)\mathrm{d}u+\beta\phi(t)+\frac{\sigma}{t}\int_0^t S(u)\mathrm{d}B(u).$$

Therefore

$$\frac{1}{t}\int_0^t I(u)\mathrm{d}u \geq \frac{d-r}{\beta(d+\mu)}\left[\left(\bar{R}_0-1\right)(d+u+c)\left(1+\alpha K\left(1-\frac{d}{r}\right)\right)+\beta\phi(t)\right.$$
$$\left.+\frac{N(t)}{t}-\frac{\ln I(t)-\ln I(0)}{t}-\alpha\frac{S(t)-S(0)}{t}\right] \tag{27}$$

where $N(t)=\sigma\int_0^t S(u)\mathrm{d}B(u)$, $N(t)$ is locally continuous martingale with initial value $N(0)=0$, and $\limsup_{t\to+\infty}\frac{\langle N(t),N(t)\rangle}{t}\leq\sigma^2K^2\left(1-\frac{d}{r}\right)^2<+\infty=\sigma\int_0^t S(u)\mathrm{d}B(u)$ *a.s.* From the law of large numbers for martingales (see [10]), we obtain $\lim_{t\to+\infty}\frac{N(t)}{t}=0$ *a.s.* By Theorem 1, we have $-\infty<\ln I(t)<\ln K\left(1-\frac{d}{r}\right)$ and $\lim_{t\to+\infty}\phi(t)=0$ *a.s.*, then from (27), we have

$$\liminf_{t\to+\infty}\frac{1}{t}\int_0^t I(u)\mathrm{d}u\geq\frac{(d-r)(d+\mu+c)\left(1+\alpha K\left(1-\frac{d}{r}\right)\right)\left(\bar{R}_0-1\right)}{\beta(d+\mu)}.$$

Therefore, the infectious diseases are persistent in the sense of mean value.

Remark 1 Theorem 3 shows that under some conditions, the infected population is persistent on average, so it can be verified that the persistence of the susceptible population is weak, in fact,

$$\limsup_{t\to+\infty}\frac{1}{t}\int_0^t S(u)\mathrm{d}u\leq K\left(1-\frac{d}{r}\right)-\frac{(d+\mu+c)\left(1+\alpha K\left(1-\frac{d}{r}\right)\right)\left(\bar{R}_0-1\right)}{\beta}\quad a.s.$$

5. Biology Interpretation and Control Measures

From Theorem 1 and Theorem 3, we know that the System (3) is persistent only under the condition of the basic reproduction number $\bar{R}_0$ being greater than 1. In order to control the spread of infectious diseases, we must adopt some strategies to make it small enough during the spread of infectious diseases. Specific control measures are as follows.

1) To strengthen the control of the source of infection, the relevant public health departments should take all measures to control the source of infection. For example, during the outbreak of infectious diseases, designated hospitals should be determined for centralized treatment to strictly prevent the spread of the disease and reduce the infection rate, so as to achieve the purpose of controlling the spread of infectious diseases;

2) To establish a direct network reporting system for infectious diseases, timely detection, timely reporting, timely treatment of infectious diseases, reduce the impact of psychological effect coefficient α on the control of the spread of infectious diseases;

3) To control the size of the intensity of Brownian motion σ^2 to ensure $\overline{R}_0 < 1$.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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