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A Family of Inertial Manifolds for a Class of Generalized Kirchhoff-Beam Equations

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Abstract

In this paper, we deal with a class of generalized Kirchhoff-Beam equations. At first, we take advantage of Hadamard's graph to get the equivalent form of the original equations. Then, the inertial manifolds are proved by using spectral gap condition. We gain main result is that the family of inertial manifolds are established under the proper assumptions of nonlinear terms $M(s)$ and $N(s)$.

Keywords

Kirchhoff-Beam Equations, Inertial Manifold, Hadamard's Graph, Spectral Gap Condition

1. Introduction

This paper mainly deals with existence of a family of inertial manifolds for a class generalized Kirchhoff-Beam equations.

$$u_{tt} + \beta(-\Delta)^{2m} u_t + M\left(\left\|D^m u\right\|_p^p\right)u_t + \alpha\Delta^{2m}u + N\left(\left\|D^m u\right\|_p^p\right)(-\Delta)^m u = f(x), \tag{1}$$

$$u(x,t) = 0, \frac{\partial^i u}{\partial v^i} = 0, i = 1,2,\cdots,m-1, x \in \partial\Omega, t > 0, \tag{2}$$

$$u(x,0) = u_0(x), u_t(x,0) = u_1(x), x \in \Omega \subset R^n. \tag{3}$$

where $m > 1$ is a positive integer, Ω is the bounded region in R^n with smooth boundary $\partial\Omega$. $f(x)$ is the external force term. $\beta(-\Delta)^{2m} u_t$ is the strongly damped term, α, β are positive constants, $\alpha \geq \frac{3}{2}$, $M\left(\left\|D^m u\right\|_p^p\right), N\left(\left\|D^m u\right\|_p^p\right)$ are the general non-negative real-valued functions, $\left\|D^m u\right\|_p^p = \int_\Omega \left|D^m u\right|^p \mathrm{d}x$, and the relevant assumptions will be given later.

Yuhuan Liao, Guoguang Lin, Jie Liu [1] has studied the existence and uniqueness of global solutions and the existence for of a family global attractors and estimate its Hausdorff dimension and Fractal dimension for the problems (1)-(3).

As well as we known, it is significant to establish inertial manifolds for the study of the long-time behavior of infinite dimensional dynamical systems. Because it is an important bridge between infinite-dimensional dynamic system and finite-dimensional dynamical system. In this article, we first take advantage of Hadamard's graph to transform problem (1)-(3) into an equivalent one-order system of form. Then, we proved the family of inertial manifolds by using spectral gap condition.

To better carry out our work, let's recall some results regarding wave equations.

Jingzhu Wu and Guoguang Lin [2] studied the following two-dimensional strong damping Boussinesq equation while $\alpha > 2$:

$$u_{tt} - \alpha\Delta u_t - \Delta u + u^{2k+1} = f(x,y),(x,y)\in\Omega,$$

$$u(x,y,0) = u_0(x,y),(x,y)\in\Omega,$$

$$u(x,y,t) = u(x+\pi,y,t) = u(x,y+\pi,t) = 0,(x,y)\in\Omega,$$

where $\Omega = (0,\pi)\times(0,\pi)\subset R\times R$, $t>0$. They obtained result that is existence of inertial manifolds.

Guigui Xu, Libo Wang and Guoguang Lin [3] investigated the strongly damped wave equation:

$$u_{tt} - \alpha\Delta u + \beta\Delta^2 u - \gamma\Delta u_t + g(u) = f(x,t),(x,t)\in\Omega\times R^+,$$

$$u(x,0) = u_0(x),u_t(x,0) = u_1(x),x\in\Omega,$$

$$u|_{\partial\Omega} = 0,\Delta u|_{\partial\Omega} = 0,(x,t)\in\partial\Omega\times R^+.$$

They gave some assumptions for the nonlinearity term $g(u)$ to satisfy the following inequalities:

(A1) $\liminf\limits_{|s|\to\infty}\frac{G(s)}{s^2}\geq 0$, $s\in R$, $G(s)=\int_0^s g(r)\mathrm{d}r$.

(A2) There is positive constant C_1 such that $\liminf\limits_{|s|\to\infty}\frac{sg(s)-C_1G(s)}{s^2}\geq 0$, $s\in R$.

According to the above assumptions, they proved the inertial manifolds by using the Hadamard's graph transformation method.

Ruijin Lou, Penhui Lv, Guoguang Lin [4] considered a class of generalized nonlinear Kirchhoff-Sine-Gordon equation:

$$u_{tt} - \beta\Delta u_t + \alpha u_t - \phi\left(\|\nabla u\|^2\right)\Delta u + g(\sin u) = f(x),$$

$$u(x,t) = 0, x\in\Omega, t\geq 0,$$

$$u(x,0)=u_0(x),u_t(x,0)=u_1(x),x\in\Omega.$$

Under some reasonable assumptions, they obtained some results that are squeezing property of the nonlinear semigroup associated with this equation and the existence of exponential attractors and inertial manifolds.

Lin Chen, Wei Wang and Guoguang Lin [5] studied higher-order Kirchhoff-type equation with nonlinear strong dissipation in n dimensional space:

$$u_{tt}+(-\Delta)^m u_t+\phi\left(\|\nabla u\|^2\right)(-\Delta)^m u+g(u)=f(x),x\in\Omega,t>0,m>1,$$

$$u(x,t)=0,\frac{\partial^i u}{\partial v^i}=0,i=1,2,\cdots,m-1,x\in\partial\Omega,t>0,$$

$$u(x,0)=u_0(x),u_t(x,0)=u_1(x).$$

For the above equation, they made some suitable assumptions about $\phi(s)$ and $g(u)$ to get existence of exponential attractors and inertial manifolds. More information on inertial manifolds can be found in the literature [6] [7] [8] [9].

2. Preliminaries

The following symbols and assumptions are introduced for the convenience of statement:

$$V_0=L^2(\Omega),V_{2m}=H^{2m}(\Omega)\cap H_0^1(\Omega),V_{2m+k}=H^{2m+k}(\Omega)\cap H_0^1(\Omega),$$
$$E_0=V_{2m}\times V_0,E_k=V_{2m+k}\times V_k,(k=0,1,2,\cdots,2m)$$

The inner product of the $L^2(\Omega)$ space is $(u,v)=\int_\Omega u(x)v(x)\mathrm{d}x$ and the norm is $\|u\|=\|u\|_{L^2}=\left(\int_\Omega|u(x)|^2\mathrm{d}x\right)^{\frac{1}{2}}$. The norm of $L^p(\Omega)$ space is called $\|u\|_p=\|u\|_{L^p(\Omega)}$.

Definition 1 [10] Assuming $S=(S(t))_{t\ge0}$ is a solution semigroup on Banach space $E_k=V_{2m+k}\times V_k$, subset $\mu_k\subset E_k$ is said to be a family of inertial manifolds, if they satisfy the following three properties:

1) w_k are a finite-dimensional Lipschitz manifold;

2) w_k is positively invariant, *i.e.*, $S(t)w_k\subseteq w_k,t\ge0,k=1,2\cdots,2m$;

3) w_k attracts exponentially all orbits of solution, that is, for any $x\in E_k$, there are constants $\eta>0,C>0$ such that

$$dist(S(t)x,w_k)\le C\mathrm{e}^{-\eta t},t\ge0.$$

Definition 1 [7] Let $A:X\to X$ be an operator and assume that $F\in C_b(X,X)$ satisfies the Lipschitz condition:

$$\|F(U)-F(V)\|_X\le l_F\|U-V\|_X,U,V\in X,$$

where $X=H_0^m(\Omega)\times H_0^m(\Omega)\times L^2(\Omega)\times L^2(\Omega)$. The operator A is said to satisfy the spectral gap condition relative to F, if the point spectrum of the operator A can be divided into two parts σ_1 and σ_2, of which σ_1 is finite, and such that, if

$$\Lambda_1=\sup\{\operatorname{Re}\lambda|\lambda\in\sigma_1\},\Lambda_2=\sup\{\operatorname{Re}\lambda|\lambda\in\sigma_2\},$$

and

$$X_i = span\{\omega_j \,|\, j \in \sigma_i\}, i = 1, 2.$$

Then

$$\Lambda_2 - \Lambda_1 > 4l_F.$$

And the orthogonal decomposition

$$X = X_1 \oplus X_2,$$

holds with continuous orthogonal projections $P_1 : X \to X_1$ and $P_1 : X \to X_2$.

Lemma 2 [7] Let the eigenvalues $\lambda_j^{\pm}, j \geq 1$ be arranged in nondecreasing order, for all $m \in N$, there is $N \geq m$ such that λ_N^- and λ_{N+1}^- are consecutive.

Theorem 1 [8] Supose dense positive definite operators A generates C^0-semigroup $S(t)$ in detachabie Hilbert space X, $F \in C_b(X, X)$ meets Lipschitz condition, A satifies the spectral condition, then the probrem
$U_t + AU = F(U), U \in X$ has an inertial manifold $w \subset X$,
$w = \text{graph}(\phi) = \{\xi + \phi(\xi) \,|\, \xi \in X_1\}$, where $\phi : X_1 \to X_1$ Lipschitz continuous function.

3. Inertial Manifold

In this section, we use the Hadamard's graph transformation method to prove the existence of inertial manifolds of problem (1)-(3) when N is sufficiently large.

Equation (1) is equivalent to the following one order evolution equation:

$$U_t + \Lambda U = F(U), \tag{4}$$

where

$$U = (u, v), v = u_t, \Lambda = \begin{pmatrix} 0 & -I \\ \alpha\Delta^{2m} & \beta(-\Delta)^{2m} \end{pmatrix}, \tag{5}$$

$$F(U) = \begin{pmatrix} 0 \\ f(x) - M\left(\|D^m u\|_p^p\right)u_t - N\left(\|D^m u\|_p^p\right)(-\Delta)^m u \end{pmatrix}, \tag{6}$$

$$D(\Lambda) = \left\{u \in H^{2m+k} \,\middle|\, u \in L^2, (-\Delta)^{2m} u \in H^{2m+k}\right\} \times H^k. \tag{7}$$

In E_k, we denote the usual graph norm, which is introduced by the scalar product, we have

$$(U, V)_{E_k} = \left(\alpha(-\Delta)^{m+k} u, (-\Delta)^{m+k} \overline{y}\right) + \left(D^k v, D^k \overline{z}\right), \tag{8}$$

$U = (u, v)$, $V = (y, z) \in E_k$, $\overline{y}, \overline{z}$ respectively express conjugate of y, z and $u, y, v, z \in H^{2m+k}(\Omega)$.

For $U \in D(\Lambda)$, we have

$$\begin{aligned} (\Lambda U, U)_{E_k} &= -\left(\alpha(-\Delta)^{m+k} v, (-\Delta)^{m+k} \overline{u}\right) + \left(\alpha(-\Delta)^{2m+2k} u + \beta(-\Delta)^{m+k} v, \overline{v}\right) \\ &= \beta\|D^{m+k} v\|^2 \geq 0, \end{aligned} \tag{9}$$

Therefore, the operator Λ in (5) is monotone, and $(HU,U)_E$ is a nonnegative and real number.

To obtain the eigenvalues of Λ, we consider the following eigenvalue equation:

$$\Lambda U=\lambda U, U=(u,v)\in E_k, \tag{10}$$

That is

$$\begin{cases}-v=\lambda u,\\ \alpha\Delta^{2m}u+\beta(-\Delta)^{2m}v=\lambda v.\end{cases} \tag{11}$$

The first equation in (11) is brought into the second equation in (11), we get

$$\begin{cases}\lambda^2 u+\alpha\Delta^{2m}u-\beta\lambda(-\Delta)^{2m}u=0,\\ u|_{\partial\Omega}=(-\Delta)^m u|_{\partial\Omega}=0,\end{cases} \tag{12}$$

Let u_j replace u in (12). And then taking $(-\Delta)^k u_j$ inner product, we obtain

$$\lambda^2\left\|D^k u\right\|^2+\alpha\left\|D^{2m+k}u\right\|^2-\beta\lambda\left\|D^{2m+k}u\right\|^2=0. \tag{13}$$

When (13) is considered a yuan quadratic equation on λ, we can get

$$\lambda_j^{\pm}=\frac{\beta\bar{\mu}_j^2\pm\sqrt{\beta^2\bar{\mu}_j^4-4\alpha\bar{\mu}_j^2}}{2}, \tag{14}$$

where μ_j is the eigenvalue of $(-\Delta)^m$ in H_0^{2m}, then $\mu_j=c_0 j^{\frac{m}{n}}$. If $\bar{\mu}_j\geq\frac{2\sqrt{\alpha}}{\beta}$, the eigenvalues of Λ are all positive and real numbers, the corresponding eigenfunction have the form $U_j^{\pm}=(u_j,-\lambda_j^{\pm}u_j)$. For (14) and future reference, we observe that for all

$$j\geq 1,\ \left\|D^{2m+k}u_j\right\|=\sqrt{\bar{\mu}_j},\ \left\|D^k u_j\right\|=1,$$

$$\left\|D^{-2m-k}u_j\right\|=\frac{1}{\sqrt{\mu_j}}=\bar{\mu}_j, k=1,2,\cdots,2m. \tag{15}$$

Lemma 3 $g=M\left(\left\|D^m u\right\|_p^p\right)u_t, h=N\left(\left\|D^m u\right\|_p^p\right)(-\Delta)^m u$, then $g,h:H_0^{2m+k}(\Omega)\to H(\Omega)$ is uniformly bounded and globally Lipschitz continuous.

Proof. $\forall u_1,u_2\in H_0^{2m+k}(\Omega)$,

$$\begin{aligned}&\left\|M\left(\left\|D^m u_1\right\|_p^p\right)u_{1t}-M\left(\left\|D^m u_2\right\|_p^p\right)u_{2t}\right\|\\ &\leq\left\|M\left(\left\|D^m u_1\right\|_p^p\right)\right\|_\infty\left\|u_{1t}-u_{2t}\right\|+M'\left(\left\|D^m\zeta\right\|_p^p\right)D^m(u_1-u_2)\left\|u_{2t}\right\|\\ &\leq\left\|M\left(\left\|D^m u_1\right\|_p^p\right)\right\|_\infty\left\|u_{1t}-u_{2t}\right\|+\left\|M'\left(\left\|D^m\zeta\right\|_p^p\right)\right\|_\infty\left\|u_{2t}\right\|D^m(u_1-u_2)\\ &\leq C_{1k}\left(\left\|u_{1t}-u_{2t}\right\|_{H^k}+\left\|u_1-u_2\right\|_{H^{2m+k}}\right),\end{aligned}$$

$$\begin{aligned}&\left\|N\left(\left\|D^m u_1\right\|_p^p\right)(-\Delta)^m u_1 - N\left(\left\|D^m u_2\right\|_p^p\right)(-\Delta)^m u_2\right\|\\&= N\left(\left\|D^m u_1\right\|_p^p\right)(-\Delta)^m (u_1-u_2) + \left(N\left(\left\|D^m u_1\right\|_p^p\right) - N\left(\left\|D^m u_2\right\|_p^p\right)\right)(-\Delta)^m u_2\\&\le N\left(\left\|D^m u_1\right\|_p^p\right)(-\Delta)^m (u_1-u_2) + N'\left(\left\|D^m \zeta\right\|_p^p\right) D^m (u_1-u_2)\left\|(-\Delta)^m u_2\right\|\\&\le \left\|N\left(\left\|D^m u_1\right\|_p^p\right)\right\|_\infty \left\|u_1-u_2\right\|_{H^{2m}} + N'\left(\left\|D^m \zeta\right\|_p^p\right)\left\|(-\Delta)^m u_2\right\|\left\|D^m (u_1-u_2)\right\|_{H^m}\\&\le C_{2k}\left\|u_1-u_2\right\|_{H^{2m+k}},\end{aligned}$$

where

$$C_{1k} = \max\left\{\left\|M\left(\left\|D^m u_1\right\|_p^p\right)\right\|_\infty, \lambda_1^{-\frac{m}{2}}\left\|M'\left(\left\|D^m \zeta\right\|_p^p\right)\right\|_\infty \left\|u_{2t}\right\|\right\},$$

$$C_{2k} = \max\left\{\left\|N\left(\left\|D^m u_1\right\|_p^p\right)\right\|_\infty, \lambda_1^{-\frac{m}{2}} N'\left(\left\|D^m \zeta\right\|_p^p\right)\left\|(-\Delta)^m u_2\right\|\right\}.$$

Lemma 3 is proved.

Theorem 2 If $\bar{\mu}_j \le \frac{2\sqrt{\alpha}}{\beta}$ holds, $l_k = \max\{C_{1k}, C_{2k}\}$ is maximum Lipschitz constant of g, h, and if $N_1 \in N^+$ is sufficiently large such that when $N \ge N_1$, the following inequality holds:

$$\beta\left(\bar{\mu}_{N+1} - \bar{\mu}_N\right)\left(\bar{\mu}_{N+1} + \bar{\mu}_N\right) \ge 8 l_k. \quad (16)$$

Then the operator Λ satisfies the spectral gap condition.

Proof.

From (8), $U = (u, \bar{u}), V = (v, \bar{v}) \in E_k$, then

$$\begin{aligned}\left\|F(U) - F(V)\right\| &= \left\|M\left(\left\|D^m u\right\|_p^p\right)u_t + N\left(\left\|D^m u\right\|_p^p\right)(-\Delta)^m u\right.\\&\quad\left. - M\left(\left\|D^m v\right\|_p^p\right)v_t - N\left(\left\|D^m v\right\|_p^p\right)(-\Delta)^m v\right\|\\&\le l_k \left\|U - V\right\|_{E_k}.\end{aligned}$$

We have $l_F \le l_k$, and take a real component $\operatorname{Re}\lambda_j^{\pm} = \frac{\beta\bar{\mu}_j^2}{2}$,

There is N_1, such that $N \ge N_1$, (16) holds. Spectra decomposition of Λ:

$$\sigma_1 = \left\{\lambda_j^{\pm} \,\middle|\, j \le N\right\}, \sigma_2 = \left\{\lambda_j^{\pm} \,\middle|\, j \ge N+1\right\}.$$

Corresponding space

$$E_{k1} = \operatorname{span}\left\{U_j^{\pm} \,\middle|\, j \le N\right\}, E_{k2} = \operatorname{span}\left\{U_j^{\pm} \,\middle|\, j \ge N+1\right\}.$$

Then

$$\begin{aligned}\Lambda_2 - \Lambda_1 &= \operatorname{Re}\left(\lambda_{N+1}^- - \lambda_N^+\right)\\&= \frac{\beta\bar{\mu}_{N+1}^2 - \beta\bar{\mu}_N^2}{2} = \frac{\beta\left(\bar{\mu}_{N+1} - \bar{\mu}_N\right)\left(\bar{\mu}_{N+1} + \bar{\mu}_N\right)}{2}\\&> 4 l_F.\end{aligned}$$

Then the operator Λ satisfies the spectral gap condition. Theorem 2 is proved.

Theorem 3 If $\bar{\mu}_j > \frac{2\sqrt{\alpha}}{\beta}$ holds, $l_k = \max\{C_1, C_2\}$ is the Lipschtiz constant of g, h.

Let $N_1 \in N^+$ big enough, $N \ge N_1$, the following inequality holds:

$$\left(\bar{\mu}_{N+1}^2 - \bar{\mu}_N^2\right)\cdot\frac{2\alpha}{\beta\left(\sqrt{J(N+1)}+\sqrt{J(N)}\right)} \ge \frac{4l}{\sqrt{\alpha+\beta-2}},$$

where $J(N) = \bar{\mu}_N^4 - \frac{4\alpha\bar{\mu}_N^2}{\beta^2}$.

Let $\frac{2\sqrt{\alpha}}{\beta} < \bar{\mu}_j < 2\sqrt{\frac{\alpha}{\beta}}$, for $\bar{\mu}_q \in \{\bar{\mu}_k\}$, such that $2\frac{\sqrt{\alpha}}{\beta} < \bar{\mu}_q < 2\sqrt{\frac{\alpha}{\beta}}$,

$$\left(\bar{\mu}_{q+1}^2 - \bar{\mu}_q^2\right)\cdot\frac{2\alpha}{\beta\left(\sqrt{J(q+1)}+\sqrt{J(q)}\right)} \ge 4l.$$

Then for any one case (1) and (2), the operator Λ satisfies the spectral gap condition.

Proof.

When $\bar{\mu}_j > \frac{2\sqrt{\alpha}}{\beta}$, all the eigenvalues of Λ are real and positive, and we can easily know that both sequences $\{\lambda_j^-\}_{j\ge1}$ and $\{\lambda_j^+\}_{j\ge1}$ are increasing.

The whole process of proof is divided into four steps.

Step 1. Since $\lambda_j^\pm$ is arranged in nondecreasing order. According to Lemma 2, given N such that λ_N^- and λ_{N+1}^- are consecutive, we separate the eigenvalue of Λ as

$$\sigma_1 = \left\{\lambda_j^-, \lambda_k^+ \,\middle|\, \max\left\{\lambda_j^-, \lambda_k^+\right\} \le \lambda_N^-\right\},$$

$$\sigma_2 = \left\{\lambda_j^-, \lambda_k^+ \,\middle|\, \lambda_j^- \le \lambda_N^- \le \min\left\{\lambda_j^-, \lambda_k^\pm\right\}\right\}.$$

Step 2. We make decomposition of E_k

$$E_{k1} = \operatorname{span}\left\{U_j^-, U_k^+ \,\middle|\, \lambda_j^-, \lambda_k^+ \in \sigma_1\right\},$$

$$E_{k2} = \operatorname{span}\left\{U_j^+, U_k^\pm \,\middle|\, \lambda_j^-, \lambda_k^\pm \in \sigma_2\right\}.$$

In order to make these two subspaces orthogonal and satisfy spectral inequality

$\Lambda_1 = \lambda_N^-$, $\Lambda_2 = \lambda_{N+1}^-$, we further decompose

$E_{k2} = E_c \oplus E_R$, with

$$E_c = \operatorname{span}\left\{U_j^+ \,\middle|\, \lambda_j^- \le \lambda_N^- < \lambda_j^+\right\},$$

$$E_R = \operatorname{span}\left\{U_R^\pm \,\middle|\, \lambda_N^- < \lambda_R^\pm\right\},$$

$$E_N = E_c \oplus E_{k1}.$$

Next, we stipulate an eigenvalue scale product of E_k such that E_{k1} and E_{k2} are orthogonal, therefore we need to introduce two functions:

Let $\Phi: E_N \to R$, $\Psi: E_R \to R$.

$$\Phi(U,V) = (\beta-\alpha)\left(D^{2m+k}u, D^{2m+k}\overline{y}\right) + \left(\overline{D^k z}, D^k u\right) + \left(\overline{D^k v}, D^k y\right) + \left(\overline{D^k z}, D^k v\right), \tag{17}$$

$$\Psi(U,V) = \beta\left(D^{2m+K}u, D^{2m+k}\overline{y}\right) + \left(D^k\overline{z}, D^{2m+k}u\right) + \left(\overline{D^k v}, D^{2m+k}y\right) + \left(\overline{D^k z}, D^k v\right), \tag{18}$$

where $U=(u,v)$, $V=(y,z)$, $\overline{y},\overline{z}$ respectively are the conjugation of y,z.

Let $U=(u,v)\in E_N$, then

$$\Phi(U,U) \ge (\beta-\alpha)\left\|D^{2m+k}u\right\|^2 - \left\|D^k v\right\|^2 - \left\|D^k u\right\|^2 + \left\|D^k v\right\|^2 \ge \left[(\beta-\alpha)\mu_1^2 - 1\right]\left\|D^k u\right\|^2$$

Since $\overline{\mu}_1 > 2\sqrt{\frac{\alpha}{\beta}}, \beta \ge \frac{4\alpha}{3}$ holds, we can know $\Phi(U,U)\ge 0$. Therefore, for all $U\in E_N$, analogously, for all $U\in E_R$, we can get

$$\Psi(U,U) \ge \beta\left\|D^{2m+k}u\right\|^2 - \left\|D^k v\right\|^2 - \left\|D^{2m+k}u\right\|^2 + \left\|D^k v\right\|^2 \ge (\beta-1)\left\|D^{2m+k}u\right\|^2.$$

That is $U=(u,v)\in E_R$, $\Psi(U,U)\ge 0$.

From above, we know that for all $U\in E_R$, then $\psi(U,U)\ge 0$ holds. So, we define a scale product with Φ and ψ in E_k.

$$\langle\langle U,V\rangle\rangle_{E_k} = \Phi(P_N U, P_N V) + \psi(P_R U, P_R V), \tag{19}$$

where P_N, P_R are respectively the projection: $E_k \to E_N$, $E_k \to E_R$.

In the inner product of E_k in (19), E_{k1} and E_{k2} are orthogonal. In fact, we need prove that E_{k1} and E_c are orthogonal.

For $U_j^+ \in E_C$, $U_j^- \in E_N$,

$$\langle\langle U_j^+, U_j^-\rangle\rangle_E = \Phi(U_j^+, U_j^-) = (\beta-\alpha)\left\|(-\Delta)^m u_j\right\|^2 - (\lambda_j^- + \lambda_j^+)\left\|u_j\right\|^2 + \lambda_j^-\lambda_j^+\left\|u_j\right\|^2.$$

According to $\left\|D^k u_j\right\|^2 = 1$, $\left\|D^{2m+k}u_j\right\|^2 = \mu_j$, $\left\|D^{-2m-k}u_j\right\|^2 = \frac{1}{\mu_j}$ and $\lambda_j^- + \lambda_j^+ = \beta\overline{\mu}_j^2$, $\lambda_j^-\lambda_j^+ = \alpha\overline{\mu}_j^2$,

then

$$\langle\langle U_j^+, U_j^-\rangle\rangle_E = \Phi(U_j^+, U_j^-) = 0.$$

Step 3. Next, we estimate the Lipschitz constant l_F of F,

$$F(U) = \left(0, h(x) - M\left(\left\|D^m u\right\|_p^p\right)u_t - N\left(\left\|D^m u\right\|_p^p\right)(-\Delta)^m u\right)^{\mathrm{T}}$$

$g,h: H^{2m+k} \to H$ are globally Lipschitz continuous with maximum.

Lipschitz constant l for arbitrarily $U=(u,v)\in E_k$, we have
Let $P_1:E_k\to E_{k1},P_2:E_k\to E_{k2}$ are the orthogonal projection.

$$U=(u,v)\in E_k,U_1=(u_1,v_1)=P_1U,U_2=(u_2,v_2)=P_2U.$$

$$P_1u=u_1,P_1v=v_1,P_2u=u_2,P_2v=v_2.$$

$$\|U\|_{E_k}^2=\tilde{\Phi}(P_1U,P_1U)+\tilde{\Psi}(P_2U,P_2U)=((\beta-\alpha)u_1^2-1+\beta-1)\|Du\|^2$$
$$\geq\left(4\alpha+\beta-2-\frac{4\alpha}{\beta^2}\right)\|D^ku\|^2\geq(\alpha+\beta-2)\|D^ku\|^2\geq 0.$$

Set $U=(u,v),V=(\hat{u},\hat{v})\in E_k$,

$$\|F(U)-F(V)\|\leq\Big\|M\left(\|D^mu\|_p^p\right)u_t-M\left(\|D^m\hat{u}\|_p^p\right)\hat{u}_t$$
$$+N\left(\|D^mu\|_p^p\right)(-\Delta)^m u-N\left(\|D^m\hat{u}\|_p^p\right)(-\Delta)^m\hat{u}\Big\|$$
$$\leq\frac{l_k}{\sqrt{\alpha+\beta-2}}\|U-V\|_{E_k},$$

Therefore

$$l_F\leq\frac{l_k}{\sqrt{\alpha+\beta-2}}.$$

Step 4. Now, we need prove the spectral gap condition holds.
From the above mentioned $\Lambda_1=\lambda_N^-$ and $\Lambda_2=\lambda_{N+1}^-$, we can get

$$\Lambda_2-\Lambda_1=\lambda_{N+1}^--\lambda_N^-$$
$$=\frac{\beta}{2}\left(\bar{\mu}_{N+1}^2-\bar{\mu}_N^2\right)+\frac{1}{2}\left(\sqrt{\beta^2\bar{\mu}_N^4-4\alpha\bar{\mu}_N^2}-\sqrt{\beta^2\bar{\mu}_{N+1}^4-4\alpha\bar{\mu}_{N+1}^2}\right)$$
$$=\frac{\beta}{2}\left(\bar{\mu}_{N+1}^2-\bar{\mu}_N^2\right)-\frac{\beta}{2}\cdot\frac{\left(\bar{\mu}_{N+1}^4-\bar{\mu}_N^4\right)-\frac{4\alpha}{\beta^2}\left(\bar{\mu}_{N+1}^2-\bar{\mu}_N^2\right)}{\sqrt{\bar{\mu}_{N+1}^4-\frac{4\alpha\bar{\mu}_{N+1}^2}{\beta^2}}+\sqrt{\bar{\mu}_N^4-\frac{4\alpha\bar{\mu}_N^2}{\beta^2}}}$$
$$=\left(\bar{\mu}_{N+1}^2-\bar{\mu}_N^2\right)\left(\frac{\beta}{2}-\frac{\beta}{2}\cdot\frac{\bar{\mu}_{N+1}^2+\bar{\mu}_N^2}{\sqrt{J(N+1)}+\sqrt{J(N)}}+\frac{2\alpha}{\beta}\cdot\frac{1}{\sqrt{J(N+1)}+\sqrt{J(N)}}\right)$$
$$>\left(\bar{\mu}_{N+1}^2-\bar{\mu}_N^2\right)\left(\frac{\beta}{2}-\frac{\beta}{2}\cdot\frac{\bar{\mu}_{N+1}^2+\bar{\mu}_N^2}{\sqrt{\bar{\mu}_{N+1}^4}+\sqrt{\bar{\mu}_N^4}}+\frac{2\alpha}{\beta}\cdot\frac{1}{\sqrt{J(N+1)}+\sqrt{J(N)}}\right)$$
$$=\frac{2\alpha\left(\bar{\mu}_{N+1}^2-\bar{\mu}_N^2\right)}{\beta\left(\sqrt{J(N+1)}+\sqrt{J(N)}\right)}.$$

we obtain

$$\Lambda_2-\Lambda_1>\left(\bar{\mu}_{N+1}^2-\bar{\mu}_N^2\right)\frac{2\alpha}{\beta\left(\sqrt{J(N+1)}+\sqrt{J(N)}\right)}\geq\frac{4l}{\sqrt{\alpha+\beta-2}}\geq 4l_F,$$

When $\bar{\mu}_j>2\sqrt{\frac{\alpha}{\beta}}$, the conclusion (1) is proved.

(2) $\frac{2\sqrt{\alpha}}{\beta} < \bar{\mu}_j < 2\sqrt{\frac{\alpha}{\beta}}$.

$$\Lambda_2 - \Lambda_1 = \lambda_{q+1}^- - \lambda_q^- = \frac{\beta}{2}\left(\bar{\mu}_{q+1}^2 - \bar{\mu}_q^2\right) + \frac{1}{2}\left(\sqrt{\Gamma(q)} - \sqrt{\Gamma(q+1)}\right),$$
$$> \frac{2\alpha\left(\bar{\mu}_{q+1}^2 - \bar{\mu}_q^2\right)}{\beta\left(\sqrt{J(q+1)} + \sqrt{J(q)}\right)} \geq 4l_k \geq 4l_F,$$

Since

$$\Lambda_2 - \Lambda_1 = \lambda_{N+1}^- - \lambda_N^- = \frac{\beta}{2}\left(\bar{\mu}_{N+1}^2 - \bar{\mu}_N^2\right)$$

Similarity the theorem 2, the conclusion (2) is proved. The theorem 3 is proved completely.

Theorem 4 Under the condition of theorem 2 and theorem 3, the initial boundary value problem (1)-(3) admits a family of inertial manifolds w_k in E_k of the form

$$w_k = \text{graph}(\rho_k) = \left\{\zeta + \rho_k(\zeta) : \zeta \in E_{k1}\right\}, k = 1, 2, \cdots, 2m$$

where E_{k1}, E_{k2} are as in theorem 2 and $\rho_k : E_{k1} \to E_{k2}$ is a Lipschitz continuous function.

Proof. It is proved directly according to the theorem 1.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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