Journal of Applied Mathematics and Physics, 2022, 10, 1918-1935
https://www.scirp.org/journal/jamp
ISSN Online: 2327-4379
ISSN Print: 2327-4352

Local Discontinuous Galerkin Method for the Time-Fractional KdV Equation with the Caputo-Fabrizio Fractional Derivative

Huanhuan Wang, Xiaoyan Xu, Junmei Dou, Ting Zhang, Leilei Wei*

College of Science, Henan University of Technology, Zhengzhou, China
Email: *leileiwei@haut.edu.cn

How to cite this paper: Wang, H.H., Xu, X.Y., Dou, J.M., Zhang, T. and Wei, L.L. (2022) Local Discontinuous Galerkin Method for the Time-Fractional KdV Equation with the Caputo-Fabrizio Fractional Derivative. *Journal of Applied Mathematics and Physics*, **10**, 1918-1935.
https://doi.org/10.4236/jamp.2022.106132

Received: May 29, 2022
Accepted: June 21, 2022
Published: June 24, 2022

Abstract

This paper studies the time-fractional Korteweg-de Vries (KdV) equations with Caputo-Fabrizio fractional derivatives. The scheme is presented by using a finite difference method in temporal variable and a local discontinuous Galerkin method (LDG) in space. Stability and convergence are demonstrated by a specific choice of numerical fluxes. Finally, the efficiency and accuracy of the scheme are verified by numerical experiments.

Keywords

Caputo-Fabrizio Fractional Derivative, Local Discontinuous Galerkin Method, Stability, Error Analysis

1. Introduction

Fractional differential equations have become increasingly important due to their deep scientific and engineering background to correctly model challenging phenomena such as long-range time memory effects, mechanical systems, control systems, etc. [1] [2]. In recent years, variable-order fractional calculus has been found in some physical processes such as algebraic structure and noise reduction. Variable-order fractional calculus is a natural choice to provide an effective mathematical framework for describing complex problems and has many advantages in describing the memory properties of systems [3]-[9].

Fractional partial differential equations can describe abnormal physical phenomena more accurately than integer partial differential equations, which have attracted more and more attention. However, it is difficult to obtain analytical solutions to fractional partial differential equations when the fractional derivatives are known. Therefore, we need to consider efficient numerical methods such as the finite element method [10] [11] [12] [13], discontinuous Galerkin method [14] [15] [16] [17] [18], spectral method [19] [20], and finite difference method [21] [22] [23], finite volume method [24] [25]. Wei [26] studied the exact numerical scheme of a class of variable-order fractional diffusion equations, using the fractional derivatives of Caputo-Fabrizio and the theoretical analysis by the local discontinuous Galerkin method. Du [27] proposed different difference schemes for multi-dimensional variable-order time fractional subdiffusion equations and found a special point approximation for the variable-order time Caputo derivative. It is proved that the resulting difference scheme is uniquely solvable. Li *et al.* [28] carried out a numerical study on three typical Caputo-type partial differential equations using the finite difference method/local discontinuous Galerkin finite element method.

The KdV equation was first proposed by Boussinesq in 1877, and it is a typical dispersion nonlinear partial differential equation. The nonlinear KdV equation was derived by Korteweg and de Vries in 1895 [29], and it describes the propagation of waves in various nonlinear dispersive media. Since then, the KdV equation has been widely used in various physical phenomena and engineering modeling, such as nonlinear wave interactions [30], interfacial electrohydrodynamics [31], plasma physics, geology, etc. Numerous numerical methods have been proposed to solve this equation, such as finite difference schemes [32] [33], pseudospectral methods [34], thermal equilibrium integration methods [35], and discontinuous Galerkin methods [36] [37]. For sufficiently smooth solutions, the following literature does some numerical work on the fractional time KdV equation. Wei *et al.* [38] proposed the LDG finite element method of the KdV-Burgers-Kuramoto equation, using variable-order Riemann-Liouville fractional derivatives, and proved the unconditional stability and convergence of the scheme. Zhang [39] constructed an efficient numerical scheme for solving linearized fractional KdV equations on unbounded spaces. The non-local fractional derivatives are obtained by exponentiating the convolution kernel and approximately evaluating the initial boundary value problem.

In this paper, the Korteweg-de Vries equation (KdV) with Caputo-Fabrizio fractional derivatives is constructed

$$\begin{cases} D_t^{1-\alpha(t)}u(x,t)+\delta u_{xxx}(x,t)+\lambda g(u)_x = f(x,t), & (x,t)\in(a,b)\times(0,T], \\ u(x,0)=u_0(x), & x\in[a,b], \end{cases} \tag{1.1}$$

where the fractional derivative orders $\alpha(t)\in(0,1)$, f, $g(u)$ and u_0 are smooth functions. δ and λ are positive constants. In addition, the solutions in this paper are periodic or compactly supported.

The Caputo-Fabrizio fractional derivative in (1.1) is defined as

$$D_t^{1-\alpha(t)}u(x,t)=\frac{1}{\alpha(t)}\int_0^t \frac{\partial u(x,s)}{\partial s}\exp\left[\frac{\alpha(t)-1}{\alpha(t)}(t-s)\right]\mathrm{d}s, \quad s\in(0,t]. \tag{1.2}$$

There are many definitions of fractional derivatives, of which the most widely used are Riemann-Liouville fractional derivatives and Caputo fractional derivatives. The Caputo-Fabrizio fractional derivative used in this paper was proposed by Caputo and Fabrizio [40] in 2015. Compared with the Caputo fractional derivative model, the Caputo-Fabrizio fractional derivative model can describe different scales and configurations of matter. The Caputo-Fabrizio fractional-order derivatives have been widely used by researchers such as Ann Al Sawoor *et al.* [41] who studied the asymptotic stability of linear and interval linear fractional-order neutral delay differential systems described by the Caputo-Fabrizio fractional derivatives.

The key to the KdV equation LDG method is to rewrite the equation into a first-order equation system by introducing two auxiliary variables. The LDG method was first introduced by Cockburn and Shu to solve the convection-diffusion equation [42]. One of its advantages is that its solution and spatial derivatives have optimal $(k+1)$ order convergence on the L^2 norm. Yan and Shu [43] developed a numerical method for LDG for general KdV-type equations involving third-order derivatives. Wei and He [44] used the LDG finite element method to solve the time-fractional KdV equation problem, discretized using finite differences in time and local discontinuous Galerkin methods in space. In [45], the authors established the L^2 conservative LDG numerical scheme and compared it with the dissipative LDG scheme of the KdV type equation to show the dissipative induced phase error. In [46], Baccouch investigated the nonlinear KdV partial differential equation LDG numerical scheme. The results show that the LDG solution is superconvergent to a special Gauss-Radau projection of the exact solution.

The structure of this paper is as follows. In Section 2, some basic notation and mathematical foundations are introduced. Section 3 mainly introduces discrete methods and constructs the LDG scheme. Section 4 presents the stability and convergence results of the scheme. In Section 5, we give numerical experiments to illustrate the accuracy of our proposed format. Finally, we summarize and discuss our results in Section 6.

2. Preliminaries

2.1. Notations and Projection

Divide the interval $\Omega=[a,b]$ as $J: a=x_{\frac{1}{2}}<x_{\frac{3}{2}}<\cdots<x_{N+\frac{1}{2}}=b$. For $j=1,\cdots,N$, define $I_j=\left[x_{j-\frac{1}{2}},x_{j+\frac{1}{2}}\right]$, and $\Delta x_j=x_{j+\frac{1}{2}}-x_{j-\frac{1}{2}}$, $h=\max\limits_{1\le j\le N}\Delta x_j$.

We divide the interval $[0,T]$ evenly into time steps $\Delta t=\frac{T}{M}=t_n-t_{n-1}$, $t_n=n\Delta t, n=0,1,\cdots,M$ are mesh points.

The left and right limits of u at $x_{j+\frac{1}{2}}$ are denoted by $u^+_{j+\frac{1}{2}}$ and $u^-_{j+\frac{1}{2}}$, respectively. Where $u^+_{j+\frac{1}{2}}$ is in the right cell I_{j+1}, and $u^-_{j+\frac{1}{2}}$ is in the left cell I_j. Define

$$[u_h]_{j+\frac{1}{2}}=u^+_{j+\frac{1}{2}}-u^-_{j+\frac{1}{2}}.$$

The associated discontinuous Galerkin space V_h^k is defined as follows

$$V_h^k = \left\{ v : v \in P^k\left(I_j\right),\, x \in I_j,\ j = 1, 2, \cdots, N \right\}.$$

In proving the error estimate, we will use two projections on the one-dimensional interval $[a,b]$.

Denoted as $\mathcal{P}$,

$$\int_{I_j} \left(\mathcal{P}\omega(x) - \omega(x)\right) v(x) = 0,\ \forall v \in P^k\left(I_j\right), \tag{2.1}$$

and $\mathcal{P}^{\pm}$,

$$\int_{I_j} \left(\mathcal{P}^{+}\omega(x) - \omega(x)\right) v(x) = 0,\ \forall v \in P^{k-1}\left(I_j\right),\ \mathcal{P}^{+}\omega\left(x^{+}_{j-\frac{1}{2}}\right) = \omega\left(x_{j-\frac{1}{2}}\right), \tag{2.2}$$

and

$$\int_{I_j} \left(\mathcal{P}^{-}\omega(x) - \omega(x)\right) v(x) = 0,\ \forall v \in P^{k-1}\left(I_j\right),\ \mathcal{P}^{-}\omega\left(x^{-}_{j+\frac{1}{2}}\right) = \omega\left(x_{j+\frac{1}{2}}\right). \tag{2.3}$$

For the above projection $\mathcal{P}, \mathcal{P}^{\pm}$, it can be obtained from the standard approximation theory [47] [48] [49] [50],

$$\|\rho\| + h\|\rho\|_{\infty} + h^{\frac{1}{2}}\|\rho\|_{\tau_h} \le Ch^{k+1}, \tag{2.4}$$

where $\rho = \mathcal{P}\omega - \omega$ or $\rho = \mathcal{P}^{\pm}\omega - \omega$. We want to denote all element boundary points in one-dimensional space by τ_h. Furthermore, we have the following definition [51]

$$\|\rho\|_{\tau_h} = \left(\frac{1}{2}\sum_{1\le i\le N}\left(\left(\rho^{+}_{i+\frac{1}{2}}\right)^2 + \left(\rho^{-}_{i+\frac{1}{2}}\right)^2\right)\right)^{\frac{1}{2}}.$$

In this paper, C is a positive constant, which may take different values in different positions. $(\cdot,\cdot)_D$ represents the scalar inner product over $L^2(D)$, $\|\cdot\|_D$ represents the correlation norm. When $D = \Omega$, we drop it.

2.2. Numerical Flux

In this paper, we will use the flux $\hat{g}\left(\psi^{-}, \psi^{+}\right)$, which is related to the discontinuous Galerkin spatial discretization. $\hat{g}\left(\psi^{-}, \psi^{+}\right)$ is a monotonic numerical flux that depends on the left and right limits of the function ψ at point $x_{j+\frac{1}{2}}$, satisfying the following conditions:

1) It is local Lipschitz continuous, so $\hat{g}\left(\psi^{-}, \psi^{+}\right)$ is bounded when the function $\psi^{\pm}$ is in a bounded region;

2) It is consistent with the flux $g(\psi)$, *i.e.*, $\hat{g}\left(\psi^{-}, \psi^{+}\right) = g(\psi)$;

3) It is a function with monotonic properties, the first parameter is a non-decreasing function, and the second parameter is a non-increasing function.

3. The LDG Schemes

This section introduces the LDG method for the time-fractional KdV Equation (1.1).

First, we discretize the fractional derivative in the time direction

$$
\begin{aligned}
D_t^{1-\alpha(t)}u(x,t_n) &= \frac{1}{\alpha(t_n)}\int_0^{t_n}\frac{\partial u(x,s)}{\partial s}\exp\left[\frac{\alpha(t_n)-1}{\alpha(t_n)}(t_n-s)\right]\mathrm{ds} \\
&= \frac{1}{(1-\alpha(t_n))\Delta t}\sum_{k=1}^{n}\Bigg(\left(u(x,t_k)-u(x,t_{k-1})\right)\Bigg(\exp\left[\frac{(\alpha(t_n)-1)\Delta t}{\alpha(t_n)}(n-k)\right] \\
&\quad -\exp\left[\frac{(\alpha(t_n)-1)\Delta t}{\alpha(t_n)}(n-k+1)\right]\Bigg)\Bigg) \\
&\quad +\frac{1}{\alpha(t_n)}\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}\left(s-t_{k-\frac{1}{2}}\right)\frac{\partial^2 u(x,c_k)}{\partial s^2}\exp\left[\frac{\alpha(t_n)-1}{\alpha(t_n)}(t_n-s)\right]\mathrm{ds} \\
&= \frac{1}{(1-\alpha(t_n))\Delta t}\sum_{k=1}^{n}\left(u(x,t_k)-u(x,t_{k-1})\right)W_k^n+R^n(x),
\end{aligned}
\tag{3.1}
$$

where $R^n(x)$ is the truncation error in the time direction,

$$
R^n(x)=\frac{1}{\alpha(t_n)}\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}\left(s-t_{k-\frac{1}{2}}\right)\frac{\partial^2 u(x,c_k)}{\partial s^2}\exp\left[\frac{\alpha(t_n)-1}{\alpha(t_n)}(t_n-s)\right]\mathrm{ds},
$$

$$
W_k^n=\exp\left[\frac{(\alpha(t_n)-1)\Delta t}{\alpha(t_n)}(n-k)\right]-\exp\left[\frac{(\alpha(t_n)-1)\Delta t}{\alpha(t_n)}(n-k+1)\right],
$$

and $c_k\in(t_{k-1},t_k)$.

By further calculation we can get

$$
\begin{aligned}
D_t^{1-\alpha(t)}u(x,t_n) &= \frac{1}{(1-\alpha(t_n))\Delta t}\Bigg(W_n^n u(x,t_n)-W_1^n u(x,t_0) \\
&\quad +\sum_{k=1}^{n-1}\left(W_k^n-W_{k+1}^n\right)u(x,t_k)\Bigg)+R^n(x).
\end{aligned}
\tag{3.2}
$$

Lemma 3.1. [52] [53] *When* $0<\alpha(t)<1$, *the truncation error* $R^n(x)$ *satisfies the following estimation*

$$
\left\|R^n(x)\right\|\le C(\Delta t)^2. \tag{3.3}
$$

W_k^n has the following properties

$$
\begin{aligned}
&0<W_1^n<W_2^n<\cdots<W_n^n, \\
&\left\|W_{k+1}^n-W_k^n\right\|<C, \quad \forall k\le n-1,
\end{aligned}
\tag{3.4}
$$

and

$$
\begin{aligned}
\sum_{n=2}^{J}W_1^n &= \sum_{n=2}^{J}\left(\exp\left[\frac{(\alpha(t_n)-1)\Delta t}{\alpha(t_n)}(n-1)\right]-\exp\left[\frac{(\alpha(t_n)-1)\Delta t}{\alpha(t_n)}(n)\right]\right) \\
&= \exp\left[\frac{(\alpha(t_n)-1)\Delta t}{\alpha(t_n)}\right]-\exp\left[\frac{(\alpha(t_n)-1)\Delta t}{\alpha(t_n)}J\right]<C.
\end{aligned}
\tag{3.5}
$$

Rewrite the Equation (1.1) as a first-order system of equations,

$$
\begin{cases}
D_t^{1-\alpha(t)}u(x,t)+\delta q_x(x,t)+\lambda g(u)_x=f(x,t), \\
p=u_x, \\
q=p_x.
\end{cases}
\tag{3.6}
$$

$u_h^n,\ p_h^n,\ q_h^n \in V_h^k$ represent approximate solutions of $u(\cdot,t_n),\ p(\cdot,t_n),\ q(\cdot,t_n)$, respectively. $f^n = f(\cdot,t_n)$. Find $u_h^n,\ p_h^n,\ q_h^n \in V_h^k$ such that for the test function $v,\ \phi,\ \varphi \in V_h^k$, we have

$$
\begin{aligned}
&W_n^n \int_\Omega u_h^n v\mathrm{d}x - \beta\delta\left(\int_\Omega q_h^n v_x \mathrm{d}x - \sum_{j=1}^{N}\left(\left(\widehat{q_h^n v^-}\right)_{j+\frac{1}{2}} - \left(\widehat{q_h^n v^+}\right)_{j-\frac{1}{2}}\right)\right)\\
&-\beta\lambda\left(\int_\Omega g\left(u_h^n\right) v_x \mathrm{d}x - \sum_{j=1}^{N}\left(\left(\widehat{g\left(u_h^n\right)v^-}\right)_{j+\frac{1}{2}} - \left(\widehat{g\left(u_h^n\right)v^+}\right)_{j-\frac{1}{2}}\right)\right)\\
&=\sum_{k=1}^{n-1}\left(W_{k+1}^n - W_k^n\right)\int_\Omega u_h^k v\mathrm{d}x + W_1^n\int_\Omega u_h^0 v\mathrm{d}x + \beta\int_\Omega f^n v\mathrm{d}x, \qquad (3.7)\\
&\int_\Omega q_h^n \phi\mathrm{d}x + \int_\Omega p_h^n \phi_x \mathrm{d}x - \sum_{j=1}^{N}\left(\left(\widehat{p_h^n \phi^-}\right)_{j+\frac{1}{2}} - \left(\widehat{p_h^n \phi^+}\right)_{j-\frac{1}{2}}\right) = 0,\\
&\int_\Omega p_h^n \varphi\mathrm{d}x + \int_\Omega u_h^n \varphi_x \mathrm{d}x - \sum_{j=1}^{N}\left(\left(\widehat{u_h^n \varphi^-}\right)_{j+\frac{1}{2}} - \left(\widehat{u_h^n \varphi^+}\right)_{j-\frac{1}{2}}\right) = 0,
\end{aligned}
$$

where $\beta = \left(1-\alpha\left(t_n\right)\right)\Delta t$.

The hat function in the element boundary term resulting from the integral by parts in (3.7) is the numerical flux. To ensure stability, we can take the following alternating numerical fluxes

$$
\widehat{u_h^n} = \left(u_h^n\right)^-, \quad \widehat{p_h^n} = \left(p_h^n\right)^+, \quad \widehat{q_h^n} = \left(q_h^n\right)^+. \qquad (3.8)
$$

The choice of flux (3.8) is not unique, only $\widehat{u_h^n}$ and $\widehat{p_h^n}, \widehat{q_h^n}$ can take the opposite sides [54].

The fluxes $\hat{g}\left(\left(u_h^n\right)^-, \left(u_h^n\right)^+\right)$ are monotonic fluxes as described in Section 2.2. Examples of monotonic fluxes suitable for local discontinuous Galerkin methods can be found [55] [56]. For example, we can use the Lax-Friedrich flux, which consists of

$$
\hat{g}^{LF}\left(\psi^-,\psi^+\right) = \frac{1}{2}\left(g\left(\psi^-\right) + g\left(\psi^+\right) - \lambda_0\left(\psi^+ - \psi^-\right)\right), \quad \lambda_0 = \max_{\psi}\left|g'(\psi)\right|.
$$

In the next section, we discuss the stability and convergence of the numerical Scheme (3.7).

4. Stability and Convergence

To simplify the notation, we consider the case of $f = 0$ in the numerical analysis.

Theorem 4.1. *Under periodic or tightly supported boundary conditions, the fully discrete LDG scheme* (*3.7*) *is unconditionally stable, and the numerical solution* u_h^n *satisfies*

$$\left\|u_h^n\right\| \le \left\|u_h^0\right\|, \quad n = 1,2,\cdots,M. \tag{4.1}$$

Proof. Add the three equations in the scheme (3.7),

$$\begin{aligned}
&W_n^n \int_\Omega u_h^n v \mathrm{d}x + \int_\Omega q_h^n \phi \mathrm{d}x + \int_\Omega p_h^n \varphi \mathrm{d}x - \beta\delta\left(\int_\Omega q_h^n v_x \mathrm{d}x - \sum_{j=1}^{N}\left(\left(\widehat{q_h^n} v^-\right)_{j+\frac{1}{2}}\right.\right.\\
&\left.\left.-\left(\widehat{q_h^n} v^+\right)_{j-\frac{1}{2}}\right)\right) - \beta\lambda\left(\int_\Omega g\left(u_h^n\right) v_x \mathrm{d}x - \sum_{j=1}^{N}\left(\left(\widehat{g\left(u_h^n\right)} v^-\right)_{j+\frac{1}{2}} - \left(\widehat{g\left(u_h^n\right)} v^+\right)_{j-\frac{1}{2}}\right)\right)\\
&+\int_\Omega p_h^n \phi_x \mathrm{d}x - \sum_{j=1}^{N}\left(\left(\widehat{p_h^n}\phi^-\right)_{j+\frac{1}{2}} - \left(\widehat{p_h^n}\phi^+\right)_{j-\frac{1}{2}}\right)\\
&+\int_\Omega u_h^n \varphi_x \mathrm{d}x - \sum_{j=1}^{N}\left(\left(\widehat{u_h^n}\varphi^-\right)_{j+\frac{1}{2}} - \left(\widehat{u_h^n}\varphi^+\right)_{j-\frac{1}{2}}\right)\\
&= \sum_{k=1}^{n-1}\left(W_{k+1}^n - W_k^n\right)\int_\Omega u_h^k v \mathrm{d}x + W_1^n \int_\Omega u_h^0 v \mathrm{d}x.
\end{aligned} \tag{4.2}$$

Substitute the test function $v = u_h^n, \phi = \beta\delta p_h^n, \varphi = -\beta\delta q_h^n$ into the scheme (4.2), using flux (3.8) and the Cauchy-Schwarz inequality, we get

$$\begin{aligned}
&W_n^n \left\|u_h^n\right\|^2 + \beta\delta\left\|q_h^n\right\|\left\|p_h^n\right\| - \beta\delta\left\|p_h^n\right\|\left\|q_h^n\right\|\\
&-\beta\delta\left(\int_\Omega q_h^n \left(u_h^n\right)_x \mathrm{d}x - \sum_{j=1}^{N}\left(\left(\left(q_h^n\right)^+\left(u_h^n\right)^-\right)_{j+\frac{1}{2}} - \left(\left(q_h^n\right)^+\left(u_h^n\right)^+\right)_{j-\frac{1}{2}}\right)\right)\\
&-\beta\lambda\left(\int_\Omega g\left(u_h^n\right)\left(u_h^n\right)_x \mathrm{d}x - \sum_{j=1}^{N}\left(\left(\widehat{g\left(u_h^n\right)}\left(u_h^n\right)^-\right)_{j+\frac{1}{2}} - \left(\widehat{g\left(u_h^n\right)}\left(u_h^n\right)^+\right)_{j-\frac{1}{2}}\right)\right)\\
&+\beta\delta\left(\int_\Omega p_h^n \left(p_h^n\right)_x \mathrm{d}x - \sum_{j=1}^{N}\left(\left(\left(p_h^n\right)^+\left(p_h^n\right)^-\right)_{j+\frac{1}{2}} - \left(\left(p_h^n\right)^+\left(p_h^n\right)^+\right)_{j-\frac{1}{2}}\right)\right)\\
&+\beta\delta\left(\int_\Omega u_h^n \left(q_h^n\right)_x \mathrm{d}x - \sum_{j=1}^{N}\left(\left(\left(u_h^n\right)^-\left(q_h^n\right)^-\right)_{j+\frac{1}{2}} - \left(\left(u_h^n\right)^-\left(q_h^n\right)^+\right)_{j-\frac{1}{2}}\right)\right)\\
&\le \sum_{k=1}^{n-1}\left(W_{k+1}^n - W_k^n\right)\left\|u_h^k\right\|\left\|u_h^n\right\| + W_1^n\left\|u_h^0\right\|\left\|u_h^n\right\|,
\end{aligned}$$

which is

$$\begin{aligned}
&W_n^n\left\|u_h^n\right\|^2 + \beta\lambda\tilde{G}\left(u_h^n\right)\\
&+\sum_{j=1}^{N}\beta\delta\left(\Psi\left(u_h^n, p_h^n, q_h^n\right)_{j+\frac{1}{2}} - \Psi\left(u_h^n, p_h^n, q_h^n\right)_{j-\frac{1}{2}}\Theta\left(u_h^n, p_h^n, q_h^n\right)_{j-\frac{1}{2}}\right)\\
&\le \sum_{k=1}^{n-1}\left(W_{k+1}^n - W_k^n\right)\left\|u_h^k\right\|\left\|u_h^n\right\| + W_1^n\left\|u_h^0\right\|\left\|u_h^n\right\|,
\end{aligned} \tag{4.3}$$

here

$$\tilde{G}\left(u_h^n\right) = -\left(\int_\Omega g\left(u_h^n\right)\left(u_h^n\right)_x \mathrm{d}x - \sum_{j=1}^{N}\left(\left(\widehat{g\left(u_h^n\right)}\left(u_h^n\right)^-\right)_{j+\frac{1}{2}} - \left(\widehat{g\left(u_h^n\right)}\left(u_h^n\right)^+\right)_{j-\frac{1}{2}}\right)\right),$$

$$\Psi\left(u_h^n, p_h^n, q_h^n\right) = -\left(u_h^n\right)^-\left(q_h^n\right)^- + \left(q_h^n\right)^+\left(u_h^n\right)^- + \frac{1}{2}\left(\left(p_h^n\right)^-\right)^2 - \left(p_h^n\right)^+\left(p_h^n\right)^- + \left(u_h^n\right)^-\left(q_h^n\right)^-,$$

$$\begin{aligned}\Theta\left(u_h^n, p_h^n, q_h^n\right) = &-\left(u_h^n\right)^-\left(q_h^n\right)^- + \left(q_h^n\right)^+\left(u_h^n\right)^- + \frac{1}{2}\left(\left(p_h^n\right)^-\right)^2 - \left(p_h^n\right)^+\left(p_h^n\right)^- \\ &+ \left(u_h^n\right)^-\left(q_h^n\right)^- + \left(u_h^n\right)^+\left(q_h^n\right)^+ - \left(q_h^n\right)^+\left(u_h^n\right)^+ - \frac{1}{2}\left(\left(p_h^n\right)^+\right)^2 \\ &+ \left(p_h^n\right)^+\left(p_h^n\right)^+ - \left(u_h^n\right)^-\left(q_h^n\right)^+.\end{aligned}$$

The above scheme can be calculated by

$$\begin{aligned}&\sum_{j=1}^{N} \beta\delta\left(\Psi\left(u_h^n, p_h^n, q_h^n\right)_{j+\frac{1}{2}} - \Psi\left(u_h^n, p_h^n, q_h^n\right)_{j-\frac{1}{2}}\right) = 0, \\ &\Theta\left(u_h^n, p_h^n, q_h^n\right)_{j-\frac{1}{2}} = \frac{1}{2}\left[p_h^n\right]_{j-\frac{1}{2}}^2.\end{aligned} \tag{4.4}$$

For nonlinear terms, let $G(u) = \int_0^u g(u)\mathrm{d}u$. Using the mean value theorem and the monotonicity of liquidity yields $\tilde{G}\left(u_h^n\right) = \left(G'(\xi) - \hat{g}\right)\left[u_h^n\right] \ge 0$, where ξ is a value between u_h^{n-} and u_h^{n+}.

Substituting (4.4) into (4.3), we get

$$W_n^n \left\|u_h^n\right\|^2 + \sum_{j=1}^{N} \frac{\beta\delta}{2}\left[p_h^n\right]_{j-\frac{1}{2}}^2 \le \sum_{k=1}^{n-1}\left(W_{k+1}^n - W_k^n\right)\left\|u_h^k\right\|\left\|u_h^n\right\| + W_1^n\left\|u_h^0\right\|\left\|u_h^n\right\|. \tag{4.5}$$

Prove Theorem 4.1 by mathematical induction. Let $n=1$ in the scheme (4.5), we have

$$W_1^1\left\|u_h^1\right\|^2 + \sum_{j=1}^{N}\frac{\beta\delta}{2}\left[p_h^1\right]_{j-\frac{1}{2}}^2 \le W_1^1\left\|u_h^0\right\|\left\|u_h^1\right\|,$$

since

$$\int_\Omega u_h^0 u_h^1 \mathrm{d}x \le \frac{1}{2}\left\|u_h^0\right\|^2 + \frac{1}{2}\left\|u_h^1\right\|^2,$$

we can get the following result

$$W_1^1\left\|u_h^1\right\|^2 \le W_1^1\left(\frac{1}{2}\left\|u_h^0\right\|^2 + \frac{1}{2}\left\|u_h^1\right\|^2\right),$$

which means

$$\left\|u_h^1\right\| \le \left\|u_h^0\right\|.$$

Suppose the following inequalities hold

$$\left\|u_h^m\right\| \le \left\|u_h^0\right\|, \quad m = 1,2,3,\cdots,n-1.$$

Next prove $\left\|u_h^n\right\| \le \left\|u_h^0\right\|$.

From (4.5) we get

$$\begin{aligned}W_n^n\left\|u_h^n\right\| &\le \sum_{k=1}^{n-1}\left(W_{k+1}^n - W_k^n\right)\left\|u_h^k\right\| + W_1^n\left\|u_h^0\right\| \\ &\le \left(\sum_{k=1}^{n-1}\left(W_{k+1}^n - W_k^n\right) + W_1^n\right)\left\|u_h^0\right\| \\ &= W_n^n\left\|u_h^0\right\|.\end{aligned} \tag{4.6}$$

Therefore, we have

$$\left\|u_h^n\right\| \le \left\|u_h^0\right\|.$$

In summary, Theorem 4.1 is proved.

Next, we will state the error estimates of the equation $g(u)=u$ in the linear case, and use (3.8) as the flux choice. We have the following theorem.

Theorem 4.2. *Let* $u(x,t_n)$ *is the exact solution of the problem* (*1.1*), $u(t)\in H^{m+1}(D)$ *is* $0\le m\le k+1$ *is smooth enough. Let* u_h^n *be the numerical solution of the fully discrete LDG scheme* (*3.7*), *then there are the following error estimates*

$$\left\|u(x,t_n)-u_h^n\right\| \le C\left(\Delta t+(\Delta t)^{-1}h^{k+1}+(\Delta t)^{-\frac{1}{2}}h^{k+\frac{1}{2}}\right), \tag{4.7}$$

C is a positive constant that depends on u,T.

Proof. Denote

$$\begin{aligned}
&e_u^n=u(x,t_n)-u_h^n=\zeta_u^n-\eta_u^n,\quad \zeta_u^n=\mathcal{P}^-e_u^n,\quad \eta_u^n=\mathcal{P}^-u(x,t_n)-u(x,t_n),\\
&e_p^n=p(x,t_n)-p_h^n=\zeta_p^n-\eta_p^n,\quad \zeta_p^n=\mathcal{P}e_p^n,\quad \eta_p^n=\mathcal{P}p(x,t_n)-p(x,t_n),\\
&e_q^n=q(x,t_n)-q_h^n=\zeta_q^n-\eta_q^n,\quad \zeta_q^n=\mathcal{P}e_q^n,\quad \eta_q^n=\mathcal{P}q(x,t_n)-q(x,t_n).
\end{aligned} \tag{4.8}$$

The above $\eta_u^n,\eta_p^n,\eta_q^n$ can be estimated by the inequality (2.4). Next, we mainly discuss $\zeta_u^n,\zeta_p^n,\zeta_q^n$.

We can easily verify that the exact solution of the partial differential Equation (1.1) satisfies the following

$$\begin{aligned}
&W_n^n\int_\Omega u(x,t_n)v\mathrm{d}x-\beta\delta\left(\int_\Omega q(x,t_n)v_x\mathrm{d}x-\sum_{j=1}^N\left(\left(q(x,t_n)v^-\right)_{j+\frac{1}{2}}-\left(q(x,t_n)v^+\right)_{j-\frac{1}{2}}\right)\right)\\
&-\beta\lambda\left(\int_\Omega u(x,t_n)v_x\mathrm{d}x-\sum_{j=1}^N\left(\left(u(x,t_n)v^-\right)_{j+\frac{1}{2}}-\left(u(x,t_n)v^+\right)_{j-\frac{1}{2}}\right)\right)\\
&=\sum_{k=1}^{n-1}\left(W_{k+1}^n-W_k^n\right)\int_\Omega u(x,t_k)v\mathrm{d}x+W_1^n\int_\Omega u(x,t_0)v\mathrm{d}x+\beta\int_\Omega f(x,t_n)v\mathrm{d}x-\beta\int_\Omega R^n(x)v\mathrm{d}x, \quad (4.9)\\
&\int_\Omega q(x,t_n)\phi\mathrm{d}x+\int_\Omega p(x,t_n)\phi_x\mathrm{d}x-\sum_{j=1}^N\left(\left(p(x,t_n)\phi^-\right)_{j+\frac{1}{2}}-\left(p(x,t_n)\phi^+\right)_{j-\frac{1}{2}}\right)=0,\\
&\int_\Omega p(x,t_n)\varphi\mathrm{d}x+\int_\Omega u(x,t_n)\varphi_x\mathrm{d}x-\sum_{j=1}^N\left(\left(u(x,t_n)\varphi^-\right)_{j+\frac{1}{2}}-\left(u(x,t_n)\varphi^+\right)_{j-\frac{1}{2}}\right)=0.
\end{aligned}$$

Select the flux (3.8), and subtract the Equations (3.7) and (4.9) to get the error equation

$$\begin{aligned}
&W_n^n\int_\Omega e_u^n v\mathrm{d}x-\beta\delta\left(\int_\Omega e_q^n v_x\mathrm{d}x-\sum_{j=1}^N\left(\left(\left(e_q^n\right)^+v^-\right)_{j+\frac{1}{2}}-\left(\left(e_q^n\right)^+v^+\right)_{j-\frac{1}{2}}\right)\right)\\
&-\beta\lambda\left(\int_\Omega e_h^n v_x\mathrm{d}x-\sum_{j=1}^N\left(\left(\left(e_h^n\right)^-v^-\right)_{j+\frac{1}{2}}-\left(\left(e_h^n\right)^-v^+\right)_{j-\frac{1}{2}}\right)\right)\\
&-\sum_{k=1}^{n-1}\left(W_{k+1}^n-W_k^n\right)\int_\Omega e_u^k v\mathrm{d}x-W_1^n\int_\Omega e_u^0 v\mathrm{d}x+\beta\int_\Omega R^n(x)v\mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
&+\int_{\Omega} e_q^n \phi \mathrm{d}x+\int_{\Omega} e_p^n \phi_x \mathrm{d}x-\sum_{j=1}^{N}\left(\left(\left(e_p^n\right)^{+} \phi^{-}\right)_{j+\frac{1}{2}}-\left(\left(e_p^n\right)^{+} \phi^{+}\right)_{j-\frac{1}{2}}\right) \\
&+\int_{\Omega} e_p^n \varphi \mathrm{d}x+\int_{\Omega} e_u^n \varphi_x \mathrm{d}x-\sum_{j=1}^{N}\left(\left(\left(e_u^n\right)^{-} \varphi^{-}\right)_{j+\frac{1}{2}}-\left(\left(e_u^n\right)^{-} \varphi^{+}\right)_{j-\frac{1}{2}}\right)=0.
\end{aligned} \tag{4.10}
$$

Substitute (4.2) into (4.10) to get

$$
\begin{aligned}
&W_n^n \int_{\Omega} \zeta_u^n v \mathrm{d}x-\beta\delta\left(\int_{\Omega} \zeta_q^n v_x \mathrm{d}x-\sum_{j=1}^{N}\left(\left(\left(\zeta_q^n\right)^{+} v^{-}\right)_{j+\frac{1}{2}}-\left(\left(\zeta_q^n\right)^{+} v^{+}\right)_{j-\frac{1}{2}}\right)\right) \\
&-\beta\lambda\left(\int_{\Omega} \zeta_h^n v_x \mathrm{d}x-\sum_{j=1}^{N}\left(\left(\left(\zeta_h^n\right)^{-} v^{-}\right)_{j+\frac{1}{2}}-\left(\left(\zeta_h^n\right)^{-} v^{+}\right)_{j-\frac{1}{2}}\right)\right) \\
&+\int_{\Omega} \zeta_q^n \phi \mathrm{d}x+\int_{\Omega} \zeta_p^n \phi_x \mathrm{d}x-\sum_{j=1}^{N}\left(\left(\left(\zeta_p^n\right)^{+} \phi^{-}\right)_{j+\frac{1}{2}}-\left(\left(\zeta_p^n\right)^{+} \phi^{+}\right)_{j-\frac{1}{2}}\right) \\
&+\int_{\Omega} \zeta_p^n \varphi \mathrm{d}x+\int_{\Omega} \zeta_u^n \varphi_x \mathrm{d}x-\sum_{j=1}^{N}\left(\left(\left(\zeta_u^n\right)^{-} \varphi^{-}\right)_{j+\frac{1}{2}}-\left(\left(\zeta_u^n\right)^{-} \varphi^{+}\right)_{j-\frac{1}{2}}\right) \\
&=\sum_{k=1}^{n-1}\left(W_{k+1}^n-W_k^n\right) \int_{\Omega} \zeta_u^k v \mathrm{d}x+W_1^n \int_{\Omega} \zeta_u^0 v \mathrm{d}x-\beta \int_{\Omega} R^n(x) v \mathrm{d}x \\
&+W_n^n \int_{\Omega} \eta_u^n v \mathrm{d}x-\beta\delta\left(\int_{\Omega} \eta_q^n v_x \mathrm{d}x-\sum_{j=1}^{N}\left(\left(\left(\eta_q^n\right)^{+} v^{-}\right)_{j+\frac{1}{2}}-\left(\left(\eta_q^n\right)^{+} v^{+}\right)_{j-\frac{1}{2}}\right)\right) \\
&-\beta\lambda\left(\int_{\Omega} \eta_h^n v_x \mathrm{d}x-\sum_{j=1}^{N}\left(\left(\left(\eta_h^n\right)^{-} v^{-}\right)_{j+\frac{1}{2}}-\left(\left(\eta_h^n\right)^{-} v^{+}\right)_{j-\frac{1}{2}}\right)\right) \\
&-\sum_{k=1}^{n-1}\left(W_{k+1}^n-W_k^n\right) \int_{\Omega} \eta_u^k v \mathrm{d}x-W_1^n \int_{\Omega} \eta_u^0 v \mathrm{d}x \\
&+\int_{\Omega} \eta_q^n \phi \mathrm{d}x+\int_{\Omega} \eta_p^n \phi_x \mathrm{d}x-\sum_{j=1}^{N}\left(\left(\left(\eta_p^n\right)^{+} \phi^{-}\right)_{j+\frac{1}{2}}-\left(\left(\eta_p^n\right)^{+} \phi^{+}\right)_{j-\frac{1}{2}}\right) \\
&+\int_{\Omega} \eta_p^n \varphi \mathrm{d}x+\int_{\Omega} \eta_u^n \varphi_x \mathrm{d}x-\sum_{j=1}^{N}\left(\left(\left(\eta_u^n\right)^{-} \varphi^{-}\right)_{j+\frac{1}{2}}-\left(\left(\eta_u^n\right)^{-} \varphi^{+}\right)_{j-\frac{1}{2}}\right).
\end{aligned} \tag{4.11}
$$

Using the projection property (2.1) - (2.3) and the test functions $v=\zeta_u^n$, $\phi=\beta\delta\zeta_p^n$, and $\varphi=-\beta\delta\zeta_q^n$ in (4.11), the following equality holds

$$
\begin{aligned}
&W_n^n \int_{\Omega}\left(\zeta_u^n\right)^2 \mathrm{d}x+\frac{\beta\delta}{2} \sum_{j=1}^{N}\left[\zeta_p^n\right]_{j-\frac{1}{2}}^2+\frac{\beta\lambda}{2} \sum_{j=1}^{N}\left[\zeta_u^n\right]_{j-\frac{1}{2}}^2 \\
&=\sum_{k=1}^{n-1}\left(W_{k+1}^n-W_k^n\right) \int_{\Omega} \zeta_u^k \zeta_u^n \mathrm{d}x+W_1^n \int_{\Omega} \zeta_u^0 \zeta_u^n \mathrm{d}x-\beta \int_{\Omega} R^n(x) \zeta_u^n \mathrm{d}x \\
&+W_n^n \int_{\Omega} \eta_u^n \zeta_u^n \mathrm{d}x+\beta\delta \sum_{j=1}^{N}\left(\left(\left(\eta_q^n\right)^{+}\left(\zeta_u^n\right)^{-}\right)_{j+\frac{1}{2}}-\left(\left(\eta_q^n\right)^{+}\left(\zeta_u^n\right)^{+}\right)_{j-\frac{1}{2}}\right) \\
&-\sum_{k=1}^{n-1}\left(W_{k+1}^n-W_k^n\right) \int_{\Omega} \eta_u^k \zeta_u^n \mathrm{d}x-W_1^n \int_{\Omega} \eta_u^0 \zeta_u^n \mathrm{d}x \\
&-\beta\delta \sum_{j=1}^{N}\left(\left(\left(\eta_p^n\right)^{+}\left(\zeta_p^n\right)^{-}\right)_{j+\frac{1}{2}}-\left(\left(\eta_p^n\right)^{+}\left(\zeta_p^n\right)^{+}\right)_{j-\frac{1}{2}}\right).
\end{aligned} \tag{4.12}
$$

Note that $\zeta_u^0 = 0$, the following equation can be obtained

$$
\begin{aligned}
&W_n^n \left\|\zeta_u^n\right\|^2 + \frac{\beta\delta}{2}\sum_{j=1}^{N}\left[\zeta_p^n\right]_{j-\frac{1}{2}}^2 + \frac{\beta\lambda}{2}\sum_{j=1}^{N}\left[\zeta_u^n\right]_{j-\frac{1}{2}}^2 \\
&\le \sum_{k=1}^{n-1}\left(W_{k+1}^n - W_k^n\right)\int_\Omega \zeta_u^k \zeta_u^n \mathrm{d}x - \beta\int_\Omega R^n(x)\zeta_u^n \mathrm{d}x + \ell \\
&\quad + \beta\delta\sum_{j=1}^{N}\left(\left(\left(\eta_q^n\right)^+\left(\zeta_u^n\right)^-\right)_{j+\frac{1}{2}} - \left(\left(\eta_q^n\right)^+\left(\zeta_u^n\right)^+\right)_{j-\frac{1}{2}}\right) \\
&\quad - \beta\delta\sum_{j=1}^{N}\left(\left(\left(\eta_p^n\right)^+\left(\zeta_p^n\right)^-\right)_{j+\frac{1}{2}} - \left(\left(\eta_p^n\right)^+\left(\zeta_p^n\right)^+\right)_{j-\frac{1}{2}}\right),
\end{aligned}
\tag{4.13}
$$

where

$$
\ell = W_n^n\int_\Omega \eta_u^n\zeta_u^n \mathrm{d}x - \sum_{k=1}^{n-1}\left(W_{k+1}^n - W_k^n\right)\int_\Omega \eta_u^k\zeta_u^n \mathrm{d}x - W_1^n\int_\Omega \eta_u^0\zeta_u^n \mathrm{d}x \le Ch^{k+1}\left\|\zeta_u^n\right\|.
$$

Using the Cauchy-Schwarz inequality, we have

$$
\begin{aligned}
&W_n^n \left\|\zeta_u^n\right\|^2 + \frac{\beta\delta}{2}\sum_{j=1}^{N}\left[\zeta_p^n\right]_{j-\frac{1}{2}}^2 + \frac{\beta\lambda}{2}\sum_{j=1}^{N}\left[\zeta_u^n\right]_{j-\frac{1}{2}}^2 \\
&\le \sum_{k=1}^{n-1}\left(W_{k+1}^n - W_k^n\right)\left\|\zeta_u^k\right\|\left\|\zeta_u^n\right\| + \beta\left\|R^n\right\|\left\|\zeta_u^n\right\| \\
&\quad + \beta\delta\sum_{j=1}^{N}\left(\left(\left(\eta_q^n\right)^+\left[\zeta_u^n\right]\right) - \left(\left(\eta_p^n\right)^+\left[\zeta_p^n\right]\right)\right)_{j-\frac{1}{2}} + Ch^{k+1}\left\|\zeta_u^n\right\|.
\end{aligned}
\tag{4.14}
$$

Use $ab \le \varepsilon a^2 + \frac{1}{4\varepsilon}b^2$,

$$
\begin{aligned}
&W_n^n \left\|\zeta_u^n\right\|^2 + \frac{\beta\delta}{2}\sum_{j=1}^{N}\left[\zeta_p^n\right]_{j-\frac{1}{2}}^2 + \frac{\beta\lambda}{2}\sum_{j=1}^{N}\left[\zeta_u^n\right]_{j-\frac{1}{2}}^2 \\
&\le \frac{W_n^n}{2}\left\|\zeta_u^n\right\|^2 + \frac{1}{2W_n^n}\left(\sum_{k=1}^{n-1}\left(W_{k+1}^n - W_k^n\right)\left\|\zeta_u^k\right\| + \beta\left\|R^n\right\| + Ch^{k+1}\right)^2 \\
&\quad + \frac{\beta\delta}{2}\sum_{j=1}^{N}\left(\left(\left(\eta_q^n\right)^+\right)^2 + \left(\left(\eta_p^n\right)^+\right)^2\right)_{j-\frac{1}{2}} + \frac{\beta\delta}{2}\sum_{j=1}^{N}\left(\left[\zeta_u^n\right]^2 + \left[\zeta_p^n\right]^2\right)_{j-\frac{1}{2}},
\end{aligned}
$$

which is

$$
\begin{aligned}
W_n^n \left\|\zeta_u^n\right\|^2 &\le \frac{W_n^n}{2}\left\|\zeta_u^n\right\|^2 + \frac{1}{2W_n^n}\left(\sum_{k=1}^{n-1}\left(W_{k+1}^n - W_k^n\right)\left\|\zeta_u^k\right\| + \beta\left\|R^n\right\| + Ch^{k+1}\right)^2 \\
&\quad + \frac{\beta\delta}{2}\sum_{j=1}^{N}\left(\left(\left(\eta_q^n\right)^+\right)^2 + \left(\left(\eta_p^n\right)^+\right)^2\right)_{j-\frac{1}{2}}.
\end{aligned}
\tag{4.15}
$$

Multiply both sides of the formula by $2W_n^n$,

$$
\begin{aligned}
\left(W_n^n\left\|\zeta_u^n\right\|\right)^2 &\le \left(\sum_{k=1}^{n-1}\left(W_{k+1}^n - W_k^n\right)\left\|\zeta_u^k\right\| + \beta\left\|R^n\right\| + Ch^{k+1}\right)^2 \\
&\quad + \beta\delta W_n^n\sum_{j=1}^{N}\left(\left(\left(\eta_q^n\right)^+\right)^2 + \left(\left(\eta_p^n\right)^+\right)^2\right)_{j-\frac{1}{2}}.
\end{aligned}
$$

According to $a^2 + b^2 \le (a+b)^2$, we can get

$$W_n^n\left\|\zeta_u^n\right\| \le \sum_{k=1}^{n-1}\left(W_{k+1}^n - W_k^n\right)\left\|\zeta_u^k\right\| + \beta\left\|R^n\right\| + Ch^{k+1} + \sqrt{\beta\delta W_n^n \sum_{j=1}^{N}\left(\left(\left(\eta_q^n\right)^+\right)^2 + \left(\left(\eta_p^n\right)^+\right)^2\right)_{j-\frac{1}{2}}}. \tag{4.16}$$

From Lemma 3.1, it can be known that $\left\|R^n\right\| \le C(\Delta t)^2$, and $\beta = \mathrm{O}(\Delta t) = C(\Delta t)$ are defined for simplicity.

Assume that the following estimates hold

$$\left\|\zeta_u^n\right\| \le Cn\left((\Delta t)^2 + h^{k+1} + (\Delta t)^{\frac{1}{2}} h^{k+\frac{1}{2}}\right). \tag{4.17}$$

We also prove it by mathematical induction. When $n=1$, the following inequality holds,

$$W_1^1\left\|\zeta_u^1\right\| \le C(\Delta t)^2 + Ch^{k+1} + \sqrt{C(\Delta t)}\left(Ch^{k+\frac{1}{2}}\right),$$

therefore

$$\left\|\zeta_u^1\right\| \le C\left((\Delta t)^2 + h^{k+1} + (\Delta t)^{\frac{1}{2}} h^{k+\frac{1}{2}}\right). \tag{4.18}$$

Then assume that

$$\left\|\zeta_u^j\right\|_\Omega \le Cj\left((\Delta t)^2 + h^{k+1} + (\Delta t)^{\frac{1}{2}} h^{k+\frac{1}{2}}\right), \quad j = 1,2,\cdots,n-1. \tag{4.19}$$

According to (4.16) and (4.18), we can get

$$W_n^n\left\|\zeta_u^n\right\| \le W_n^n(n-1)C\left((\Delta t)^2 + h^{k+1} + (\Delta t)^{\frac{1}{2}} h^{k+\frac{1}{2}}\right) + C(\Delta t)^2 + Ch^{k+1} + C(\Delta t)^{\frac{1}{2}} h^{k+\frac{1}{2}},$$

so

$$\left\|\zeta_u^n\right\| \le Cn\left((\Delta t)^2 + h^{k+1} + (\Delta t)^{\frac{1}{2}} h^{k+\frac{1}{2}}\right). \tag{4.20}$$

Since $j\Delta t \le n\Delta t = T$,

$$\begin{aligned}\left\|\zeta_u^n\right\| &\le Cn\Delta t\left(\Delta t + (\Delta t)^{-1} h^{k+1} + (\Delta t)^{-\frac{1}{2}} h^{k+\frac{1}{2}}\right)\\ &= CT\left(\Delta t + (\Delta t)^{-1} h^{k+1} + (\Delta t)^{-\frac{1}{2}} h^{k+\frac{1}{2}}\right)\\ &\le C\left(\Delta t + (\Delta t)^{-1} h^{k+1} + (\Delta t)^{-\frac{1}{2}} h^{k+\frac{1}{2}}\right).\end{aligned} \tag{4.21}$$

Combining the triangle inequality and the projection property (2.4), it can be seen that the Theorem 4.2 holds.

5. Numerical Experiment

In this section, discussing the effectiveness of the above scheme for solving KdV equations, we consider the following numerical example with initial values and periodic boundary conditions

$$\begin{cases} D_t^{1-\alpha(t)}u((x,t)+\delta u_{xxx}(x,t)+\lambda g(u)_x = f(x,t), & (x,t)\in(0,1)\times(0,1], \\ u(x,0)=\sin(2\pi x), & x\in[0,1], \end{cases} \tag{5.1}$$

where $\delta=2, \lambda=9, g(u)=\frac{1}{3}u^2$,

$$f(x,t)=\exp[t]\left(1-\exp\left[\frac{-t}{\alpha(t)}\right]\right)\sin(2\pi x)-16\pi^2\exp[t]\cos(2\pi x) \\ +6\pi\exp[2t]\sin(4\pi x).$$

Now we can check that the exact solution is

$$u(x,t)=e^t\sin(2\pi x).$$

In the following numerical calculations, we will provide the results of the above examples under different $\alpha(t)$ conditions using piecewise P^k polynomials to validate our method. The detailed results for the time and space directions are listed below, with $h=1/N$, $\Delta t=1/M$ for time step and space step, respectively.

In order to reflect the spatial accuracy of the scheme, **Figure 1** and **Figure 2** adopt a fixed small time step $\Delta t=\frac{1}{1000}$ and the variable space step $N=5,10,20,40$. Selecting different $\alpha(t)$, the accuracy of L^2 norm and L^∞ norm of piecewise P^k polynomial can reach the optimal order. **Table 1** examines the convergence rate in the time direction of the LDG method, we choose a sufficiently small space step $h=\frac{1}{1000}$ and a variable time step $\Delta t=5,10,20,40$. It can be seen from **Table 1** that it has first-order convergence in time, which is also consistent with the theoretical results.

Table 1. For different order $\alpha(t)$ when $N=1000$, $T=1$, use the piecewise P^2 polynomial to test the time accuracy.

	M	L^2-error	order	L^∞-error	order
$\alpha(t)=\frac{2\cos(t)}{9}$	5	2.342556643646625e−02	-	3.844443535521065e−02	-
	10	1.287283762777858e−02	0.86	1.784166238464656e−02	1.01
	20	6.828570270036006e−03	0.91	9.105420610539252e−03	0.97
	40	3.497426279254679e−03	0.97	4.694853168636008e−03	0.96
$\alpha(t)=\frac{2+t}{7}$	5	4.662548602534104e−02	-	6.552106547740656e−02	-
	10	2.479318097735330e−02	0.91	3.451685864768518e−02	0.92
	20	1.270402131562550e−02	0.96	1.726411528972149e−02	0.99
	40	6.405110600169740e−03	0.99	8.833848912488482e−03	0.97

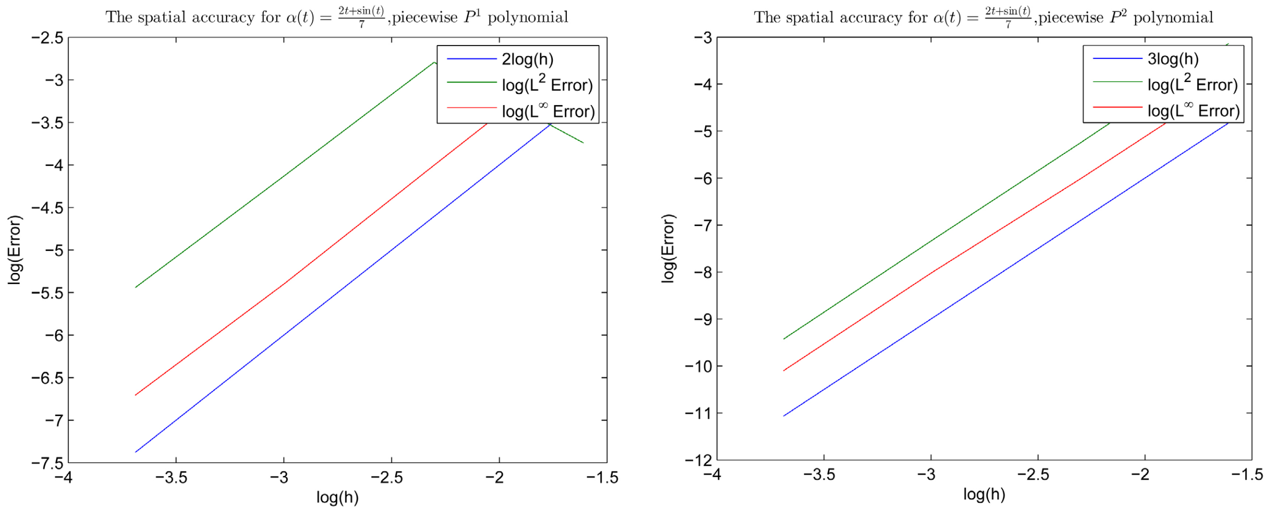

Figure 1. L^2 errors and L^∞ errors VS h, order for $\alpha(t)=\dfrac{2t+\sin(t)}{7}$, $M=1000$, piecewise P^1 and P^2 polynomial.

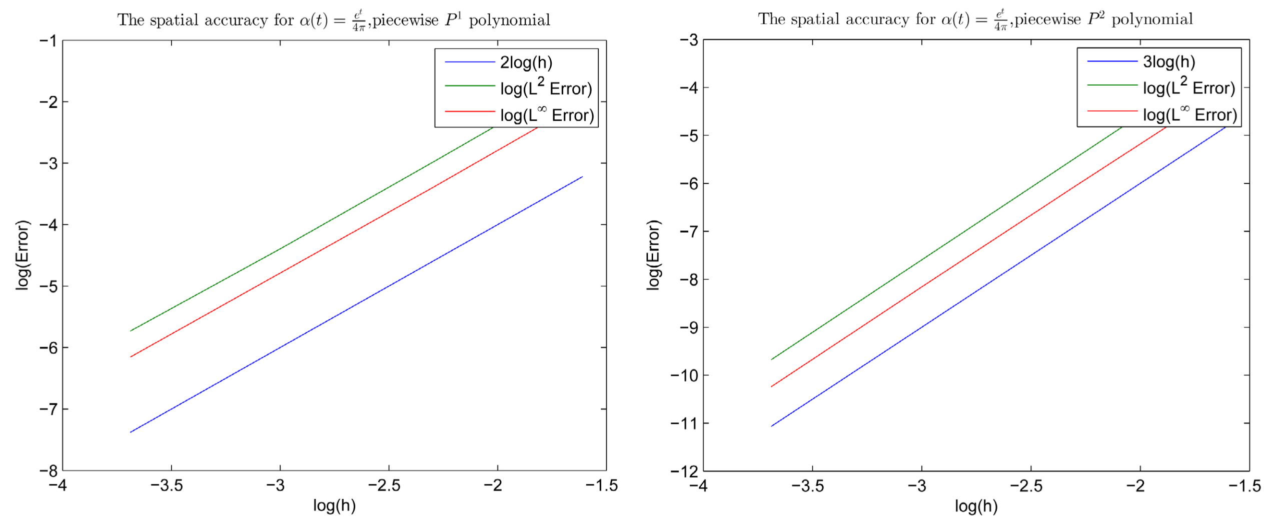

Figure 2. L^2 errors and L^∞ errors VS h, order for $\alpha(t)=\dfrac{e^t}{4\pi}$, $M=1000$, piecewise P^1 and P^2 polynomial.

6. Conclusion

This paper discusses the solution of a class of time-fractional KdV equations by the LDG method under the Caputo-Fabrizio fractional derivative. We derive the stability and error estimates of the proposed scheme. Numerical results demonstrate the effectiveness and good numerical performance of the method. In the future, we will consider generalizing this scheme to two-dimensional or high-dimensional cases.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

References

[1] Lan, K. and Lin, W. (2013) Positive Solutions of Systems of Caputo Fractional Differential Equations. *Communications in Applied Analysis*, **17**, 61-86.

[2] Meerschaert, M.M. and Sikorskii, A. (2011) Stochastic Models for Fractional Calculus. De Gruyter, Berlin/Boston. https://doi.org/10.1515/9783110258165

[3] Chen, Y., Liu, L., Li, B. and Sun, Y. (2014) Numerical Solution for the Variable Order Linear Cable Equation with Bernstein Polynomials. *Applied Mathematics and Computation*, **238**, 329-341. https://doi.org/10.1016/j.amc.2014.03.066

[4] Gu, X., Sun, H., Zhao, Y. and Zheng, X. (2021) An Implicit Difference Scheme for Time-Fractional Diffusion Equations with a Time-Invariant Type Variable Order. *Applied Mathematics Letters*, **120**, Article ID: 107270. https://doi.org/10.1016/j.aml.2021.107270

[5] Heydari, M.H. and Atangana, A. (2019) A Cardinal Approach for Nonlinear Variable-Order Time Fractional Schrodinger Equation Defined by Atangana-Baleanu-Caputo Derivative. *Chaos, Solitons & Fractals*, **128**, 339-348. https://doi.org/10.1016/j.chaos.2019.08.009

[6] Samko, S.G. and Ross, B. (1993) Integration and Differentiation to a Variable Fractional-Order. *Integral Transforms and Special Functions*, **1**, 277-300. https://doi.org/10.1080/10652469308819027

[7] Shen, S., Liu, F., Chen, J., Turner, I. and Anh, V. (2012) Numerical Techniques for the Variable Order Time Fractional Diffusion Equation. *Applied Mathematics and Computation*, **218**, 10861-10870. https://doi.org/10.1016/j.amc.2012.04.047

[8] Shyu, J.J., Pei, S.C. and Chan, C.H. (2009) An Iterative Method for the Design of Variable Fractional-Order FIR Differintegrators. *Signal Process*, **89**, 320-327. https://doi.org/10.1016/j.sigpro.2008.09.009

[9] Sun, H.G., Chen, W., Wei, H. and Chen, Y.Q. (2011) A Comparative Study of Constant-Order and Variable-Order Fractional Models in Characterizing Memory Property of Systems. *European Physical Journal: Special Topics*, **193**, 185-192. https://doi.org/10.1140/epjst/e2011-01390-6

[10] Jin, B., Lazarov, R.D. and Zhou, Z. (2016) An Analysis of the L1 Scheme for the Subdiffusion Equation with Nonsmooth Data. *IMA Journal of Numerical Analysis*, **36**, 197-221. https://doi.org/10.1093/imanum/dru063

[11] Li, M., Huang, C.M. and Jiang, F.Z. (2017) Galerkin Finite Element Method for Higher Dimensional Multi-Term Fractional Diffusion Equation on Non-Uniform Meshes. *Applicable Analysis*, **96**, 1269-1284. https://doi.org/10.1080/00036811.2016.1186271

[12] Zhao, Y.M. (2015) Finite Element Method for Two-Dimensional Space-Fractional Advection-Dispersion Equations. *Applied Mathematics and Computation*, **257**, 553-565. https://doi.org/10.1016/j.amc.2015.01.016

[13] Cao, Y. (2018) Crank-Nicolson WSGI Difference Scheme with Finite Element Method for Multi-Dimensional Time-Fractional Wave Problem. *Computational and Applied Mathematics*, **37**, 5126-5145. https://doi.org/10.1007/s40314-018-0626-2

[14] Deng, W.H. and Hesthaven, J.S. (2015) Local Discontinuous Galerkin Methods for Fractional Ordinary Differential Equations. *BIT Numerical Mathematics*, **55**, 967-985. https://doi.org/10.1007/s10543-014-0531-z

[15] Liu, Y. (2017) High-Order Local Discontinuous Galerkin Method Combined with WSGD-Approximation for a Fractional Subdiffusion Equation. *Computers & Mathematics with Applications*, **73**, 1298-1314.

https://doi.org/10.1016/j.camwa.2016.08.015

[16] Guo, L., Wang, Z.B. and Vong, S.W. (2016) Fully Discrete Local Discontinuous Galerkin Methods for Some Time-Fractional Fourth-Order Problems. *International Journal of Computer Mathematics*, **93**, 1665-1682. https://doi.org/10.1080/00207160.2015.1070840

[17] Wei, L., Zhang, X.D. and He, Y.N. (2013) Analysis of a Local Discontinuous Galerkin Method for Time-Fractional Advection-Diffusion Equations. *International Journal of Numerical Methods for Heat & Fluid Flow*, **23**, 634-648. https://doi.org/10.1108/09615531311323782

[18] Bi, H. and Chen, Y. (2020) The Error Estimates of Direct Discontinuous Galerkin Methods Based on Upwind-Baised Fluxes. *Journal of Applied Mathematics and Physics*, **8**, 2964-2970. https://doi.org/10.4236/jamp.2020.812219

[19] Lin, Y.M. and Xu, C.J. (2007) Finite Difference/Spectral Approximations for the Time-Fractional Diffusion Equation. *Journal of Computational Physics*, **225**, 1533-1552. https://doi.org/10.1016/j.jcp.2007.02.001

[20] Song, F.Y. and Xu, C.J. (2015) Spectral Direction Splitting Methods for Two-Dimensional Space Fractional Diffusion Equations. *Journal of Computational Physics*, **299**, 196-214. https://doi.org/10.1016/j.jcp.2015.07.011

[21] Sun, Z. and Wu, X. (2006) A Fully Discrete Difference Scheme for a Diffusion-Wave System. *Applied Numerical Mathematics*, **56**, 193-209. https://doi.org/10.1016/j.apnum.2005.03.003

[22] Li, C.P. and Ding, H.F. (2014) Higher Order Finite Difference Method for the Reaction and Anomalous-Diffusion Equation. *Applied Mathematical Modelling*, **38**, 3802-3821. https://doi.org/10.1016/j.apm.2013.12.002

[23] Ding, H.F. and Li, C.P. (2016) High-Order Compact Difference Schemes for the Modified Anomalous Subdiffusion Equation. *Numerical Methods for Partial Differential Equations*, **32**, 213-242. https://doi.org/10.1002/num.21992

[24] Liu, F. (2014) A New Fractional Finite Volume Method for Solving the Fractional Diffusion Equation. *Applied Mathematical Modelling*, **38**, 3871-3878. https://doi.org/10.1016/j.apm.2013.10.007

[25] Cheng, A.J., Wang, H. and Wang, K.X. (2015) A Eulerian-Lagrangian Control Volume Method for Solute Transport with Anomalous Diffusion. *Numerical Methods for Partial Differential Equations*, **31**, 253-267. https://doi.org/10.1002/num.21901

[26] Wei, L. and Li, W. (2021) Local Discontinuous Galerkin Approximations to Variable-Order Time-Fractional Diffusion Model Based on the Caputo-Fabrizio Fractional Derivative. *Mathematics and Computers in Simulation*, **188**, 280-290. https://doi.org/10.1016/j.matcom.2021.04.001

[27] Du, R., Alikhanov, A.A. and Sun, Z. (2020) Temporal Second Order Difference Schemes for the Multi-Dimensional Variable-Order Time Fractional Sub-Diffusion Equations. *Computers & Mathematics with Applications*, **79**, 2952-2972. https://doi.org/10.1016/j.camwa.2020.01.003

[28] Li, C.P. and Wang, Z. (2019) The Local Discontinuous Galerkin Finite Element Methods for Caputo-Type Partial Differential Equations: Numerical Analysis. *Applied Numerical Mathematics*, **140**, 1-22.

[29] Korteweg, D.J. and Vries, G. (1895) On the Change of Form of Long Waves Advancing in a Rectangular Canal, and on a New Type of Long Stationary Waves. *Philosophical Magazine*, **39**, 422-443. https://doi.org/10.1080/14786449508620739

[30] Zabusky, N. and Kruskal, M. (1965) Interactions of Solitons in a Collisionless Plasma and the Recur Rence of Initial States. *Physical Review Letters*, **15**, 240-243.

https://doi.org/10.1103/PhysRevLett.15.240

[31] Fung, M. (1997) Kdv Equation as an Euler-Poincare Equation. *Chinese Journal of Physics*, **35**, 789-796.

[32] Goda, K. (1975) On Stability of Some Finite Difference Schemes for the Korteweg-de Vries Equation. *Journal of the Physical Society of Japan*, **39**, 229-236. https://doi.org/10.1143/JPSJ.39.229

[33] Vliegenthart, A.C. (1971) On Finite-Difference Methods for the Korteweg-de Vries Equation. *Journal of Engineering Mathematics*, **5**, 137-155. https://doi.org/10.1007/BF01535405

[34] Fornberg, B. and Whitham, G.B. (1978) A Numerical and Theoretical Study of Certain Nonlinear Wave Phenomena. *Philosophical Transactions of the Royal Society of London. Series A*, *Mathematical and Physical Sciences*, **289**, 373-404. https://doi.org/10.1098/rsta.1978.0064

[35] Kutluay, S., Bahadir, A.R. and Ozdes, A. (2000) A Small Time Solutions for the Korteweg-de Vries Equation. *Applied Mathematics and Computation*, **107**, 203-210. https://doi.org/10.1016/S0096-3003(98)10119-4

[36] An, N., Huang, C. and Yu, X. (2020) Error Analysis of Discontinuous Galerkin Method for the Time Fractional KdV Equation with Weak Singularity Solution. *Discrete and Continuous Dynamical Systems-Series B*, **25**, 321-334. https://doi.org/10.3934/dcdsb.2019185

[37] Zhang, Q. and Xia, Y. (2019) Conservative and Dissipative Local Discontinuous Galerkin Methods for Korteweg-de Vries Type Equations. *Communications in Computational Physics*, **25**, 532-563. https://doi.org/10.4208/cicp.OA-2017-0204

[38] Wei, L., Wei, X. and Tang, B. (2022) Numerical Analysis of Variable-Order Fractional KdV-Burgers-Kuramoto Equation. *Electronic Research Archive*, **30**, 1263-1281. https://doi.org/10.3934/era.2022066

[39] Zhang, Q., Zhang, J., Jiang, S. and Zhang, Z. (2018) Numerical Solution to a Linearized Time Fractional KdV Equation on Unbounded Domains. *Mathematics of Computation*, **87**, 693-719. https://doi.org/10.1090/mcom/3229

[40] Caputo, M. and Fabrizio, M. (2015) A New Definition of Fractional Derivative without Singular Kernel. *Progress in Fractional Differentiation and Applications*, **1**, 73-85.

[41] Sawoor, A. and Sadkane, M. (2020) Asymptotic Analysis of Linear and Interval Linear Fractional-Order Neutral Delay Differential Systems Described by the Caputo-Fabrizio Derivative. *Applied Mathematics*, **11**, 1229-1242. https://doi.org/10.4236/am.2020.1112084

[42] Cockburn, B. and Shu, C.W. (1998) The Local Discontinuous Galerkin Method for Time-Dependent Convection-Diffusion Systems. *SIAM Journal on Numerical Analysis*, **35**, 2440-2463. https://doi.org/10.1137/S0036142997316712

[43] Yan, J. and Shu, C.W. (2002) A Local Discontinuous Galerkin Method for KdV Type Equations. *SIAM Journal on Numerical Analysis*, **40**, 769-791. https://doi.org/10.1137/S0036142901390378

[44] Wei, L., He, Y. and Zhang, X. (2015) Analysis of an Implicit Fully Discrete Local Discontinuous Galerkin Method for the Time-Fractional KdV Equation. *Advances in Applied Mathematics and Mechanics*, **7**, 510-527. https://doi.org/10.4208/aamm.2013.m220

[45] Karakashian, O. and Xing, Y.L. (2016) A Posteriori Error Estimates for Conservative local Discontinuous Galerkin Methods for the Generalized Korteweg-de Vries Equation. *Communications in Computational Physics*, **20**, 250-278. https://doi.org/10.4208/cicp.240815.301215a

[46] Baccouch, M. (2019) Superconvergence of the Semi-Discrete Local Discontinuous Galerkin Method for Nonlinear KdV-Type Problems. *Discrete and Continuous Dynamical Systems-B*, **24**, 19-54. https://doi.org/10.3934/dcdsb.2018104

[47] Shao, L., Feng, X. and He, Y. (2011) The Local Discontinuous Galerkin Finite Element Method for Burger's Equation. *Mathematical and Computer Modelling*, **54**, 2943-2954. https://doi.org/10.1016/j.mcm.2011.07.016

[48] Xia, Y., Yan, Y. and Shu, C.W. (2008) Application of the Local Discontinuous Galerkin Method for the Allen Cahn/Cahn-Hilliard System. *Communications in Computational Physics*, **5**, 821-835. https://doi.org/10.21236/ADA464873

[49] Xu, Y. and Shu, C.W. (2008) A Local Discontinuous Galerkin Method for the Camassa-Holm Equation. *SIAM Journal on Numerical Analysis*, **46**, 1998-2021. https://doi.org/10.1137/070679764

[50] Cockburn, B., Kanschat, G. and Perugia, I. (2001) Superconvergence of the Local Discontinuous Galerkin Method for Elliptic Problems on Cartesian Grids. *SIAM Journal on Numerical Analysis*, **39**, 264-285. https://doi.org/10.1137/S0036142900371544

[51] Cheng, Y., Meng, X. and Zhang, Q. (2017) Application of Generalized Gauss-Radau Projections for the Local Discontinuous Galerkin Method for Linear Convection-Diffusion Equations. *Mathematics of Computation*, **86**, 1233-1267. https://doi.org/10.1090/mcom/3141

[52] Zhang, M., Liu, Y. and Li, H. (2019) High-Order Local Discontinuous Galerkin Method for a Fractal Mobile/Immobile Transport Equation with the Caputo-Fabrizio Fractional Derivative. *Numerical Methods for Partial Differential Equations*, **35**, 1588-1612. https://doi.org/10.1002/num.22366

[53] Liu, Z.G., Cheng, A.J. and Li, X.L. (2018) A Second Order Finite Difference Scheme for Quasilinear Time Fractional Parabolic Equation Based on New Fractional Derivative. *International Journal of Computer Mathematics*, **95**, 396-411. https://doi.org/10.1080/00207160.2017.1290434

[54] Wang, H. and Zheng, X. (2019) Analysis and Numerical Solution of a Nonlinear Variable-Order Fractional Differential Equation. *Advances in Computational Mathematics*, **45**, 2647-2675. https://doi.org/10.1007/s10444-018-9603-y

[55] Cockburn, B. and Shu, C.W. (1989) TVB Runge-Kutta Local Projection Discontinuous Galerkin Finite Element Method for Conservation Laws II: General Framework. *Mathematics of Computation*, **52**, 411-435. https://doi.org/10.1090/S0025-5718-1989-0983311-4

[56] LeVeque, R. (1990) Numerical Methods for Conservation Laws. Birkhauser Verlag, Basel. https://doi.org/10.1007/978-3-0348-5116-9