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Dirac Quantum Beats in Extreme Relativistic Diffraction in Time

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Abstract

We solve the relativistic Dirac equation for the shutter problem. We prove that, at intermediate relativistic energies, the probability density oscillates in time in a similar way to the optical Fresnel oscillations in a straight edge. However, for extreme-relativistic beams, the Fresnel oscillations turn into quantum damped beats.

Keywords

Dirac Shutter Problem, Relativistic Diffraction in Time, Quantum Beats

1. Introduction

Similarities between optics and quantum mechanics have long been recognized. One example of this symmetry was obtained by Moshinsky [1] who addressed the following quantum shutter problem. Consider a monoenergetic beam of free particles, moving parallel to the *x*-axis. For negative times, the beam is interrupted at $x=0$ by a perfectly absorbing shutter perpendicular to the beam. Suddenly, at time $t=0$, the shutter is opened, allowing for $t>0$ the free time-evolution of the beam of particles. What is the time-dependent density observed at a distance *x* from the shutter?

The shutter problem implies solving the time-dependent Schrödinger equation with an initial condition given by

$$\psi(x,0)=\mathrm{e}^{ikx}\theta(-x), \tag{1}$$

where $\theta(x)$ denotes the step function. For $t>0$, Moshinsky proved that the free propagation of the beam has a probability density, $\rho(x,t)=|\psi(x,t)|^2$, given by:

$$\rho(x,t)=\frac{1}{2}\left[C(\xi)+\frac{1}{2}\right]^2+\frac{1}{2}\left[S(\xi)+\frac{1}{2}\right]^2, \tag{2}$$

here, $C(\xi)+iS(\xi)\equiv\int_0^{\xi}\exp\left(i\pi u^2/2\right)\mathrm{d}u$, denotes the complex Fresnel function and the argument ξ is given by

$$\xi(x,t)\equiv\sqrt{\frac{m}{\pi\hbar t}}\left(\frac{\hbar k}{m}t-x\right). \tag{3}$$

The right-hand side in Equation (2) looks similar to the mathematical expression for the light intensity in the optical Fresnel diffraction by a straight-edge, Born-Wolf [2]. For a fixed position, $X\equiv px/\hbar=1$, the plot of the probability density $\rho(X,T)$ as a function of time, $T\equiv Et/\hbar$, is shown in **Figure 1**.

A good measure of the "width" in time of this diffraction effect, can be obtained from the difference $\Delta t\equiv t_2-t_1$ between the first two times at which ρ takes the (classical) mean value. We obtain, for thermal neutrons, $\Delta t\approx 10^{-9}\ \mathrm{s}$. The experimental evidence of this quantum prediction has been confirmed until very recently by Szriftigiser, Guéry-Odelin, Arndt, and Dalibard [3].

These transient oscillations are a pure quantum phenomenon, and similar oscillations arise at the moment of closing and opening gates in nanoscopic circuits [4]. For a review on the subject see [5] [6]. With adequate potentials added to the model, it has been used to study transient dynamics of tunneling matter waves [7] [8] [9] [10], and the transient response to abrupt changes of the interaction potential in semiconductor structures and quantum dots [11] [12]. There is, in summary, a strong motivation for a thorough understanding of transient time oscillation in beams of matter.

The analogy between the quantum shutter problem and optical diffraction raises the question of whether the transient densities for other types of wave equations show this analogy. Using the Dirac equation for the shutter problem at low energies, we found in a previous paper [13], that the algebraic expression for the relativistic quantum density $\rho(x,t)$ no longer resembles the algebraic Fresnel expression for the optical diffraction. In spite of this, when the exact relativistic density is plotted versus time, the plots show transient oscillations which prove that diffraction in time is present in the relativistic realm.

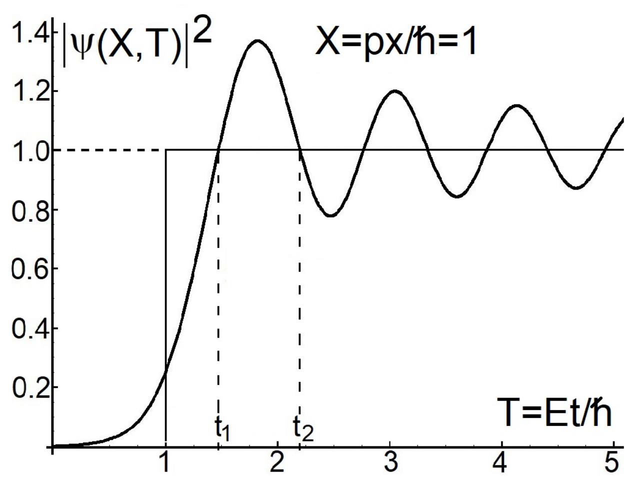

Figure 1. Schrödinger diffraction in time, $X\equiv px/\hbar=1$.

The main contribution of this paper is to show the existence of transient quantum beats in extreme-relativistic diffraction in time. In Section 2, we solve the Dirac equation for the shutter problem and in Section 3 we prove, as expected, that for intermediate energies, at internuclear distances, the plot of the probability density looks similar to the optical Fresnel oscillations derived in the Schrödinger equation. Finally, in Section 4, as we gradually increase the energy until we reach extreme-relativistic energies, the Dirac diffraction changes gradually from a Fresnel pattern into quantum damped beats! For 1/2-spin particles, this has never been reported before. We conclude with the application of the present result to the quantum problem of extreme-relativistic particles suddenly released from an infinite potential well.

2. The Dirac Shutter Problem

For 1/2-spin particles, we assume for all negative times, $t \le 0$, a beam of right-moving free particles in the left side of a perfectly absorbing shutter located at the origin, and none to the right. For the case of propagation into the direction of the *z*-axis, we want to calculate the spinor wave function, $\psi(z,t) = (\psi_1, \psi_2, \psi_3, \psi_4)^{\mathrm{T}}$, which is the solution of the one-dimensional Dirac equation:

$$\left\{ \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \frac{1}{i} \frac{\partial}{\partial(ct)} + \begin{pmatrix} 0 & \sigma_z \\ \sigma_z & 0 \end{pmatrix} \frac{1}{i} \frac{\partial}{\partial z} + \frac{1}{\lambda_c} \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \right\} \psi = 0, \tag{4}$$

here σ_z is the Pauli matrix and, $\lambda_c \equiv \hbar / m_0 c$, the reduced Compton wave-length. Three quantum numbers are needed to classify the Dirac free-particle solutions, namely, the momentum, $\boldsymbol{p} \equiv \hbar \boldsymbol{k}$, positive or negative energies, $E = \pm \hbar\omega$, where $\omega = c\left(k^2 + \lambda_c^{-2}\right)^{1/2}$, and the helicity, $\Lambda_S = \boldsymbol{S} \cdot \boldsymbol{p} / p$. In our case, we assume that the initial condition corresponds to a right moving plane wave propagating along the z-direction, $\boldsymbol{k} = (0,0,k)$, a positive energy, $E = +\hbar\omega$, and the spin parallel to the direction of motion, $S_z = +1/2$. The incident plane wave is then given by:

$$\psi_{k,+E,+1/2}(z,t) = N\left(1, 0, \frac{k}{\lambda_c^{-1} + \omega/c}, 0\right)^{\mathrm{T}} \mathrm{e}^{i(kz - \omega t)} \theta(-z) \quad (t \le 0), \tag{5}$$

where N is the normalization factor, and $\theta(z)$ denotes the Heaviside step function defined in Equation (1).

For free particles, the helicity Λ_S is a constant of motion. The initial direction of spin, $S_z = 1/2$, will be conserved. Therefore, the two components of the wave function: (ψ_2 and ψ_4) which are zero at the initial time will remain zero for all positive times: $\psi_2(z,t) = \psi_4(z,t) = 0 \ (t \ge 0)$. The two remaining components: $\psi \equiv (\psi_1, \psi_3)^{\mathrm{T}}$ evolve in time as solutions of the equation,

$$\left(I \frac{\partial}{\partial(ct)} + \sigma_x \frac{\partial}{\partial z} + \frac{i}{\lambda_c} \sigma_z \right) \psi(z,t) = 0 \tag{6}$$

with the initial condition:

$$\psi(z,0) \equiv \left(\psi_1(z,0) \quad \psi_3(z,0)\right)^{\mathrm{T}} = \left(N \quad \gamma\right)^{\mathrm{T}} \mathrm{e}^{ikz} \theta(-z), \tag{7}$$

where $\gamma \equiv Nk/\left(\lambda_c^{-1} + \omega/c\right)$. Using the condition: $N^2 + \gamma^2 = 1$, the normalization factor becomes: $N^2 = \left(1 + c/\lambda_c \omega\right)/2$.

For the sake of conciseness, we denote, as usual $\beta \equiv v/c$, and use the Compton wave-length λ_c, to define dimensionless variables:

$$Z \equiv \frac{z}{\lambda_c}, \quad T \equiv \frac{ct}{\lambda_c}, \quad \kappa \equiv k\lambda_c = \frac{p}{m_0 c}, \quad \Omega \equiv \frac{\omega \lambda_c}{c} = \frac{E}{m_0 c^2}$$

$$N^2 = \left(1 + \sqrt{1-\beta^2}\right)\Big/2, \quad \gamma^2 = \left(1 - \sqrt{1-\beta^2}\right)\Big/2. \tag{8}$$

Using these variables we derive in **Appendix A**, for the Dirac shutter, the *exact* transmitted wave, $\psi_>(Z,T)$, valid for ($Z \ge 0$):

$$\frac{2y_>(Z,T)}{\theta(T-Z)\mathrm{e}^{-i\Omega T}} = \begin{pmatrix} \gamma + N \\ \gamma + N \end{pmatrix} \mathrm{e}^{+i\Omega Z} + \begin{pmatrix} -\gamma\left[iG_0 + G_2\right] - NZG_1 \\ +N\left[iG_0 - G_2\right] - \gamma ZG_1 \end{pmatrix} (Z \ge 0) \tag{9}$$

The functions $G_0(Z,T)$, $G_1(Z,T)$ and $G_2(Z,T)$ are defined in Appendix A. Notice the step function $\theta(T-Z)$ which shows the correct relativistic prediction that no traveling wave arrives at position z until $ct \ge z$.

3. Intermediate-Energy Diffraction in Time

In this section the case of an incoming monochromatic beam with intermediate relativistic velocities, $\beta = 0.4$, and the particle detector located at $Z = 20$ ($z = \lambda_c Z = 4.2 \times 10^{-15}$ m, for neutrons), is investigated. We use the Dirac solution: $\psi_>(z,t)$, given in Equation (9), to calculate the probability density ρ:

$$\rho(Z,T) \equiv \left|\psi_>\right|^2 = \left|\psi_1\right|^2 + \left|\psi_3\right|^2 \equiv \rho_1 + \rho_3. \tag{10}$$

For a position, $Z = 20$, we show in **Figure 2** a plot of the probability density as a function of time. We find, as expected, damped diffraction oscillations which look similar to the ones predicted by the Schrödinger theory (see **Figure 1**). However, the present relativistic diffraction differs from the Schrödinger one by the presence of a small amplitude oscillation superposed on the main one. Understanding these double oscillations is important. For $\beta = 0.4$, the initial condition for (ψ_1, ψ_3) given in Equation (7) has constant amplitudes (N, γ) which

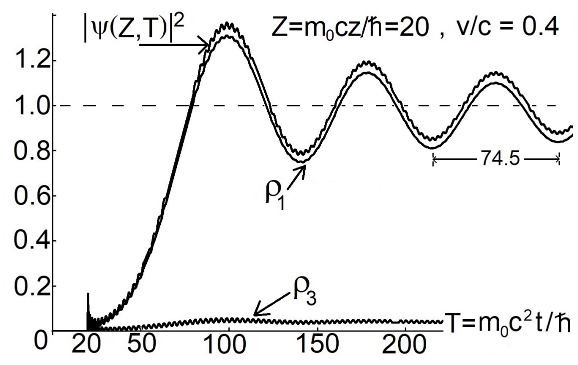

Figure 2. Dirac transient oscillations with $Z = 20$ and $v/c = 0.4$.

are quite different in magnitude: $N^2 \approx 1-\beta^2/4 = 0.96$ and $\gamma^2 \approx \beta^2/2 = 0.04$. This explains the plots shown in **Figure 2**, where the amplitude oscillations of ρ_3 is very small in comparison with those of ρ_1. For low energies the relativistic density ρ_1 looks like the Schrödinger diffraction oscillations and ρ_3 is just a small oscillatory perturbation. In the plot of **Figure 2** we measure the half-period of one initial oscillation, $\Delta T = 74.5$ ($\Delta t = \lambda_c \Delta T/c = 1.0\times10^{-22}$ sec. for neutrons).

Extreme-Relativistic Diffraction in Time

The next question is: for 1/2-spin particles, how does diffraction in time look like for an extreme-relativistic value of β? In this section we show the main contribution of this paper, namely that as we gradually increase the velocity β, the Dirac diffraction changes gradually from a Fresnel pattern into quantum damped beats!

To illustrate this result we take the case of an extreme-relativistic velocity: $\beta = 0.998$, and the particle detector at a fixed distance: $Z = 1$, ($z \equiv Z\lambda_c = 2.1\times10^{-16}$ m, for neutrons). For this case we plot in **Figure 3**, as a function of time *T*, the exact Dirac density $\rho(Z,T)$ given in Equation (9).

The origin of these quantum beats is easy to understand. For an extreme-relativistic value of β we have: First, two Dirac components (ψ_1,ψ_3) which have similar amplitudes: $N^2 = 0.53$ and $\gamma^2 = 0.47$, (in fact, if $\beta \to 1$, then: $N^2 \to \gamma^2 = 0.5$). Second, looking at the plot on **Figure 3**, we measure the *half-periods* of oscillations for ρ_1 and ρ_3, we get: $\Delta T_1 = 0.42$ and $\Delta T_3 = 0.38$, ($\Delta t_1 = \lambda_c \Delta T_1/c = 3\times10^{-25}$ sec, for neutrons). These corresponds to angular frequencies: $\Omega_1 = \pi/\Delta T_1 = 7.4$ and $\Omega_3 = 8.2 = \Omega_1 + 0.8$. So, we have two density oscillations with similar amplitudes and similar angular frequencies. When we add them together we get quantum beats! This result is well known using trigonometric functions: $\sin(A_1)+\sin(A_1+\varepsilon) = 2\cos(\varepsilon/2)\sin(A_1+\varepsilon/2)$. Therefore, for the extreme-relativistic, $\beta = 0.998$, the total probability density may be *interpreted* as the product of two oscillations, one with slow frequency:

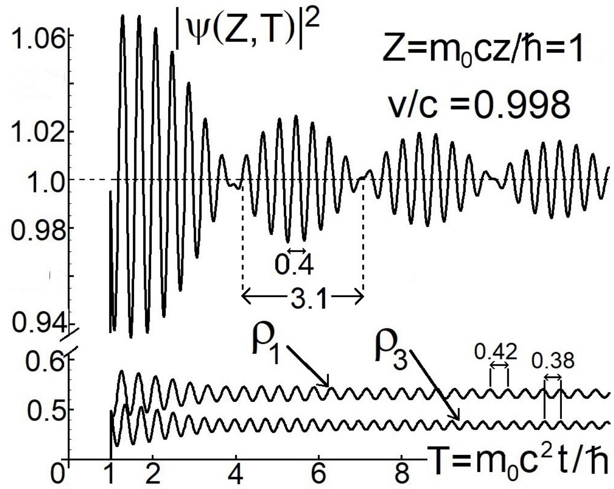

Figure 3. Quantum beats for $Z = 1$ and extreme-relativistic $v/c = 0.998$.

$\Omega_{slow} = (\Omega_3 - \Omega_1)/2 = 0.4$ and the other with a fast one: $\Omega_{fast} = (\Omega_3 + \Omega_1)/2 = 7.4 + 0.4$. The slow frequency oscillation modulates the amplitude of the fast one. Density quantum beats is a pure relativistic quantum result!

4. Conclusions

Exact Dirac diffraction in time solution, what for? We claim that the exact Dirac diffraction in time solution can be used as a basic building block to solve a more interesting problem: Consider a relativistic particle which, for all negative times, was inside an infinite potential well with walls at $z = (0, a)$. Suddenly, at time $t = 0$, the particle is released, allowing for $t > 0$ the free time-evolution of the particle. For $t > 0$, what is the time-dependent density observed at a distance $z > a$ or $z < 0$?

To solve this problem, for 1/2-spin particles, we need the time-dependent solution of a free-particle Dirac equation, with the following initial condition:

$$\psi(z,0) = (N \quad \gamma)^{\mathrm{T}} \left(\frac{\mathrm{e}^{ikz} - \mathrm{e}^{-ikz}}{2i} \right) \left[\theta(a-z) - \theta(-z) \right]. \tag{11}$$

Here k is a quantized, relativistic, wave-number. It is evident that the time-evolution of this initial condition is the superposition of four similar solutions for the Dirac's diffraction in time solutions.

At first glance we expect, at low and high energies, a sort of Fraunhofer and Fresnel diffraction density patterns which resemble the optical diffraction of light by a narrow slit. The important fact is that, at extreme-relativistic energies, the expected density wave pattern will be a pure relativistic quantum result: outgoing density quantum beats at right and left directions! As far as we know, this has never been reported before. How useful is this model? That remains to be seen. We will get the exact analytic solution of this problem in the near future.

As for the verification of this theoretical prediction, we make clear the difficulty of doing so. For neutrons at internuclear distances, $z \sim 10^{-16}$ m, and velocities $v/c = 0.998$, our Dirac shutter solution predicts beats oscillations with periods of the order $\Delta t \sim 10^{-25}$ sec! As far as we know, these times order of magnitude is out of reach of the present technology.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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Appendix

A1. The Dirac Solution

Using dimensionless variables: $Z \equiv z/\lambda_c$, $T \equiv ct/\lambda_c$, $\kappa \equiv k\lambda_c$ and $\Omega \equiv \omega\lambda_c/c$, the Dirac Equation (6) and the initial condition, Equation (7) become:

$$\left(I\frac{\partial}{\partial T} + \sigma_x \frac{\partial}{\partial Z} + i\sigma_z \right)\psi(Z,T) = 0, \tag{12}$$

$$\psi(Z,0) \equiv \begin{pmatrix}\psi_1(Z,0) & \psi_3(Z,0)\end{pmatrix}^{\mathrm{T}} = \begin{pmatrix}N & \gamma\end{pmatrix}^{\mathrm{T}} \mathrm{e}^{i\kappa Z}\theta(-Z). \tag{13}$$

Taking the Laplace transform in Equation (12), and denoting,

$$\phi(Z,s) \equiv \mathcal{L}\left[\psi(Z,T)\right] \equiv \int_0^\infty \mathrm{e}^{-sT}\psi(Z,T)\mathrm{d}T,$$

Equation (12) becomes:

$$\frac{\mathrm{d}\phi}{\mathrm{d}Z} + \left(\sigma_x s + \sigma_y\right)\phi = \psi(Z,0), \quad (-\infty < Z < \infty). \tag{14}$$

This is a linear matrix differential equation in the Z variable, and the solution is readily obtained. Due to the presence of the step function $\theta(-Z)$, the origin $Z = 0$ is a singular point, where we demand that the function $\phi(Z,s)$ must be continuous. This fact suggest breaking the infinite range $(-\infty < Z < \infty)$ into the left ($Z < 0$) and right ($Z > 0$) ranges.

For the left-side of the shutter, $Z < 0$, we define the function $\phi_<(Z,s)$ as the solution of the differential equation:

$$\frac{\mathrm{d}\phi_<}{\mathrm{d}Z} + \left(\sigma_x s + \sigma_y\right)\phi_< = \begin{pmatrix}N & \gamma\end{pmatrix}^{\mathrm{T}} \mathrm{e}^{i\kappa Z}, \quad (Z < 0) \tag{15}$$

and for the right-side, $Z > 0$, we define $\phi_>(Z,s)$ as the solution of

$$\frac{\mathrm{d}\phi_>}{\mathrm{d}Z} + \left(\sigma_x s + \sigma_y\right)\phi_> = 0, \quad (Z > 0). \tag{16}$$

Both functions $\phi_<$ and $\phi_>$ must be bounded, ($\phi_<$ at $Z = -\infty$) and ($\phi_>$ at $Z = +\infty$), and be continuous at the interface, $Z = 0$.

The Hermitian matrix: $\sigma_x s + \sigma_y$ has eigenvalues: $\lambda_1 = \sqrt{s^2+1} = -\lambda_2$, with associated orthonormal eigenvectors:

$$u_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & \sqrt{\frac{s+i}{s-i}}\end{pmatrix}^{\mathrm{T}}, \quad u_2 = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & -\sqrt{\frac{s+i}{s-i}}\end{pmatrix}^{\mathrm{T}}. \tag{17}$$

Taking into account the boundary conditions, we have the general solutions for both differential equations:

$$\phi_>(Z,s) = \frac{A}{\sqrt{2}}\begin{pmatrix}1 & \sqrt{\frac{s+i}{s-i}}\end{pmatrix}^{\mathrm{T}} \mathrm{e}^{-Z\sqrt{s^2+1}} \quad (Z > 0), \tag{18}$$

$$\phi_<(Z,s) = \frac{B}{\sqrt{2}}\begin{pmatrix}1 & -\sqrt{\frac{s+i}{s-i}}\end{pmatrix}^{\mathrm{T}} \mathrm{e}^{+Z\sqrt{s^2+1}} + \begin{pmatrix}N & \gamma\end{pmatrix}^{\mathrm{T}} \frac{\mathrm{e}^{i\kappa Z}}{s+i\Omega} \quad (Z < 0). \tag{19}$$

The constants A and B are fixed by the continuity condition: $\phi_{<}(0,s)=\phi_{>}(0,s)$, we have:

$$A=\frac{1}{\sqrt{2}}\frac{1}{s+i\Omega}\left[\gamma\sqrt{\frac{s-i}{s+i}}+N\right],\quad B=\frac{1}{\sqrt{2}}\frac{1}{s+i\Omega}\left[\gamma\sqrt{\frac{s-i}{s+i}}-N\right] \tag{20}$$

Substituting (A, and B) into Equations (18) and (19) we get the solution for Dirac shutter problem in the (Z,s) space.

$$\psi_{>}(Z,s)=\frac{\mathrm{e}^{-Z\sqrt{s^2+1}}}{2(s+i\Omega)}\begin{pmatrix}\gamma\sqrt{\frac{s-i}{s+i}}+N\\ \gamma+N\sqrt{\frac{s+i}{s-i}}\end{pmatrix}\quad(Z>0)$$

$$\psi_{<}(Z,s)=\frac{\mathrm{e}^{+Z\sqrt{s^2+1}}}{2(s+i\Omega)}\begin{pmatrix}\gamma\sqrt{\frac{s-i}{s+i}}-N\\ -\gamma+N\sqrt{\frac{s+i}{s-i}}\end{pmatrix}+\begin{pmatrix}N\\ \gamma\end{pmatrix}\frac{\mathrm{e}^{i\kappa Z}}{s+i\Omega}\quad(Z<0) \tag{21}$$

Notice the simple pole, $s=-i\Omega$, and the branch points, $s=\pm i$, all of them located in the imaginary axis. By the Nyquist stability criterion, the time dependent solution $\psi(Z,T)$ will be an oscillatory bounded solution.

Using the convolution theorem and Laplace Transforms Tables [14] we find the following results, valid only for $Z\geq 0$.

$$\mathcal{L}^{-1}\left[\frac{\mathrm{e}^{-Z\sqrt{s^2+1}}}{s+i\Omega}\right]=\theta(T-Z)\mathrm{e}^{-i\Omega T}\left[\mathrm{e}^{+i\Omega Z}-ZG_1\right]\quad(Z\geq 0), \tag{22}$$

$$\mathcal{L}^{-1}\left[\sqrt{\frac{s-i}{s+i}}\frac{\mathrm{e}^{-Z\sqrt{s^2+1}}}{s+i\Omega}\right]=\theta(T-Z)\mathrm{e}^{-i\Omega T}\left[-iG_0-G_2+\mathrm{e}^{+i\Omega Z}\right]\quad(Z\geq 0). \tag{23}$$

Here, to simplify the notation, we have denoted the integral-defined, complex functions, valid only for ($0<Z\leq T$):

$$G_0(Z,T;\Omega)\equiv\int_Z^T \mathrm{d}u\,\mathrm{e}^{i\Omega u}J_0\left(\sqrt{u^2-Z^2}\right), \tag{24}$$

$$G_1(Z,T;\Omega)\equiv\int_Z^T \mathrm{d}u\,\mathrm{e}^{i\Omega u}\frac{J_1\left(\sqrt{u^2-Z^2}\right)}{\sqrt{u^2-Z^2}}, \tag{25}$$

$$G_2(Z,T;\Omega)\equiv\int_Z^T \mathrm{d}u\,\mathrm{e}^{i\Omega u}u\frac{J_1\left(\sqrt{u^2-Z^2}\right)}{\sqrt{u^2-Z^2}} \tag{26}$$

where $J_0(u)$ and $J_1(u)$ are Bessel functions of first kind and order 0 and 1, respectively.

Therefore, for $Z\geq 0$, the exact transmitted wave is given by:

$$\frac{2\psi_{>}(Z,T)}{\theta(Z-T)\mathrm{e}^{-i\Omega T}}=\begin{pmatrix}N+\gamma\\ N+\gamma\end{pmatrix}\mathrm{e}^{i\Omega Z}-\begin{pmatrix}\gamma\left[iG_0+G_2\right]+NZG_1\\ \gamma ZG_1-N\left[iG_0-G_2\right]\end{pmatrix} \tag{27}$$

In the same way we obtain, as expected, for $Z\leq 0$ the incident and reflected wave:

$$\frac{2\psi_{<}(Z,T)}{\theta(Z+T)\mathrm{e}^{-i\Omega T}} = 2\begin{pmatrix} N \\ \gamma \end{pmatrix}\mathrm{e}^{i\kappa Z} + \begin{pmatrix} \gamma - N \\ N - \gamma \end{pmatrix}\mathrm{e}^{i\Omega|Z|} - \begin{pmatrix} \gamma\left[iG_0(|Z|) + G_2(|Z|)\right] + NZG_1(|Z|) \\ \gamma ZG_1(|Z|) - N\left[iG_0(|Z|) - G_2(|Z|)\right] \end{pmatrix} \tag{28}$$

We see that the 1D shutter problem is, in fact, a particular time-dependent scattering problem, and here we have the exact relativistic Dirac solution for spin-1/2 particles.