

Non Perturbative Approach to the Time Evolution of a Simple Excited Coupled Quantum System in a Cavity

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Abstract

We consider a system modeled by a harmonic oscillator of frequency $\bar{\omega}$, coupled to the scalar potential inside a reflecting sphere of radius R . We use *dressed* states introduced originally in [1] and recently employed in [2] to present a non-perturbative unified description of the decay process of the system, in free space and in the case of the system being confined in a finite cavity. In the situation that we start from the initial condition that the system is in the first excited state, we give exact formulas to describe its time evolution for a cavity of arbitrary size. In the particular case of a very large cavity (free space), we recover the behaviour expected from perturbation theory in the limit of the small coupling constant. In the case of a very small cavity, our results are in good agreement with experimental observations.

Keywords

Radiation Process in Cavities, Dressed Coordinates

1. Introduction

Exact solutions to problems in physical sciences are known to researchers since a long time ago to be a rather rare situation. It is a common feature of different branches of physical sciences, such as celestial mechanics, field theory and statistical physics, that the exact solution of coupled equations describing that the physics of interacting bodies is a very hard problem. In statistical physics and constructive field theory, general theorems can be derived using cluster-like expansions and other related methods [3]. In some cases, these methods allow the rigorous construction of field theoretical models (see for instance [4] and other references therein), but they are not of great usefulness in calculations of a pre-

dictive character. Actually, apart from computer calculations in lattice field theory, the only available method to solve this kind of problem, except for a few special cases, is perturbation theory. The method, originally introduced to deal with orbital problems in celestial mechanics, has been extremely successful since the discovery of Neptune, treated as a “perturbation” to the orbit of Uranus by Le Verrier in 1846. In modern physics, a prototype situation, for instance in abelian gauge theories, is a system composed of a charged particle described by a matter field interacting with a neutral (radiation) field through some (in general non-linear) coupling characterized by some parameter g , usually named the coupling constant or the charge of the particle. The perturbative solution to this situation is obtained by means of the introduction of bare, non-interacting matter and radiation fields, which are associated with bare quanta, the interaction being introduced order by order in powers of the coupling constant in the perturbative expansion for the observables. Perturbative Quantum Field Theory gives remarkably accurate results in Quantum Electrodynamics and in Weak interactions. In high energy physics, asymptotic freedom allows applying Quantum Chromodynamics in its perturbative form and very important results have been obtained in this way in the last decades. However, as a matter of principle, due to the non-vanishing of the coupling constant, the idea of a bare particle associated with a bare matter field is actually an artifact of perturbation theory and is physically meaningless. A charged physical particle is always coupled to the gauge field, in other words, it is always “dressed” by a cloud of quanta of the gauge field (photons, in the case of Electrodynamics). In perturbation theory, this dressing of the charged particle is done by the renormalization procedure, order by order in powers of the renormalized coupling constant. In practice, we are limited to relatively small orders, calculations becoming very involved at higher orders. In spite of its wide applicability, there are situations where the use of perturbation theory is not possible, as in the low energy domain of Quantum Chromodynamics, where confinement of quarks and gluons takes place. In this particular situation, no analytical approach in the context of Quantum Field Theory is available up to the present moment. There are other situations in the domain of Atomic Physics, Cavity Electrodynamics and Quantum Optics, where perturbation methods are of little usefulness, for instance, in resonant effects associated with the coupling of atoms with strong radiofrequency fields [5]. The theoretical understanding of these effects on perturbative grounds requires the calculation of very high-order terms in perturbation series, which makes the standard Feynman diagrams technique practically unreliable in those cases [5]. The trials of treating non-perturbative such kinds of systems have led to the idea of *dressed* atom, introduced in [6] [7]. Since then this concept has been used to investigate several situations involving the interaction of atoms and electromagnetic fields [8] [9] [10]. The core of theoretical difficulties is the non-linear character of the problem, which implies, as noted above, very hard mathematical problems to be dealt with. A way to circumvent these mathematical difficulties is to assume that under certain conditions the coupled atom-electromagnetic field

system may be approximated by the system composed of a harmonic oscillator coupled *linearly* to the field through some effective coupling constant g . This is the case in the context of the general QED linear response theory, where the electric dipole interaction gives the leading contribution to the radiation process [11] [12] [13]. Linear approximations of this type have been applied in cavity QED, in particular to the theoretical investigation of higher-generation Schrodinger cat-states in high-Q cavities, as has been done for instance in [14]. Also, approaches of this type have been used in condensed matter physics for studies of Brownian motion and in quantum optics to study decoherence, by assuming a linear coupling between a cavity harmonic mode and a thermal bath of oscillators at zero temperature [15] [16]. In fact, such an idea of confining the system in a finite volume is present since a long time ago in the literature as a kind of regularization mechanism to avoid divergences present in the continuous mathematical language used in free space computations [17]. This device is introduced to make the eigenvalue problem be mathematically well defined, but the limit of taking afterwards an infinite volume is not trivial [1] [17]. In particular, as is stressed in the appendix of [17], the states in a continuous formulation cannot simply be considered as the infinite volume limit of confined eigenstates. More recently very similar ideas have been employed in radiation theory [18].

In former works [2] [19] [20], we have presented separate studies for a quantum small system in a cavity; in [19], the conditions for the existence of such a static system were studied and in the second one [20], we have investigated the time evolution of the system starting from a given excited state. In the present manuscript, we intend to present a unified treatment of both situations, in a small cavity and a very large sphere (free space).

We consider a system composed of an atom (approximated by a harmonic oscillator) coupled linearly to the scalar potential, the whole system being confined inside a reflecting sphere of radius R . From a mathematical point of view the structure of our problem is basically the same as that of the diagonalization of the Lee-Friedrichs Hamiltonian, which describes a two-level atom interacting with a scalar field, that has been studied, for instance in [21]. This means that the eigenfrequencies spectra of both coupled systems are basically the same. This similarity reflects a formal relation between our system and previous results in the literature for the interaction of a two-level atom with a field in the rotating wave approximation (RWA). We give a non-perturbative treatment to the field-atom system by introducing some *dressed* coordinates that allow dividing the coupled system into two parts, the *dressed* atom and the *dressed* field, which makes it unnecessary to work directly with the concepts of the bare atom, bare field and interaction between them. For instance, to describe the radiation process, having as initial condition that only the mechanical oscillator (the atom), q_0 be excited, the usual procedure is to consider the interaction term in the Hamiltonian written in terms of q_0 and the field modes q_i as a perturbation, which induces transitions among the eigenstates of the free Hamiltonian. In this way, it is possible to treat approximately the problem having as initial

condition that only the bare mechanical oscillator (the atom) be excited. But as is well known this initial condition is physically not consistent in reason of the divergence of the bare oscillator frequency, due to the interaction with the field. The traditional way to circumvent this difficulty is by the renormalization procedure, introducing perturbative order by order corrections to the oscillator frequency.

In this paper we adopt an alternative procedure, as in [1] and [2], we do not make explicit use of the concepts of interacting bare oscillator and field, described by the coordinates q_0 and $\{q_i\}$. We introduce *dressed* coordinates q'_0 and $\{q'_i\}$ for, respectively the *dressed* atom and the *dressed* field modes. In terms of these new coordinates a non-perturbative approach to the radiation process and the distribution of energy inside the cavity is possible. For completeness in the next section, we review the formalism and results of [2].

2. The Dressed State Approach

We consider an atom approximated by a harmonic oscillator $q_0(t)$ of frequency ω_0 (we will introduce below a *renormalized* frequency $\bar{\omega}$ which is physically meaningful) coupled linearly to the scalar potential ϕ , the whole system being confined in a sphere of radius R centered at the oscillator position. The equations of motion are,

$$\ddot{q}_0(t) + \omega_0^2 q_0(t) = 2\pi\sqrt{gc} \int_0^R d^3r \phi(\mathbf{r}, t) \delta(\mathbf{r}) \tag{1}$$

$$\frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} - \nabla^2 \phi(\mathbf{r}, t) = 2\pi\sqrt{gc} q_0(t) \delta(\mathbf{r}). \tag{2}$$

Using a basis of spherically symmetric Bessel functions defined in the domain $0 < |\mathbf{r}| < R$, the equations above can be written as a set of equations coupling the atom to the harmonic field modes,

$$\ddot{q}_0(t) + \omega_0^2 q_0(t) = \eta \sum_{i=1}^{\infty} \omega_i q_i(t) \tag{3}$$

$$\ddot{q}_i(t) + \omega_i^2 q_i(t) = \eta \omega_i q_0(t). \tag{4}$$

In the above equations, g is a coupling constant (with dimension of frequency), $\eta = \sqrt{2g\Delta\omega}$ and $\Delta\omega = \pi c/R$ is the interval between two neighbouring field frequencies, $\omega_{i+1} - \omega_i = \Delta\omega = \pi c/R$ and q_i stands for the harmonic modes of the field. Using the coordinate transformation $q_\mu = t_\mu^r Q_r$ in the equations of motion and explicitly making use of the normalization condition $\sum_{\mu=0}^N (t_\mu^r)^2 = 1$, we get,

$$t_k^r = \frac{\eta \omega_k}{\omega_k^2 - \Omega_r^2} t_0^r, \tag{5}$$

$$t_0^r = \left[1 + \sum_{k=1}^N \frac{\eta^2 \omega_k^2}{(\omega_k^2 - \Omega_r^2)^2} \right]^{-\frac{1}{2}} \tag{6}$$

with the subsidiary condition giving the eigenfrequencies spectrum,

$$\omega_0^2 - N\eta^2 - \Omega^2 = \eta^2 \sum_{k=1}^N \frac{\Omega^2}{\omega_k^2 - \Omega^2} \quad (7)$$

We consider for a moment as in [1], the problem of a harmonic oscillator q_0 coupled to N other oscillators. In the limit $N \rightarrow \infty$ we recover our original situation of the coupling oscillator-field after redefinition of divergent quantities, in a manner analogous as renormalization is done in field theories. In terms of the cutoff N the coupled Equations (3) and (4) are simply rewritten taking the upper limit N instead of ∞ for the summation in the right hand side of Equation (3). Our system of $N+1$ coupled oscillators $q_0, \{q_i\}$ is described by the Hamiltonian,

$$H = \frac{1}{2} \left[p_0^2 + \omega_0^2 q_0^2 + \sum_{k=1}^N (p_k^2 + \omega_k^2 q_k^2 - 2\eta\omega_k q_0 q_k) \right], \quad (8)$$

which can be turned to principal axis by means of a point transformation, $q_\mu = t_\mu^r Q_r$, $p_\mu = t_\mu^r P_r$, performed by an orthonormal matrix $T = (t_\mu^r)$, $\mu = (0, k)$, $k = 1, 2, \dots, N$, $r = 0, \dots, N$. The subscript 0 and k refer respectively to the atom and the harmonic modes of the field and r refers to the normal modes. The transformed Hamiltonian in principal axis reads,

$$H = \frac{1}{2} \sum_{r=0}^N (P_r^2 + \Omega_r^2 Q_r^2), \quad (9)$$

where the Ω_r 's are the normal frequencies corresponding to the possible collective oscillation modes of the coupled system. Using the coordinate transformation $q_\mu = t_\mu^r Q_r$ in the equations of motion and explicitly making use of the normalization condition $\sum_{\mu=0}^N (t_\mu^r)^2 = 1$, we get,

$$t_k^r = \frac{\eta\omega_k}{\omega_k^2 - \Omega_r^2} t_0^r, \quad (10)$$

$$t_0^r = \left[1 + \sum_{k=1}^N \frac{\eta^2 \omega_k^2}{(\omega_k^2 - \Omega_r^2)^2} \right]^{-\frac{1}{2}} \quad (11)$$

with the subsidiary condition giving the eigenfrequencies spectrum,

$$\omega_0^2 - N\eta^2 - \Omega^2 = \eta^2 \sum_{k=1}^N \frac{\Omega^2}{\omega_k^2 - \Omega^2} \quad (12)$$

There are $N+1$ solutions Ω_r to Equation (12), corresponding to the $N+1$ normal collective oscillation modes. It is easily seen that if $\omega_0^2 > N\eta^2$, Equation (12) yields only positive solutions for Ω^2 , what means that the system oscillates harmonically in all its modes. Indeed, in this case the left hand term of Equation (12) is positive for negative values of Ω^2 . Conversely the right hand term is negative for those values of Ω^2 . Thus there is no negative solution of that equation when $\omega_0^2 > N\eta^2$. On the other hand, it can be shown that if $\omega_0^2 < N\eta^2$, Equation (12) has a single negative solution Ω_-^2 . This means that there is an oscillation mode whose amplitude varies exponentially and that does not allow stationary configurations. We will not care about this last situation. Nevertheless

it is interesting to note that in a different context, it is precisely this negative squared frequency solution (runaway solution) that is related to the existence of a bound state in the Lee-Friedrichs model. This solution is considered in [22] in the framework of a model to describe qualitatively the existence of bound states in particle physics.

Thus we take $\omega_0^2 > N\eta^2$ and define the *renormalized* oscillator frequency $\bar{\omega}$, $\bar{\omega} = \sqrt{\omega_0^2 - N\eta^2}$. In the limit $N \rightarrow \infty$ the meaning of the frequency renormalization becomes clear. It is exactly the analogous of a mass renormalization in field theory, the infinite ω_0 being chosen in such a way as to make the renormalized frequency $\bar{\omega}$ finite and equal to the observed oscillator frequency. In terms of the renormalized frequency Equation (12) can be written, after some manipulations, in the form [1],

$$\cot\left(\frac{R\Omega}{c}\right) = \frac{\Omega}{\pi g} + \frac{c}{R\Omega}\left(1 - \frac{\bar{\omega}^2 R}{\pi g c}\right). \tag{13}$$

The solutions of Equation (13) with respect to Ω give the spectrum of eigenfrequencies Ω_r corresponding to the collective normal modes. The transformation matrix elements turning the oscillator-field system to principal axis is obtained taking the limit $N \rightarrow \infty$, after some rather long but straightforward manipulations in [1]. They read,

$$t_0^r = \frac{\Omega_r}{\sqrt{\frac{R}{2\pi g c}(\Omega_r^2 - \bar{\omega}^2)^2 + \frac{1}{2}(3\Omega_r^2 - \bar{\omega}^2) + \frac{\pi g R}{2c}\Omega_r^2}}; \quad t_k^r = \frac{\eta\omega_k}{\omega_k^2 - \Omega_r^2}t_0^r. \tag{14}$$

3. Dressed States and the Distribution of Energy inside the Cavity

We define below some coordinates q'_0 , q'_i associated to the *dressed* atom and the *dressed* field. These coordinates will reveal themselves to be appropriate to give an appealing non-perturbative description of the atom-field system. We start from the eigenstates of our system, represented by the normalized eigenfunctions,

$$\phi_{n_0 n_1 n_2 \dots}(Q, t) = \prod_s \left[\left(2^{-n_s} n_s! \right)^{-\frac{1}{2}} H_{n_s} \left(\sqrt{\frac{\Omega_s}{\hbar}} Q_s \right) \right] \Gamma_0 e^{-i\sum_s n_s \Omega_s t}, \tag{15}$$

where H_{n_s} stands for the n_s -th Hermite polynomial and Γ_0 is the normalized vacuum eigenfunction. Let us introduce *dressed* coordinates q'_0 and $\{q'_i\}$ for, respectively the *dressed* atom and the *dressed* field, defined by [1],

$$\sqrt{\frac{\bar{\omega}_\mu}{\hbar}} q'_\mu = \sum_r t_\mu^r \sqrt{\frac{\Omega_r}{\hbar}} Q_r, \tag{16}$$

valid for arbitrary R and where $\bar{\omega}_\mu = \{\bar{\omega}, \omega_i\}$. In terms of the bare coordinates the dressed coordinates are expressed as,

$$q'_\mu = \sum_\nu \alpha_{\mu\nu} q_\nu; \quad \alpha_{\mu\nu} = \frac{1}{\sqrt{\bar{\omega}_\mu}} \sum_r t_\mu^r t_\nu^r \sqrt{\Omega_r}. \tag{17}$$

Let us define for a fixed instant the complete orthonormal set of functions [1],

$$\psi_{\kappa_0 \kappa_1 \dots}(q') = \prod_{\mu} \left[\left(2^{-\kappa_{\mu}} \kappa_{\mu}! \right)^{\frac{1}{2}} H_{\kappa_{\mu}} \left(\sqrt{\frac{\bar{\omega}_{\mu}}{\hbar}} q'_{\mu} \right) \right] \Gamma_0, \tag{18}$$

where $q'_{\mu} = q'_0, q'_i$, $\bar{\omega}_{\mu} = \{\bar{\omega}, \omega_i\}$. Note that the ground state Γ_0 in the above equation is the same as in Equation (15). The invariance of the ground state is due to our definition of *dressed* coordinates given by Equation (16). Each function $\psi_{\kappa_0 \kappa_1 \dots}(q')$ describes a state in which the *dressed* oscillator q'_{μ} is in its κ_{μ} -th excited state. Using Equation (16) the functions (18) can be expressed in terms of the normal coordinates Q_r . But since (15) is a complete set of orthonormal functions, the functions (18) may be written as linear combinations of the eigenfunctions of the coupled system (we take $t = 0$ for the moment),

$$\psi_{\kappa_0 \kappa_1 \dots}(q') = \sum_{n_0 n_1 \dots} T_{\kappa_0 \kappa_1 \dots}^{n_0 n_1 \dots}(0) \phi_{n_0 n_1 n_2 \dots}(Q, 0), \tag{19}$$

where the coefficients are given by,

$$T_{\kappa_0 \kappa_1 \dots}^{n_0 n_1 \dots}(0) = \int dQ \psi_{\kappa_0 \kappa_1 \dots} \phi_{n_0 n_1 n_2 \dots}, \tag{20}$$

the integral extending over the whole Q -space.

We consider the particular configuration ψ in which only one dressed oscillator q'_{μ} is in its N -th excited state, all other being in the ground state,

$$\psi_{0 \dots N(\mu) 0 \dots}(q') = \left(2^{-N} N! \right)^{\frac{1}{2}} H_N \left(\sqrt{\frac{\bar{\omega}_{\mu}}{\hbar}} q'_{\mu} \right) \Gamma_0. \tag{21}$$

The coefficients (20) can be calculated in this case using Equations (20), (18) and (16) with the help of the theorem [23],

$$\begin{aligned} & \frac{1}{N!} \left[\sum_r (t_{\mu}^r)^2 \right]^{\frac{N}{2}} H_N \left(\frac{\sum_r t_{\mu}^r \sqrt{\frac{\Omega_r}{\hbar}} Q_r}{\sqrt{\sum_r (t_{\mu}^r)^2}} \right) \\ &= \sum_{m_0 + m_1 + \dots = N} \frac{(t_{\mu}^0)^{m_0} (t_{\mu}^1)^{m_1} \dots}{m_0! m_1! \dots} H_{m_0} \left(\sqrt{\frac{\Omega_0}{\hbar}} Q_0 \right) H_{m_1} \left(\sqrt{\frac{\Omega_1}{\hbar}} Q_1 \right) \dots \end{aligned} \tag{22}$$

We get,

$$T_{0 \dots N(\mu) 0 \dots}^{n_0 n_1 \dots} = \left(\frac{N!}{n_0! n_1! \dots} \right)^{\frac{1}{2}} (t_{\mu}^0)^{n_0} (t_{\mu}^1)^{n_1} \dots, \tag{23}$$

where the subscripts $\mu = 0, i$ refer respectively to the dressed atom and the harmonic modes of the field and the quantum numbers satisfy the constraint $n_0 + n_1 + \dots = N$.

In the following we focus our attention on the behaviour of the system with the initial condition that only one dressed oscillator q'_{μ} (the dressed atom or one of the modes of the dressed field) be in the N -th excited state. We will study in detail the particular case $N = 1$, which will be enough to have a clear understanding of our approach. Let us call Γ_1^{μ} the configuration in which only the

dressed oscillator q'_μ is in the first excited level. We have from Equations (21), (19), (23) and (16) the following expression for the time evolution of the first-level excited dressed oscillator q'_μ ,

$$\Gamma_1^\mu(t) = \sum_\nu f^{\mu\nu}(t)\Gamma_1^\nu(0), \tag{24}$$

where the coefficients $f^{\mu\nu}(t)$ are given by

$$f^{\mu\nu}(t) = \sum_s t_{\mu^s\nu^s}^s e^{-i\Omega_s t}. \tag{25}$$

From Equation (24) we see that the initially excited dressed oscillator naturally distributes its energy among itself and all other dressed oscillators as time goes on, with probability amplitudes given by Equation (25). If the dressed oscillator q'_0 (the atom) is in its first excited state at $t = 0$, its decay rate may be evaluated from the time evolution equation,

$$\Gamma_1^0(t) = \sum_\nu f^{0\nu}(t)\Gamma_1^\nu(0). \tag{26}$$

In Equation (26) the coefficients $f^{0\nu}(t)$ have a simple interpretation: $f^{00}(t)$ and $f^{0i}(t)$ are respectively the probability amplitudes that at time t the dressed atom still be excited or have radiated a photon of frequency ω_i . We see that this formalism allows a quite natural description of the radiation process as a simple exact time evolution of the system. We consider in the following the time evolution of the excited atom, in the cases of a very large and a very small cavity.

1) A very large cavity

In the case of a very large cavity our method generalizes what can be obtained from perturbation theory. The probability that the atom be still excited at time t can be obtained in continuous language from the amplitude given by Equation (25),

$$f^{00}(t) = \int_0^\infty \frac{2g\Omega^2 e^{-i\Omega t} d\Omega}{(\Omega^2 - \omega^2)^2 + \pi^2 g^2 \Omega^2}. \tag{27}$$

For large t ($t \gg \frac{1}{\omega}$), but for in principle arbitrary coupling g , we obtain for the probability of finding the atom still excited at time t , the result [1],

$$|f^{00}(t)|^2 = e^{-\pi g t} \left(1 + \frac{\pi^2 g^2}{4\bar{\omega}^2} \right) + e^{-\pi g t} \frac{8\pi g}{\pi \bar{\omega}^4 t^3} \left(\sin \tilde{\omega} t + \frac{\pi g}{2\langle \bar{\omega} \rangle} \cos \tilde{\omega} t \right) + \frac{16\pi^2 g^2}{\pi^2 \bar{\omega}^8 t^6}, \tag{28}$$

where $\tilde{\omega} = \sqrt{\bar{\omega}^2 - \frac{\pi^2 g^2}{4}}$. In the above expression the approximation $t \gg \frac{1}{\omega}$ plays a role only in the two last terms, due to the difficulties to evaluate exactly the integral in Equation (27) along the imaginary axis using Cauchy's theorem. The first term comes from the residue at $\Omega = \tilde{\omega} + i\frac{\pi g}{2}$ and would be the same if we have done an exact calculation. If we consider in Equation (28) $g \ll \bar{\omega}$, which corresponds in electromagnetic theory to the fact that the fine structure

constant α is small compared to unity (for explicit calculations we take below $g/\bar{\omega} = \alpha$), we obtain the well known perturbative exponential decay law.

2) A very small cavity

Let us now consider the atom placed at the center of a very small cavity, *i.e.* that satisfies the condition that its radius be much smaller than the coherence length, $R \ll c/g$. To obtain the eigenfrequencies spectrum, we remark that from a numerical analysis of Equation (7) it can be seen that in the case of a small cavity radius R , its solutions are near the frequency values corresponding to the asymptots of the curve $\cot\left(\frac{R\Omega}{c}\right)$, which correspond to the field modes $\omega_i = i\pi c/R$. The smallest solution departs more from the first asymptot than the other larger solutions depart from their respective nearest asymptot. As we take larger and larger solutions, they are nearer and nearer to the values corresponding to the asymptots. For instance, for a cavity radius R of the order of 10^{-2}m and $\bar{\omega} \sim 10^{10}/\text{s}$, only the lowest eigenfrequency Ω_0 is significantly different from the field frequency corresponding to the first asymptot, all the other eigenfrequencies $\Omega_k, k = 1, 2, \dots$ being very close to the field modes $k\pi c/R$. For higher values of $\bar{\omega}$ (and lower values of R) the differences between the eigenfrequencies and the field modes frequencies are still smaller.

Thus to solve Equation (13) for the larger eigenfrequencies we expand the function $\cot\left(\frac{R\Omega}{c}\right)$ around the values corresponding to the asymptots. We write,

$$\Omega_k = \frac{\pi c}{R}(k + \varepsilon_k), \quad k = 1, 2, \dots \quad (29)$$

with $0 < \varepsilon_k < 1$, satisfying the equation,

$$\cot(\pi\varepsilon_k) = \frac{c}{gR}(k + \varepsilon_k) + \frac{1}{k + \varepsilon_k} \left(1 - \frac{\bar{\omega}^2 R}{\pi g c}\right). \quad (30)$$

But since for a small cavity every ε_k is much smaller than 1, Equation (30) may be linearized in ε_k , giving,

$$\varepsilon_k = \frac{\pi g c R k}{\pi^2 c^2 k^2 - \bar{\omega}^2 R^2}. \quad (31)$$

Equations (29) and (31) give approximate solutions to the eigenfrequencies $\Omega_k, k = 1, 2, \dots$.

To solve Equation (13) with respect to the lowest eigenfrequency Ω_0 , let us assume that it satisfies the condition $\Omega_0 R/c \ll 1$ (we will see below that this condition is compatible with the condition of a small cavity as defined above). Inserting the condition $\Omega_0 R/c \ll 1$ in Equation (13) and keeping up to quadratic terms in Ω the solution for the lowest eigenfrequency Ω_0 can be written,

$$\Omega_0 = \frac{\bar{\omega}}{\sqrt{1 + \frac{\pi g R}{c}}}. \quad (32)$$

Consistency between Equation (32) and the condition $\Omega_0 R/c \ll 1$ gives a condition on the cavity radius,

$$R \ll \frac{c}{g} \frac{\pi}{2} \left(\frac{g}{\bar{\omega}} \right)^2 \left(1 + \sqrt{1 + \frac{4}{\pi^2} \left(\frac{\bar{\omega}}{g} \right)^2} \right). \tag{33}$$

Let us define the coupling constant g to be such that $g = \bar{\omega}\alpha$, where α is the fine structure constant, $\alpha = 1/137$. Then the factor multiplying c/g in the above equation is ~ 0.07 and the condition $R \ll c/g$ is replaced by a more restrictive assumption $R \ll 0.07(c/g)$. For a typical infrared frequency, for instance $\bar{\omega} \sim 2.0 \times 10^{11}/s$, our calculations are valid for a radius $R \ll 10^{-3}$ m.

From Equation (14) and using the above expressions for the eigenfrequencies in a small cavity, we obtain the matrix elements,

$$\left(t_0^0 \right)^2 = \frac{1}{\sqrt{1 + \pi R \frac{g}{c} + \frac{\pi g \bar{\omega}^4 R^3}{2c^3} + 1}} \approx 1 - \frac{\pi g R}{c}; \quad \left(t_0^k \right)^2 \approx \frac{2gR}{\pi c k^2} \tag{34}$$

To obtain the above equations we have neglected the corrective term ε_k , from the expressions for the eigenfrequencies Ω_k . Nevertheless, corrections in ε_k should be included in the expressions for the matrix elements t_k^k , in order to avoid spurious singularities due to our approximation.

Let us consider the situation where the dressed atom is initially in its first excited level. Then from Equation (25) we obtain the probability that it will still be excited after a elapsed time t ,

$$\begin{aligned} |f^{00}(t)|^2 &= \left(t_0^0 \right)^4 + 2 \sum_{k=1}^{\infty} \left(t_0^0 \right)^2 \left(t_0^k \right)^2 \cos(\Omega_k - \Omega_0)t \\ &+ \sum_{k,l=1}^{\infty} \left(t_0^k \right)^2 \left(t_0^l \right)^2 \cos(\Omega_k - \Omega_l)t. \end{aligned} \tag{35}$$

Using Equations (34) in Equation (35), we obtain

$$\begin{aligned} |f^{00}(t)|^2 &= 1 - \pi\delta + \pi^2\delta^2 + 4 \left(\frac{\delta}{\pi} - \delta^2 \right) \sum_{k=1}^{\infty} \frac{1}{k^2} \cos(\Omega_k - \Omega_0)t \\ &+ \frac{4}{\pi^2} \delta^2 \sum_{k,l=1}^{\infty} \frac{1}{k^2 l^2} \cos(\Omega_k - \Omega_l)t, \end{aligned} \tag{36}$$

where we have introduced the adimensional parameter $\delta = Rg/c \ll 1$, corresponding to a small cavity and we remember that the eigenfrequencies are given by Equations (29) and (31). As time goes on, the probability that the atom be excited attains periodically a minimum value which has a lower bound given by,

$$\text{Min} \left(|f^{00}(t)|^2 \right) = 1 - \frac{5\pi}{3} \delta + \frac{14\pi^2}{9} \delta^2. \tag{37}$$

For a frequency $\bar{\omega}$ of the order $\bar{\omega} \sim 4.00 \times 10^{14}/s$ (in the red visible), which corresponds to $\delta \sim 0.005$ and $2R \sim 1.0 \times 10^{-6}$ m, we see from Equation (37) that the probability that the atom be at any time excited will never fall below a value ~ 0.97 , or a decay probability that is never higher that a value ~ 0.03 . In other words, atoms having such emission frequency, placed in a such a small

cavity in the first excited level, will be stable in the excited state to the order of 97%. It is interesting to compare this result with experimental observations in [24] [25], where stability is found for atoms emitting in the visible range placed between two parallel mirrors a distance $L = 1.1 \times 10^{-6}$ m apart from one another. For lower frequencies the size of the cavity ensuring quasi-stability of the same order as above, for the excited atom may be considerably larger. For instance, for $\bar{\omega}$ in a typical microwave value, $\bar{\omega} \sim 2.00 \times 10^{10}/\text{s}$ and taking also $\delta \sim 0.005$, the probability that the atom remain in the first excited level at any time will be larger than a value of the order of 97%, for a cavity radius $R \sim 1.0 \times 10^{-2}$ m. The probability that the atom remain excited as time goes on oscillates with time between a maximum and a minimum values and never departs significantly from the situation of stability of the atom in the excited state. Indeed for an emission frequency $\bar{\omega} \sim 4.00 \times 10^{14}/\text{s}$ (in the red visible) considered above and $R \sim 1.0 \times 10^{-6}$ m, the period of oscillation between the minimum and maximum values of the probability that the atom be excited, is

$$T \sim \frac{1}{12} \times 10^{-14} \text{ s}, \text{ while for } \bar{\omega} \sim 2.00 \times 10^{10}/\text{s}, \text{ and } R \sim 1.4 \times 10^{-2} \text{ m}, \text{ the period is}$$

$$T \sim \frac{1.4}{6} \times 10^{-10} \text{ s}.$$

4. Concluding Remarks

We have used in this paper a formalism that allows a unified approach to the radiation process by an atom, in rather different situations, as the atom is confined in a very small cavity or in free space. The behaviour of atoms confined in small cavities is completely different from the behaviour of an atom in free space or in a large cavity. In the first case, the emission process is very sensitive to the presence of boundaries, a fact that has been pointed out since a long time ago in the literature [26] [27] [28]. Our *dressed* states approach gives a unified description for the dressing of the atom by the field modes and the emission process in a cavity of arbitrary size, which includes microcavities and very large cavities (free space emission). We recover here with our formalism the experimental observation that excited states of atoms in sufficiently small cavities are stable. We are able to give formulas for the probability of an atom to remain excited for an infinitely long time, provided it is placed in a cavity of appropriate size. For an emission frequency in the visible red, the size of such cavity is in good agreement with experimental observations [24] [25]. Also, our approach gives results in good agreement with previous theoretical results for the emission in free space, generalizing the well-known exponential decay law. Moreover, the detailed behaviours which we obtain with our formalism are very different in the two situations: The atom in a very large cavity has a probability decay rate weekly oscillating and monotonically varying with time (the probability that the atom be excited decreases almost exponentially with increasing time). In the case of an excited atom placed in the center of a very small cavity, the probability that it remains excited as time goes on oscillates very rapidly with time and never de-

parts significantly from the situation of stability of the atom in the excited state.

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Conflicts of Interest

The author has no conflicts to disclose.

Data Availability

The data that supports the findings of this study are available within the article and its quoted references.

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