Scientific Research Publishing

Journal of Applied Mathematics and Physics, 2022, 10, 1651-1677
https://www.scirp.org/journal/jamp
ISSN Online: 2327-4379
ISSN Print: 2327-4352

The Family of Global Attractors of Coupled Kirchhoff Equations

Guoguang Lin, Fumei Chen

Department of Mathematics, Yunnan University, Kunming, China
Email: 15925159599@163.com, 1006423087@qq.com

How to cite this paper: Lin, G.G. and Chen, F.M. (2022) The Family of Global Attractors of Coupled Kirchhoff Equations. *Journal of Applied Mathematics and Physics*, **10**, 1651-1677.
https://doi.org/10.4236/jamp.2022.105115

Received: April 1, 2022
Accepted: May 23, 2022
Published: May 26, 2022

Abstract

In this paper, we study the initial boundary value problem of coupled generalized Kirchhoff equations. Firstly, the rigid term and nonlinear term of Kirchhoff equation are assumed appropriately to obtain the prior estimates of the equation in E_0 and E_k space, and then the existence and uniqueness of solution is verified by Galerkin's method. Then, the solution semigroup $S(t)$ is defined, and the bounded absorptive set B_k is obtained on the basis of prior estimation. Through using Rellich-Kondrachov compact embedding theorem, it is proved that the solution semigroup $S(t)$ has the family of the global attractors A_k in space E_k. Finally, by linearizing the equation, it is proved that the solution semigroup $S(t)$ is Frechet differentiable on E_k, and the family of global attractors A_k have finite Hausdroff dimension and Fractal dimension.

Keywords

Kirchhoff Equation, Prior Estimation, Existence and Uniqueness of Solutions, The Family of Global Attractors, Dimension Estimation

1. Introduction

This paper mainly studies the initial boundary value problem of the coupled generalized Kirchhoff equations:

$$\begin{cases} u_{tt}+M\left(\left\|\nabla^m u\right\|_p^p+\left\|\nabla^m v\right\|^2\right)(-\Delta)^{2m}u+\beta(-\Delta)^{2m}u_t+g_1(u_t,v_t)=f_1(x), & (1)\\ v_{tt}+M\left(\left\|\nabla^m u\right\|_p^p+\left\|\nabla^m v\right\|^2\right)(-\Delta)^{2m}v+\beta(-\Delta)^{2m}v_t+g_2(u_t,v_t)=f_2(x), & (2)\\ u(x,0)=u_0(x),u_t(x,0)=u_1(x),x\in\Omega, & (3)\\ v(x,0)=v_0(x),v_t(x,0)=v_1(x),x\in\Omega, & (4)\\ \dfrac{\partial^i u}{\partial n^i}=0,\dfrac{\partial^i v}{\partial n^i}=0,(i=0,1,2,\cdots,2m-1),x\in\partial\Omega. & (5)\end{cases}$$

where $\Omega \subseteq R^n\,(n \ge 1)$ is a bounded domain with a smooth boundary $\partial\Omega$, $g_1(u_t, v_t), g_2(u_t, v_t)$ are nonlinear source terms, $M\left(\|\nabla^m u\|_p^p + \|\nabla^m v\|^2\right)(-\Delta)^{2m} u$, $M\left(\|\nabla^m u\|_p^p + \|\nabla^m v\|^2\right)(-\Delta)^{2m} v$ are the rigid terms which $M\left(\|\nabla^m u\|_p^p + \|\nabla^m v\|^2\right)$ is real function, $f_1(x), f_2(x)$ are the external force disturbance, and $\beta(-\Delta)^{2m} u_t$, $\beta(-\Delta)^{2m} v_t\,(\beta \ge 0)$ are strong dissipative terms. Assumption of rigid and nonlinear source term will be presented at the back, to get the equation of the long time behavior of some theoretical results.

Kirchhoff equation is an important nonlinear wave equation. In 1883, when studying the free vibration of elastic strings, Kirchhoff [1] proposed a physical model

$$\rho h \frac{\partial^2 u}{\partial^2 t} + \delta u_t = P_0 + \frac{Eh}{2L}\left(\int_0^L |u_x|^2 \mathrm{d}x\right) + f(x), 0 < x < L, t > 0.$$

where t is the time variable, E is the elastic modulus, h is the cross-sectional area, L is the length of the string, ρ is the mass density, P_0 is the initial axial tension, δ is the drag coefficient, $f(x)$ is the external force term, and $u = u(x,t)$ is the transverse displacement of space time t and coordinates x. This equation describes the movement of the elastic rod more accurately than the classical wave equation. Subsequently, many scholars have studied the existence, regularity, decay, and asymptotic behavior of global solutions of Kirchhoff equations with strong damping or dissipative terms.

Masamro [2] studied the initial boundary value problem for a class of Kirchhoff type equations with dissipative and damping terms

$$\begin{cases} u_{tt} - M\left(\|\nabla u\|^2\right)\Delta u + \delta |u|^\rho u + \gamma u_t = f(x), x \in \Omega, t > 0, \\ u(x,t) = 0, x \in \partial\Omega, t \ge 0, \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), x \in \Omega. \end{cases}$$

By using Galerkin's method, the existence of global solution of the equation under initial boundary value condition is proved, where $\Omega \subset R^n$ is a bounded domain with smooth Boundary $\partial\Omega$, and $\delta > 0$, $\alpha \ge 0$, $\forall \gamma \ge 0$, $M(\gamma) \in C^1[0,\infty)$.

In reference [3], the initial boundary value problem of high-order strongly damped Kirchhoff equation is understood by studying the paper of Guoguang Lin and Chunmeng Zhou,

$$\begin{cases} u_{tt} + M\left(\|\nabla^m u\|_p^p\right)(-\Delta)^{2m} u + \beta(-\Delta)^{2m} u_t + |u|^\rho u = f(x), \alpha > 0, \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), x \in \Omega, \\ u(x,t) = \dfrac{\partial^j u}{\partial v^j}, j = 1, 2, \cdots, 2m-1, x \in \partial\Omega. \end{cases}$$

where $m > 0$, $p \ge 2$, $\Omega \subset R^n$ is the bounded domain with smooth boundary $\partial\Omega$, $\beta > 0$ is the dissipation coefficient, $\beta(-\Delta)^{2m} u_t$ is the strong dissipative term, $|u|^\rho u$ is the nonlinear term among $\rho \ge -1$, and $f(x)$ is the external term. The existence and uniqueness of the global solution and its continuous dependence on the initial value are proved by Galerkin's method, and the existence and dimension of the global attractor are obtained.

Based on the above references, Guoguang Lin and Lingjuan Hu [4] studied nonlinear coupled Kirchhoff equations with strong damping

$$\begin{cases} u_{tt} + M\left(\|\nabla^m u\|^2 + \|\nabla^m v\|^2\right)(-\Delta)^m u + \beta(-\Delta)^m u_t + g_1(u,v) = f_1(x), \\ v_{tt} + M\left(\|\nabla^m u\|^2 + \|\nabla^m v\|^2\right)(-\Delta)^m v + \beta(-\Delta)^m v_t + g_2(u,v) = f_2(x), \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), x \in \Omega, \\ v(x,0) = v_0(x), v_t(x,0) = v_1(x), x \in \Omega, \\ \dfrac{\partial^i u}{\partial n^i} = 0, \dfrac{\partial^i v}{\partial n^i} = 0, (i = 0,1,2,\cdots,m-1), x \in \partial\Omega. \end{cases}$$

where $\Omega \subseteq R^n (n \geq 1)$ is a bounded domain with a smooth boundary $\partial\Omega$, $g_1(u,v), g_2(u,v)$ are nonlinear source term, $f_1(x), f_2(x)$ is the external force disturbance, $M\left(\|\nabla^m u\|^2 + \|\nabla^m v\|^2\right)(-\Delta)^m u$, $M\left(\|\nabla^m u\|^2 + \|\nabla^m v\|^2\right)(-\Delta)^m v$ are rigid terms which $M\left(\|\nabla^m u\|^2 + \|\nabla^m v\|^2\right)$ is real function, $\beta(-\Delta)^m u_t$, and $\beta(-\Delta)^m v_t\ (\beta \geq 0)$ are strong dissipative terms. The existence and uniqueness of the global solution and its continuous dependence on the initial value are proved by Galerkin's method, and the existence and dimension of the global attractor are obtained.

On the basis of previous studies, this paper further improves the order of the strong dissipative term and the rigid term in Guoguang Lin and Lingjuan Hu [4], where the coefficient of the rigid term is extended from $M\left(\|\nabla^m u\|^2 + \|\nabla^m v\|^2\right)$ to $M\left(\|\nabla^m u\|_p^p + \|\nabla^m v\|^2\right)$.

With the progress of science and technology and the continuous development of mathematical physics equations, since the 1980s, Kirchhoff equations have been applied in lots of many fields such as Newtonian mechanics, ocean acoustics, cosmic physics, especially in engineering physics, measuring bridge vibration has played a huge role. Therefore, more and more scholars began to pay attention to and study the Kirchhoff equation in depth, including the existence and uniqueness of global solutions, a family of the global attractor, Hausdroff dimension and Fractal dimension, the existence of random attractors, energy decay and explosion of solutions, exponential attractors and inertia manifold, etc. And the relevant specific theoretical basis and research results can be found in the literature [5]-[17].

The main research ideas of this paper are that the existence and uniqueness of solution is verified by Galerkin's method in E_0 and E_k space. Then the solution semigroup $S(t)$ is defined, and the bounded absorptive set B_k is obtained on the basis of prior estimation. Through using Rellich-Kondrachov compact embedding theorem, it is proved that the solution semigroup $S(t)$ has the family of the global attractors A_k in space E_k. Finally, by linearizing the equation, it is proved that the solution semigroup $S(t)$ is Frechet differentiable on E_k. So the family of global attractors A_k has finite Hausdroff dimension and Fractal dimension and $d_H(A_k)<\frac{3}{7}N, d_F(A_k)<\frac{6}{7}N$.

2. Existence and Uniqueness of Solutions

The following symbols and assumptions are introduced for the convenience of statement:

$$V_0=L^2(\Omega), V_{2m}=H^{2m}(\Omega)\cap H_0^1(\Omega), V_{2m+k}=H^{2m+k}(\Omega)\cap H_0^1(\Omega),$$
$$V_k=H^k(\Omega)\cap H_0^1(\Omega), E_0=V_{2m}\times V_0\times V_{2m}\times V_0,$$
$$E_k=V_{2m+k}\times V_k\times V_{2m+k}\times V_k\ (k=1,2,\cdots,2m).$$

The inner product of the $L^2(\Omega)$ space is $(u,v)=\int_\Omega u(x)v(x)\mathrm{d}x$ and the norm is $\|u\|=\|u\|_{L^2}=\left(\int_\Omega |u(x)|^2\mathrm{d}x\right)^{\frac{1}{2}}$, the norm of $L^p(\Omega)$ space is called $\|u\|_p=\|u\|_{L^p(\Omega)}$, A_k are the family of global attractors, B_k is the bounded absorption set of E_k, where $k=1,2,\cdots,2m$. C_i and $C(R_l)(l=1,2,\cdots,k)$ are constant.

$w=u_t+\varepsilon u, q=v_t+\varepsilon v$, where u,v is the solution of the problem (1)-(5);

$\bar{u}=u_1-u_2, \bar{v}=v_1-v_2$, where u_1,u_2,v_1,v_2 are the two solutions of the problem (1)-(5);

$W=U_t+\varepsilon U, Q=V_t+\varepsilon V$, where U,V is the solution of the linear initial boundary value problem (66);

$y=\underline{u}-u-U$ and $z=\underline{v}-v-V$ is the solution of the problem (67)-(68).

The rigid term and nonlinear source term are assumed as follows:

(H1) $M(s)=M\left(\|\nabla^m u\|_p^p+\|\nabla^m v\|^2\right)$ and $M(s)\in C^2(R^+)$;

(H2) $m^*\ge M(s)\ge m_*\ge 1$、 $\mu=\begin{cases} m_* & \frac{\mathrm{d}}{\mathrm{d}t}\|\nabla^{2m}u\|\ge 0 \\ m^* & \frac{\mathrm{d}}{\mathrm{d}t}\|\nabla^{2m}u\|\le 0 \end{cases}$;

(H3) $g_1(u_t,v_t), g_2(u_t,v_t)\in C^k\ (k=1,2,\cdots,2m)$,
$(g_1(u_t,v_t),w)+(g_2(u_t,v_t),q)\ge\alpha\left(\|w\|^2+\|q\|^2\right)-\varepsilon\left(\|u\|^2+\|v\|^2\right)-\gamma\left(\|u_t\|^2+\|v_t\|^2\right)$, where γ is related to α and ε;

(H4) $\frac{\beta}{2}-\frac{\gamma}{\lambda_1^{2m}}\ge 0$, $\frac{\beta}{2}-\frac{C_0}{2\varepsilon\lambda_1^{2m}}\ge 0$;

(H5)
$$0<\varepsilon<\min\left\{\frac{1}{3}\alpha, \frac{-\beta\lambda_1^{2m}+\sqrt{\left(\beta\lambda_1^{2m}\right)^2-16\left(1-m_*\lambda_1^{2m}\right)}}{2}, \frac{4m_*\lambda_1^{2m}}{\beta\lambda_1^{2m}-2}, \frac{-(4-C_0)+\sqrt{(4-C_0)^2+\beta\lambda_1^{2m}}}{2}\right\}.$$

Lemma 1. Assuming (H1)-(H5) are true, letting $(u_0,w_0,v_0,q_0)\in E_0$,

$f_1(x), f_2(x) \in L^2(\Omega)$, then there is a solution (u,w,v,q) for problem (1)-(5), which has the following properties:

(i) $(u,w,v,q) \in L^\infty((0,+\infty);E_0)$;

(ii) $\|(u,w,v,q)\|_{E_0} = \|\nabla^{2m}u\|^2 + \|w\|^2 + \|\nabla^{2m}v\|^2 + \|q\|^2 \le \overline{y(0)}e^{-\alpha_0 t} + \frac{C_0}{2\alpha_0\varepsilon}$

$\frac{\beta}{2}\int_0^T \|\nabla^{2m}w\|^2 + \|\nabla^{2m}q\|^2 \, dt \le \frac{C_0}{2\varepsilon}T + \overline{y(0)}$;

(iii) There are normal numbers $C(R_0)$ and $t \ge t_0$, such that

$$\|(u,w,v,q)\|_{E_0}^2 = \|\nabla^{2m}u\|^2 + \|w\|^2 + \|\nabla^{2m}v\|^2 + \|q\|^2 \le C(R_0),$$

where $\overline{y(0)} = \left(\mu + \frac{\varepsilon\beta}{2}\right)\left(\|\nabla^{2m}u_0\|^2 + \|w_0\|^2 + \|\nabla^{2m}v_0\|^2 + \|q_0\|^2\right)$.

Proof: Let $w = u_t + \varepsilon u$ inner product with Equation (1),

$$\left(u_{tt} + M\left(\|\nabla^m u\|_p^p + \|\nabla^m v\|^2\right)(-\Delta)^{2m} u + \beta(-\Delta)^{2m} u_t + g_1(u_t,v_t), w\right) = (f_1(x), w). \quad (6)$$

Some items are treated as follows:

$$(u_{tt}, w) = \frac{1}{2}\frac{d}{dt}\|w\|^2 - \varepsilon\|w\|^2 + \varepsilon^2(u,w), \quad (7)$$

$$\begin{aligned}\left(M(s)(-\Delta)^{2m} u, w\right) &= \frac{M(s)}{2}\frac{d}{dt}\|\nabla^{2m}u\|^2 + \varepsilon M(s)\|\nabla^{2m}u\|^2 \\ &\ge \frac{\mu}{2}\frac{d}{dt}\|\nabla^{2m}u\|^2 + \varepsilon m_*\|\nabla^{2m}u\|^2,\end{aligned} \quad (8)$$

$$\begin{aligned}\left(\beta(-\Delta)^{2m} u_t, w\right) &= \left(\frac{\beta}{2}\nabla^{2m}u_t, \nabla^{2m}u_t + \varepsilon\nabla^{2m}u\right) + \frac{\beta}{2}\left(\nabla^{2m}w - \varepsilon\nabla^{2m}u, \nabla^{2m}w\right) \\ &= \frac{\beta}{2}\|\nabla^{2m}u_t\|^2 + \frac{\varepsilon\beta}{4}\frac{d}{dt}\|\nabla^{2m}u\|^2 + \frac{\beta}{2}\|\nabla^{2m}w\|^2 - \frac{\beta\varepsilon}{2}\left(\nabla^{2m}u, \nabla^{2m}w\right),\end{aligned} \quad (9)$$

$$(g_1(u_t,v_t), w) + (g_2(u_t,v_t), q) \ge \alpha\left(\|w\|^2 + \|q\|^2\right) - \varepsilon\left(\|u\|^2 + \|v\|^2\right) - \gamma\left(\|u_t\|^2 + \|v_t\|^2\right). \quad (10)$$

Similarly, letting $q = v_t + \varepsilon v$ inner product with Equation (2),

$$\left(v_{tt} + M\left(\|\nabla^m u\|_p^p + \|\nabla^m v\|^2\right)(-\Delta)^{2m} v + \beta(-\Delta)^{2m} v_t + g_2(u_t,v_t),\ q\right) = (f_2(x), q). \quad (11)$$

The treatment of each item is similar to (7)-(10), and the above results are sorted out,

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left[\left(\mu_1 + \frac{\varepsilon\beta}{2}\right)\left(\|\nabla^{2m}u\|^2 + \|\nabla^{2m}v\|^2\right) + \|w\|^2 + \|q\|^2\right] + (\alpha - \varepsilon)\left(\|w\|^2 + \|q\|^2\right) \\ &+\varepsilon m_*\left(\|\nabla^{2m}u\|^2 + \|\nabla^{2m}v\|^2\right) + \varepsilon^2\left[(u,w) + (v,q)\right] - \varepsilon\left(\|u\|^2 + \|v\|^2\right) \\ &+\frac{\beta}{2}\left(\|\nabla^{2m}u_t\|^2 + \|\nabla^{2m}v_t\|^2\right) - \gamma\left(\|u_t\|^2 + \|v_t\|^2\right) + \frac{\beta}{2}\left(\|\nabla^{2m}w\|^2 + \|\nabla^{2m}q\|^2\right) \\ &-\frac{\beta\varepsilon}{2}\left[\left(\nabla^{2m}u, \nabla^{2m}w\right) + \left(\nabla^{2m}v, \nabla^{2m}q\right)\right] \le (f_1(x), w) + (f_2(x), q).\end{aligned} \quad (12)$$

Some terms are treated as follows by using Holder's inequality, Young's inequality and Poincare's inequality

$$\varepsilon^2\left[(u,w)+(v,q)\right]-\varepsilon\left(\|u\|^2+\|v\|^2\right)\geq-\varepsilon\left(\|w\|^2+\|q\|^2\right)-\left(\frac{\varepsilon^3}{4}+\varepsilon\right)\left(\|u\|^2+\|v\|^2\right), \quad (13)$$

$$-\left(\frac{\varepsilon^3}{4}+\varepsilon\right)\left(\|u\|^2+\|v\|^2\right)\geq-\frac{1}{\lambda_1^{2m}}\left(\frac{\varepsilon^3}{4}+\varepsilon\right)\left(\|\nabla^{2m}u\|^2+\|\nabla^{2m}v\|^2\right), \quad (14)$$

$$-\gamma\left(\|u_t\|^2+\|v_t\|^2\right)\geq-\frac{\gamma}{\lambda_1^{2m}}\left(\|\nabla^{2m}u_t\|^2+\|\nabla^{2m}v_t\|^2\right), \quad (15)$$

$$\begin{aligned}&-\frac{\beta\varepsilon}{2}\left[\left(\nabla^{2m}u,\nabla^{2m}w\right)+\left(\nabla^{2m}v,\nabla^{2m}q\right)\right]\\&\geq-\frac{\beta}{4}\left(\|\nabla^{2m}w\|^2+\|\nabla^{2m}q\|^2\right)-\frac{\beta\varepsilon^2}{4}\left(\|\nabla^{2m}u\|^2+\|\nabla^{2m}v\|^2\right),\end{aligned} \quad (16)$$

$$\left(f_1(x),w\right)+\left(f_2(x),q\right)\leq\varepsilon\left(\|w\|^2+\|q\|^2\right)+\frac{1}{4\varepsilon}\left(\|f_1(x)\|^2+\|f_2(x)\|^2\right). \quad (17)$$

Insert the inequality (13)-(17) into Equation (12), and get

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\left(\mu+\frac{\varepsilon\beta}{2}\right)\left(\|\nabla^{2m}u\|^2+\|\nabla^{2m}v\|^2\right)+\|w\|^2+\|q\|^2\right]+(\alpha-3\varepsilon)\left(\|w\|^2+\|q\|^2\right)\\&+\left(\varepsilon m_*-\frac{\beta\varepsilon^2}{4}-\frac{\varepsilon^3+4\varepsilon}{4\lambda_1^{2m}}\right)\left(\|\nabla^{2m}u\|^2+\|\nabla^{2m}v\|^2\right)+\frac{\beta}{4}\left(\|\nabla^{2m}w\|^2+\|\nabla^{2m}q\|^2\right)\\&+\left(\frac{\beta}{2}-\frac{\gamma}{\lambda_1^{2m}}\right)\left(\|\nabla^{2m}u_t\|^2+\|\nabla^{2m}v_t\|^2\right)\leq\frac{1}{4\varepsilon}\left(\|f_1(x)\|^2+\|f_2(x)\|^2\right),\end{aligned} \quad (18)$$

letting $0<\varepsilon<\min\left\{\frac{1}{3}\alpha,\frac{-\beta\lambda_1^{2m}+\sqrt{\left(\beta\lambda_1^{2m}\right)^2-16\left(1-m_*\lambda_1^{2m}\right)}}{2}\right\}$, $\frac{\beta}{2}-\frac{\gamma}{\lambda_1^{2m}}\geq0$, $\beta\geq0$ and finally the above Equation (18) is simplified as:

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left[\left(\mu+\frac{\varepsilon\beta}{2}\right)\left(\|\nabla^{2m}u\|^2+\|\nabla^{2m}v\|^2\right)+\|w\|^2+\|q\|^2\right]+2(\alpha-3\varepsilon)\left(\|w\|^2+\|q\|^2\right)\\&+2\left(\varepsilon m_*-\frac{\beta\varepsilon^2}{4}-\frac{\varepsilon^3+4\varepsilon}{4\lambda_1^{2m}}\right)\left(\|\nabla^{2m}u\|^2+\|\nabla^{2m}v\|^2\right)+\frac{\beta}{2}\left(\|\nabla^{2m}w\|^2+\|\nabla^{2m}q\|^2\right)\\&\leq\frac{1}{2\varepsilon}\left(\|f_1(x)\|^2+\|f_2(x)\|^2\right),\end{aligned} \quad (19)$$

then (19) is finally simplified as:

$$\frac{\mathrm{d}}{\mathrm{d}t}\overline{y(t)}+\alpha_0\overline{y(t)}+\frac{\beta}{2}\left(\|\nabla^{2m}w\|^2+\|\nabla^{2m}q\|^2\right)\leq\frac{C_0}{2\varepsilon}, \quad (20)$$

where $\overline{y(t)}=\left(\mu+\frac{\varepsilon\beta}{2}\right)\left(\|\nabla^{2m}u\|^2+\|\nabla^{2m}v\|^2\right)+\|w\|^2+\|q\|^2$,

$\alpha_0=\min\left\{2(\alpha-3\varepsilon),\frac{4\varepsilon m_*\lambda_1^{2m}-\beta\varepsilon^2\lambda_1^{2m}-\varepsilon^3+4\varepsilon}{(2\mu+\varepsilon\beta)\lambda_1^{2m}}\right\}$.

By using the Gronwall's inequality,

$$\overline{y(t)}\leq\overline{y(0)}\mathrm{e}^{-\alpha_0 t}+\frac{C_0}{2\varepsilon\alpha_0}, \quad (21)$$

$$\frac{\beta}{2}\int_0^T \left\|\nabla^{2m}w\right\|^2+\left\|\nabla^{2m}q\right\|^2 \mathrm{d}t \le \frac{C_0}{2\varepsilon}T+\overline{y(0)}, \tag{22}$$

where $\overline{y(0)}=\left(\mu+\frac{\varepsilon\beta}{2}\right)\left(\left\|\nabla^{2m}u_0\right\|^2+\left\|w_0\right\|^2+\left\|\nabla^{2m}v_0\right\|^2+\left\|q_0\right\|^2\right)$, so there are normal numbers $C(R_0)=\frac{C_0}{\alpha_0\varepsilon}$ and $t_0=\left|-\frac{1}{\varepsilon}\ln\frac{C}{\left|2\varepsilon\alpha_0\right|\left|\overline{y(0)}\right|}\right|$, when $t>t_0$,

$$\left\|(u,w,v,q)\right\|_{E_0}^2=\left\|\nabla^{2m}u\right\|^2+\left\|w\right\|^2+\left\|\nabla^{2m}v\right\|^2+\left\|q\right\|^2\le\frac{\overline{y(t)}}{\min\left\{\mu+\frac{\varepsilon\beta}{2},1\right\}}\le C(R_0). \tag{23}$$

Lemma 1 is proved.

Lemma 2. Assuming (H1)-(H5) are true, letting $(u_0,w_0,v_0,q_0)\in E_k$, $f_1(x),f_2(x)\in H^k(\Omega)$, $g_1(u_t,v_t),g_2(u_t,v_t)\in C^k$ where $k=1,2,\cdots,2m$, then there is a solution (u,w,v,q) for problem (1)-(5), which has the following properties:

(i) $(u,w,v,q)\in L^\infty\left((0,+\infty);E_k\right)$, $(k=1,2,\cdots,2m)$;

(ii)

$$\left\|(u,w,v,q)\right\|_{E_k}=\left\|\nabla^{2m+k}u\right\|^2+\left\|\nabla^k w\right\|^2+\left\|\nabla^{2m+k}v\right\|^2+\left\|\nabla^k q\right\|^2\le\overline{y_1(0)}\mathrm{e}^{-\alpha_1 t}+\frac{C_2}{2\alpha_1\varepsilon},$$

$$\frac{\beta}{4}\int_0^T\left\|\nabla^{2m+k}w\right\|^2+\left\|\nabla^{2m+k}q\right\|^2\mathrm{d}t\le\frac{C_2}{2\varepsilon}T+\overline{y_1(0)};$$

(iii) There are normal numbers $C(R_k)$ and $t\ge t_{0k}$, such that

$$\left\|(u,w,v,q)\right\|_{E_k}^2=\left\|\nabla^{2m+k}u\right\|^2+\left\|\nabla^k w\right\|^2+\left\|\nabla^{2m+k}v\right\|^2+\left\|\nabla^k q\right\|^2\le C(R_k),$$

where $\overline{y_1(0)}=\left(\mu+\frac{\varepsilon\beta}{2}\right)\left(\left\|\nabla^{2m+k}u_0\right\|^2+\left\|\nabla^k w_0\right\|^2+\left\|\nabla^{2m+k}v_0\right\|^2+\left\|\nabla^k q_0\right\|^2\right)$.

Proof: Let $(-\Delta)^k w=(-\Delta)^k u_t+\varepsilon(-\Delta)^k u$ inner product with Equation (1),

$$\begin{aligned}&\left(u_{tt}+M\left(\left\|D^m u\right\|_p^p+\left\|D^m v\right\|^2\right)(-\Delta)^{2m}u+\beta(-\Delta)^{2m}u_t+g_1(u_t,v_t),(-\Delta)^k w\right)\\&=\left(f_1(x),(-\Delta)^k w\right).\end{aligned} \tag{24}$$

Some items are treated as follows:

$$\left(u_{tt},(-\Delta)^k w\right)=\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\nabla^k w\right\|^2-\varepsilon\left\|\nabla^k w\right\|^2+\varepsilon^2\left(\nabla^k u,\nabla^k w\right), \tag{25}$$

$$\begin{aligned}\left(M(s)(-\Delta)^{2m}u,(-\Delta)^k w\right)&=\frac{M(s)}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\nabla^{2m+k}u\right\|^2+\varepsilon M(s)\left\|\nabla^{2m+k}u\right\|^2\\&\ge\frac{\mu}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\nabla^{2m+k}u\right\|^2+\varepsilon m_*\left\|\nabla^{2m+k}u\right\|^2,\end{aligned} \tag{26}$$

$$\begin{aligned}&\left(\beta(-\Delta)^{2m}u_t,(-\Delta)^k w\right)\\&=\left(\frac{\beta}{2}\Delta^{2m}u_t,\nabla^{2m+k}u_t+\varepsilon\nabla^{2m+k}u\right)+\frac{\beta}{2}\left(\nabla^{2m+k}w-\varepsilon\nabla^{2m+k}u,\nabla^{2m+k}w\right)\\&=\frac{\beta}{2}\left\|\nabla^{2m+k}u_t\right\|^2+\frac{\varepsilon\beta}{4}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\nabla^{2m+k}u\right\|^2+\frac{\beta}{2}\left\|\nabla^{2m+k}w\right\|^2-\frac{\beta\varepsilon}{2}\left(\nabla^{2m+k}u,\nabla^{2m+k}w\right),\end{aligned} \tag{27}$$

$$\begin{aligned}&\left(g_1\left(u_t,v_t\right),(-\Delta)^k w\right)+\left(g_2\left(u_t,v_t\right),(-\Delta)^k q\right)\\&\le\left\|\nabla^k g_1\left(u_t,v_t\right)\right\|\left\|\nabla^k w\right\|+\left\|\nabla^k g_2\left(u_t,v_t\right)\right\|\left\|\nabla^k q\right\|\\&\le C_1\left(\left\|\nabla^k u_t\right\|\left\|\nabla^k w\right\|+\left\|\nabla^k v_t\right\|\left\|\nabla^k q\right\|\right)\\&\le\frac{C_1}{2\lambda_1^{2m}\varepsilon}\left(\left\|\nabla^{2m+k}u_t\right\|^2+\left\|\nabla^{2m+k}v_t\right\|^2\right)+\frac{\varepsilon C_1}{2}\left(\left\|\nabla^k w\right\|^2+\left\|\nabla^k q\right\|^2\right).\end{aligned}\tag{28}$$

Similarly, letting $(-\Delta)^k q=(-\Delta)^k v_t+\varepsilon(-\Delta)^k v$ inner product with Equation (2),

$$\begin{aligned}&\left(v_{tt}+M\left(\left\|\nabla^m u\right\|_p^p+\left\|\nabla^m v\right\|^2\right)(-\Delta)^{2m}v+\beta(-\Delta)^{2m}v_t+g_2\left(u_t,v_t\right),(-\Delta)^k q\right)\\&=\left(f_2(x),(-\Delta)^k q\right).\end{aligned}\tag{29}$$

The treatment of each item is similar to (25)-(28), and the above results are sorted out,

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\left(\mu+\frac{\varepsilon\beta}{2}\right)\left(\left\|\nabla^{2m+k}u\right\|^2+\left\|\nabla^{2m+k}v\right\|^2\right)+\left\|\nabla^k w\right\|^2+\left\|\nabla^k q\right\|^2\right]\\&-\left(\varepsilon+\frac{\varepsilon C_0}{2}\right)\left(\left\|\nabla^k w\right\|^2+\left\|\nabla^k q\right\|^2\right)+\varepsilon m_*\left(\left\|\nabla^{2m+k}u\right\|^2+\left\|\nabla^{2m+k}v\right\|^2\right)\\&+\varepsilon^2\left[\left(\nabla^k u,\nabla^k w\right)+\left(\nabla^k v,\nabla^k q\right)\right]+\left(\frac{\beta}{2}-\frac{C_0}{2\lambda_1^{2m}\varepsilon}\right)\left(\left\|\nabla^{2m+k}u_t\right\|^2+\left\|\nabla^{2m+k}v_t\right\|^2\right)\\&+\frac{\beta}{2}\left(\left\|\nabla^{2m+k}w\right\|^2+\left\|\nabla^{2m+k}q\right\|^2\right)-\frac{\beta\varepsilon}{2}\left[\left(\nabla^{2m+k}u,\nabla^{2m+k}w\right)+\left(\nabla^{2m+k}v,\nabla^{2m+k}q\right)\right]\\&\le\left(f_1(x),(-\Delta)^k w\right)+\left(f_2(x),(-\Delta)^k q\right).\end{aligned}\tag{30}$$

Some terms are treated as follows by using Holder's inequality, Young's inequality and Poincare's inequality

$$\begin{aligned}&-\varepsilon^2\left[\left(\nabla^k u,\nabla^w w\right)+\left(\nabla^k v,\nabla^k q\right)\right]\\&\ge-\frac{\varepsilon^2}{2}\left(\left\|\nabla^k w\right\|^2+\left\|\nabla^k q\right\|^2\right)-\frac{\varepsilon^2}{2}\left(\left\|\nabla^k u\right\|^2+\left\|\nabla^k v\right\|^2\right)\\&\ge-\frac{\varepsilon^2}{2}\left(\left\|\nabla^k w\right\|^2+\left\|\nabla^k q\right\|^2\right)-\frac{\varepsilon^2}{2\lambda_1^{2m}}\left(\left\|\nabla^{2m+k}u\right\|^2+\left\|\nabla^{2m+k}v\right\|^2\right),\end{aligned}\tag{31}$$

$$\begin{aligned}&-\frac{\beta\varepsilon}{2}\left[\left(\nabla^{2m+k}u,\nabla^{2m+k}w\right)+\left(\nabla^{2m+k}v,\nabla^{2m+k}q\right)\right]\\&\ge-\frac{\beta}{4}\left(\left\|\nabla^{2m+k}w\right\|^2+\left\|\nabla^{2m+k}q\right\|^2\right)-\frac{\beta\varepsilon^2}{4}\left(\left\|\nabla^{2m+k}u\right\|^2+\left\|\nabla^{2m+k}v\right\|^2\right),\end{aligned}\tag{32}$$

$$\begin{aligned}&\left(f_1(x),(-\Delta)^k w\right)+\left(f_2(x),(-\Delta)^k q\right)\\&\le\varepsilon\left(\left\|\nabla^k w\right\|^2+\left\|\nabla^k q\right\|^2\right)+\frac{1}{4\varepsilon}\left(\left\|\nabla^k f_1(x)\right\|^2+\left\|\nabla^k f_2(x)\right\|^2\right).\end{aligned}\tag{33}$$

Insert the inequality (31)-(33) into Equation (30), get

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\left(\mu+\frac{\varepsilon\beta}{2}\right)\left(\left\|\nabla^{2m+k}u\right\|^2+\left\|\nabla^{2m+k}v\right\|^2\right)+\left\|\nabla^k w\right\|^2+\left\|\nabla^k q\right\|^2\right]\\&+\left(\varepsilon m_*-\frac{\beta\varepsilon^2}{4}-\frac{\varepsilon^2}{2\lambda_1^{2m}}\right)\left(\left\|\nabla^{2m+k}u\right\|^2+\left\|\nabla^{2m+k}v\right\|^2\right)\end{aligned}$$

$$+\left(\frac{\beta}{2}-\frac{C_1}{2\varepsilon\lambda_1^{2m}}\right)\left(\left\|\nabla^{2m+k}u_t\right\|^2+\left\|\nabla^{2m+k}v_t\right\|^2\right)$$
$$+\left(\frac{\beta\lambda_1^{2m}}{8}-2\varepsilon-\frac{\varepsilon^2-C_1\varepsilon}{2}\right)\left(\left\|\nabla^k w\right\|^2+\left\|\nabla^k q\right\|^2\right) \tag{34}$$
$$+\frac{\beta}{8}\left(\left\|\nabla^{2m+k}w\right\|^2+\left\|\nabla^{2m+k}q\right\|^2\right)\le\frac{1}{4\varepsilon}\left(\left\|\nabla^k f_1(x)\right\|^2+\left\|\nabla^k f_2(x)\right\|^2\right),$$

assuming $0<\varepsilon<\min\left\{\frac{4m_*\lambda_1^{2m}}{\beta\lambda_1^{2m}-2},\frac{-(4-C_1)+\sqrt{(4-C_1)^2+\beta\lambda_1^{2m}}}{2}\right\}$,

$\frac{\beta}{2}-\frac{C_1}{2\varepsilon\lambda_1^{2m}}\ge 0$, $\beta\ge 0$ and finally the above Equation (18) is simplified as:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\left(\mu+\frac{\varepsilon\beta}{2}\right)\left(\left\|\nabla^{2m+k}u\right\|^2+\left\|\nabla^{2m+k}v\right\|^2\right)+\left\|\nabla^k w\right\|^2+\left\|\nabla^k q\right\|^2\right]$$
$$+2\left(\varepsilon m_*-\frac{\beta\varepsilon^2}{4}-\frac{\varepsilon^2}{2\lambda_1^{2m}}\right)\left(\left\|\nabla^{2m+k}u\right\|^2+\left\|\nabla^{2m+k}v\right\|^2\right)$$
$$+2\left(\frac{\beta\lambda_1^{2m}}{8}-2\varepsilon-\frac{\varepsilon^2-C_1\varepsilon}{2}\right)\left(\left\|\nabla^k w\right\|^2+\left\|\nabla^k q\right\|^2\right) \tag{35}$$
$$+\frac{\beta}{4}\left(\left\|\nabla^{2m+k}w\right\|^2+\left\|\nabla^{2m+k}q\right\|^2\right)\le\frac{1}{2\varepsilon}\left(\left\|\nabla^k f_1(x)\right\|^2+\left\|\nabla^k f_2(x)\right\|^2\right),$$

then (35) is finally simplified as:

$$\frac{\mathrm{d}}{\mathrm{d}t}\overline{y_1(t)}+\alpha_1\overline{y_1(t)}+\frac{\beta}{4}\left(\left\|\nabla^{2m+k}w\right\|^2+\left\|\nabla^{2m+k}q\right\|^2\right)\le\frac{C_2}{2\varepsilon}, \tag{36}$$

where $\overline{y_1(t)}=\left(\mu+\frac{\varepsilon\beta}{2}\right)\left(\left\|\nabla^{2m+k}u\right\|^2+\left\|\nabla^{2m+k}v\right\|^2\right)+\left\|\nabla^k w\right\|^2+\left\|\nabla^k q\right\|^2$,

$\alpha_1=\min\left\{\frac{4\lambda_1^{2m}\varepsilon m_*-\beta\varepsilon^2\lambda_1^{2m}-\varepsilon^2}{(2\mu+\varepsilon\beta)\lambda_1^{2m}},2\left(\frac{\beta\lambda_1^{2m}}{8}-2\varepsilon-\frac{\varepsilon^2-C_1\varepsilon}{2}\right)\right\}$.

By using the Gronwall's inequality,

$$\overline{y_1(t)}\le\overline{y_1(0)}\mathrm{e}^{-\alpha_1 t}+\frac{C_2}{2\varepsilon\alpha_1}, \tag{37}$$

$$\frac{\beta}{4}\int_0^T\left\|\nabla^{2m+k}w\right\|^2+\left\|\nabla^{2m+k}q\right\|^2\mathrm{d}t\le\frac{C_2}{2\varepsilon}T+\overline{y_1(0)}, \tag{38}$$

where $\overline{y_1(0)}=\left(\mu+\frac{\varepsilon\beta}{2}\right)\left(\left\|\nabla^{2m+k}u_0\right\|^2+\left\|\nabla^k w_0\right\|^2+\left\|\nabla^{2m+k}v_0\right\|^2+\left\|\nabla^k q_0\right\|^2\right)$, so there are normal numbers $C(R_k)=\frac{C_2}{\varepsilon\alpha_1}$ and $t_{0k}=\left|-\frac{1}{\alpha_1}\ln\frac{C_2}{\left|2\varepsilon\alpha_1\right|\left|\overline{y_1(0)}\right|}\right|$, when $t\ge t_{0k}$,

$$\left\|(u,w,v,q)\right\|_{\mathrm{E}_k}^2=\left\|\nabla^{2m+k}u\right\|^2+\left\|\nabla^k w\right\|^2+\left\|\nabla^{2m+k}v\right\|^2+\left\|\nabla^k q\right\|^2$$
$$\le\frac{\overline{y_1(t)}}{\min\left\{\mu+\frac{\varepsilon\beta}{2},1\right\}}\le C(R_k) \tag{39}$$

Lemma 2 is proved.

Theorem 1. Assuming (H$_1$)-(H$_5$) is true, $(u_0,w_0,v_0,q_0)\in E_k$, $f_1(x),f_2(x)\in H^k(\Omega)$, $g_1(u_t,v_t),g_2(u_t,v_t)\in C^2$, then the initial boundary value problem (1)-(5) has a unique solution

$$(u(x,t),w(x,t),v(x,t),q(x,t))\in L^\infty((0,+\infty);\mathrm{E}_k)\quad (k=0,1,2,\cdots,2m).$$

Proof: First, the existence of the solution is proved by Galerkin's method:

Step 1: Approximate solution

$(-\Delta)^{2m+k}\omega_j=\lambda_j^{2m+k}\omega_j\,(k=0,1,2,\cdots,2m)$, where λ_j represents the eigenvalue of $(-\Delta)$ with homogeneous Dirichlet boundary on Ω, ω_j represents the eigenfunction determined by corresponding eigenvalue λ_j, and $\omega_1,\omega_2,\cdots,\omega_h,\cdots$ constitute the normal orthonormal basis of $H^{2m+k}(\Omega)$.

Assuming $u_h=u_h(t)=\sum_{j=1}^{h}g_{jh}(t)\omega_j$, $v_h=v_h(t)=\sum_{j=1}^{h}f_{jh}(t)\omega_j$ are approximate solutions of the initial boundary value problem (1)-(5), it is obvious that (u_h,v_h) is dense in $H^{2m+k}\times H^{2m+k}$, where $g_{jh}(t)$, $f_{jh}(t)$ is determined by the following conditions

$$\begin{cases}\left(u_h''+M\left(\left\|\nabla^m u_h\right\|_p^p+\left\|\nabla^m v_h\right\|^2\right)(-\Delta)^{2m}u_h+\beta(-\Delta)^{2m}u_h'+g_1(u_h',v_h'),(-\Delta)^k\omega_j\right)=\left(f_1(x),(-\Delta)^k\omega_j\right), & (40)\\ \left(v_h''+M\left(\left\|\nabla^m u_h\right\|_p^p+\left\|\nabla^m v_h\right\|^2\right)(-\Delta)^{2m}v_h+\beta(-\Delta)^{2m}v_h'+g_2(u_h',v_h'),(-\Delta)^k\omega_j\right)=\left(f_2(x),(-\Delta)^k\omega_j\right), & (41)\end{cases}$$

where $j=1,2,\cdots,h$. And the nonlinear ordinary differential Equations (40) and (41) satisfy the initial conditions

$$u_h(0)=u_{0h}=\sum_{j=1}^{h}\alpha'_{jh}\omega_j\to u_0,(h\to\infty)\ \text{ in }\ H_0^{2m+k}(\Omega)\cap L^2(\Omega),\qquad(42)$$

$$u_h'(0)=u_{1h}=\sum_{j=1}^{h}\beta'_{jh}\omega_j\to u_1,(h\to\infty)\ \text{ in }\ H_0^{k}(\Omega)\cap L^2(\Omega),\qquad(43)$$

$$v_h(0)=v_{0h}=\sum_{j=1}^{h}\alpha''_{jh}\omega_j\to v_0,(h\to\infty)\ \text{ in }\ H_0^{2m+k}(\Omega)\cap L^2(\Omega),\qquad(44)$$

$$v_h'(0)=v_{1h}=\sum_{j=1}^{h}\beta''_{jh}\omega_j\to v_1,(h\to\infty)\ \text{ in }\ H_0^{k}(\Omega)\cap L^2(\Omega),\qquad(45)$$

so when $h\to\infty$, $(u_{oh},u_{1h},v_{oh},u_{1h})\to(u_o,u_1,v_o,u_1)$ is in E_k.

The general results of nonlinear ordinary differential equations guarantee the existence of approximate solutions on the interval $[0,t_h]$.

Step 2: prior estimation

Multiply both sides of the Equation (40) by $g'_{jh}(t)+\varepsilon g_{jh}(t)$ and sum over *j*; in the same way, multiply both sides of the Equation (41) by $f'_{jh}(t)+\varepsilon f_{jh}(t)$ and sum over *j*. Let $w_h(t)=u_h'(t)+\varepsilon u_h(t)$, $q_h(t)=v_h'(t)+\varepsilon v_h(t)$, and according to the prior estimation in Lemma 2, can be obtained

$$\left\|(u_h,w_h,v_h,q_h)\right\|_{E_k}=\left\|\nabla^{2m+k}u_h\right\|^2+\left\|\nabla^k w_h\right\|^2+\left\|\nabla^{2m+k}v_h\right\|^2+\left\|\nabla^k q_h\right\|^2\le C(R_k),\qquad(46)$$

$$\frac{\beta}{4}\int_0^T\left\|\nabla^{2m+k}w\right\|^2+\left\|\nabla^{2m+k}q\right\|^2\mathrm{d}t\le\frac{C_2}{2\varepsilon}T+\overline{y_1(0)}.\qquad(47)$$

According to (46), (u_h, w_h, v_h, q_h) is bounded in $L^\infty([0,+\infty);E_k)$ and it can be seen from (47) that w_h is bounded in $L^2((0,T);H_0^{2m+k})$ and q_h is bounded in $L^2((0,T);H_0^{2m+k})$.

Step 3: Limit process

According to the Danford-Pttes theorem, $L^\infty([0,+\infty);E_k)$ is conjugate to $L^1([0,+\infty);E'_k)$, $L^2((0,T);H_0^{2m+k})$ is conjugate to $L^2((0,T);H_0^{2m+k'})$, and we can pick subsequence $\{u_s\}$ from sequence $\{u_h\}$ and subsequence $\{v_s\}$ from sequence $\{v_h\}$, such that

$$(u_s, w_s, v_s, q_s) \to (u, w, v, q) \text{ is weak * convergence in } L^\infty([0,+\infty);E_k). \quad (48)$$

By the Rellich-Kondrachov compact embedding theorem, E_k is compactly embedding in E_0, $(u_s, w_s, v_s, q_s) \to (u, w, v, q)$ is strong convergence in $L^2([0,+\infty);E_0)$ almost everywhere.

In Equations (40) and (41) above, letting $h = s$, and taking the limit, for fixed j and $s > j$, get

$$\begin{cases} \left(u''_s + M\left(\|\nabla^m u_s\|_p^p + \|\nabla^m v_s\|^2\right)(-\Delta)^{2m} u_s + \beta(-\Delta)^{2m} u'_s + g_1(u'_s, v'_s), (-\Delta)^k \omega_j\right) = \left(f_1(x), (-\Delta)^k \omega_j\right), & (49) \\ \left(v''_s + M\left(\|\nabla^m u_s\|_p^p + \|\nabla^m v_s\|^2\right)(-\Delta)^{2m} v_s + \beta(-\Delta)^{2m} v'_s + g_2(u'_s, v'_s), (-\Delta)^k \omega_j\right) = \left(f_2(x), (-\Delta)^k \omega_j\right). & (50) \end{cases}$$

According to Equation (48) above, it can be concluded that

$$\left(u_s(t), (-\Delta)^k \omega_j\right) \to \left(u(t), \lambda_j^k \omega_j\right) \text{ is weak * convergence in } L^\infty[0,+\infty),$$

$$\left(u'_s(t), (-\Delta)^k \omega_j\right) \to \left(u'(t), \lambda_j^k \omega_j\right) \text{ is weak * convergence in } L^\infty[0,+\infty).$$

So $\left(u''_s(t), (-\Delta)^k \omega_j\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left(u'_s(t), (-\Delta)^k \omega_j\right) \to \left(u''(t), \lambda_j^k \omega_j\right)$ converges in $D'[0,+\infty)$, and similarly

$\left(v''_s(t), (-\Delta)^k \omega_j\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left(v'_s(t), (-\Delta)^k \omega_j\right) \to \left(v''(t), \lambda_j^k \omega_j\right)$ converges in $D'[0,+\infty)$, where $D'[0,+\infty)$ is the dual space of $D[0,+\infty)$;

$$\left(M\left(\|\nabla^m u_s\|_p^p + \|\nabla^m v_s\|^2\right)(-\Delta)^{2m} u_s, (-\Delta)^k \omega_j\right) \to \left(M\left(\|\nabla^m u\|_p^p + \|\nabla^m v\|^2\right)(\nabla)^{2m+k} u, \lambda_j^{\frac{2m+k}{2}} \omega_j\right)$$
is weak * convergence in $L^\infty[0,+\infty)$, similarly
$$\left(M\left(\|\nabla^m u_s\|_p^p + \|\nabla^m v_s\|^2\right)(-\Delta)^{2m} v_s, (-\Delta)^k \omega_j\right) \to \left(M\left(\|\nabla^m u\|_p^p + \|\nabla^m v\|^2\right)(\nabla)^{2m+k} v, \lambda_j^{\frac{2m+k}{2}} \omega_j\right)$$
is weak * convergence in $L^\infty[0,+\infty)$;

$$\begin{aligned} &\left(\beta(-\Delta)^{2m} u'_s, (-\Delta)^k \omega_j\right) \\ &= \left(\beta(-\Delta)^{2m} w_s, (-\Delta)^k \omega_j\right) - \left(\beta(-\Delta)^{2m} \varepsilon u_s, (-\Delta)^k \omega_j\right) \\ &= \left(\beta(-\Delta)^{\frac{k}{2}} w_s, (-\Delta)^{2m+\frac{k}{2}} \omega_j\right) - \beta\varepsilon\left((-\Delta)^{\frac{2m+k}{2}} u_s, (-\Delta)^{\frac{2m+k}{2}} \omega_j\right), \end{aligned}$$

so $\left(\beta(-\Delta)^{2m}u_s',(-\Delta)^k\omega_j\right)\to\beta\left((-\Delta)^{\frac{k}{2}}w,\lambda_j^{2m+\frac{k}{2}}\omega_j\right)-\beta\varepsilon\left((-\Delta)^{\frac{2m+k}{2}}u,\lambda_j^{\frac{2m+k}{2}}\omega_j\right)$ is weak * convergence in $L^\infty[0,+\infty)$, similarly $\left(\beta(-\Delta)^{2m}v_s',(-\Delta)^k\omega_j\right)\to\beta\left((-\Delta)^{\frac{k}{2}}q,\lambda_j^{2m+\frac{k}{2}}\omega_j\right)-\beta\varepsilon\left((-\Delta)^{\frac{2m+k}{2}}v,\lambda_j^{\frac{2m+k}{2}}\omega_j\right)$ is weak * convergence in $L^\infty[0,+\infty)$; $\left(f_1(x),(-\Delta)^k\omega_j\right)\to\left((-\Delta)^{\frac{k}{2}}f_1(x),\lambda_j^k\omega_j\right)$ is weak * convergence in $L^\infty[0,+\infty)$, similarly $\left(f_2(x),(-\Delta)^k\omega_j\right)\to\left((-\Delta)^{\frac{k}{2}}f_2(x),\lambda_j^k\omega_j\right)$ is weak * convergence in $L^\infty[0,+\infty)$.

So Equations (49) and (50) above can be deduced that

$$\begin{cases}\left(u''+M\left(\|\nabla^m u\|_p^p+\|\nabla^m v\|^2\right)(-\Delta)^{2m}u+\beta(-\Delta)^{2m}u'+g_1(u',v'),(-\Delta)^k\omega_j\right)=\left(f_1(x),(-\Delta)^k\omega_j\right),\\ \left(v''+M\left(\|\nabla^m u\|_p^p+\|\nabla^m v\|^2\right)(-\Delta)^{2m}v+\beta(-\Delta)^{2m}v'+g_2(u',v'),(-\Delta)^k\omega_j\right)=\left(f_2(x),(-\Delta)^k\omega_j\right).\end{cases}$$

It's true for any *j*, and from the density of $\omega_1,\omega_2,\cdots,\omega_h,\cdots$ can obtain

$$\begin{cases}\left(u''+M\left(\|\nabla^m u\|_p^p+\|\nabla^m v\|^2\right)(-\Delta)^{2m}u+\beta(-\Delta)^{2m}u'+g_1(u',v'),(-\Delta)^k w\right)=\left(f_1(x),(-\Delta)^k w\right),\\ \left(v''+M\left(\|\nabla^m u\|_p^p+\|\nabla^m v\|^2\right)(-\Delta)^{2m}v+\beta(-\Delta)^{2m}v'+g_2(u',v'),(-\Delta)^k q\right)=\left(f_2(x),(-\Delta)^k q\right),\end{cases}$$

$$\forall w\in H^k\cap H_0^1,\quad \forall q\in H^k\cap H_0^1.$$

The existence is proved.

Then, prove the uniqueness of the solution:

Assuming u_1,u_2,v_1,v_2 are the two solutions of the problem (1)-(5), letting $\overline{u}=u_1-u_2,\overline{v}=v_1-v_2$, from the initial boundary value problem (1)-(5), obtain that

$$\begin{cases}\overline{u_{tt}}+M(s_1)(-\Delta)^{2m}u_1-M(s_2)(-\Delta)^{2m}u_2-\beta(-\Delta)^{2m}\overline{u_t}=g_1(u_{2t},v_{2t})-g_1(u_{1t},v_{1t}),\\ \overline{v_{tt}}+M(s_1)(-\Delta)^{2m}v_1-M(s_2)(-\Delta)^{2m}v_2-\beta(-\Delta)^{2m}\overline{v_t}=g_2(u_{2t},v_{2t})-g_2(u_{1t},v_{1t}),\\ \overline{u}(x,0)=0,\ \overline{u_t}(x,0)=0,\quad x\in\Omega,\\ \overline{v}(x,0)=0,\ \overline{v_t}(x,0)=0,\quad x\in\Omega,\\ \dfrac{\partial^i\overline{u}}{\partial^i n}=0,\ \dfrac{\partial^i\overline{v}}{\partial^i n}=0,\ (i=0,1,2,\cdots,2m-1),\end{cases}\tag{51}$$

where $s_1=M\left(\|\nabla^m u_1\|_p^p+\|\nabla^m v_1\|^2\right),s_2=M\left(\|\nabla^m u_2\|_p^p+\|\nabla^m v_2\|^2\right)$.

Let $\overline{u_t}$ inner product with the first equation of the above (51) and obtain

$$\begin{aligned}&\left(\overline{u_{tt}}+M(s_1)(-\Delta)^{2m}u_1-M(s_2)(-\Delta)^{2m}u_2-\beta(-\Delta)^{2m}\overline{u_t},\overline{u_t}\right)\\&=\left(g_1(u_{2t},v_{2t})-g_1(u_{1t},v_{1t}),\overline{u_t}\right).\end{aligned}\tag{52}$$

Some items are treated as follows

$$\left(\overline{u_{tt}},\overline{u_t}\right)=\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\overline{u_t}\right\|, \tag{53}$$

$$\left(\beta(-\Delta)^{2m}\overline{u_t},\overline{u_t}\right)=\beta\left\|\nabla^{2m}\overline{u_t}\right\|^2, \tag{54}$$

$$\begin{aligned}&\left|\left(M(s_1)(-\Delta)^{2m}u_1-M(s_2)(-\Delta)^{2m}u_2,\overline{u_t}\right)\right|\\&=\left|\left(M(s_1)(-\Delta)^{2m}\overline{u},\overline{u_t}\right)+\left(M(s_1)(-\Delta)^{2m}u_2-M(s_2)(-\Delta)^{2m}u_2,\overline{u_t}\right)\right|\\&\geq\frac{\mu}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\nabla^{2m}\overline{u}\right\|^2-\left|\left(M(s_1)-M(s_2)\right)\left((-\Delta)^{2m}u_2,\overline{u_t}\right)\right|.\end{aligned} \tag{55}$$

Part of the above formula (55) is treated as follows

$$\begin{aligned}&\left|\left(M(s_1)-M(s_2)\right)\left((-\Delta)^{2m}u_2,\overline{u_t}\right)\right|\\&\leq M'(\chi)\left|\left\|\nabla^m u_1\right\|_p^p-\left\|\nabla^m u_2\right\|_p^p+\left\|\nabla^m v_1\right\|^2-\left\|\nabla^m v_2\right\|^2\right|\left\|\nabla^{2m}u_2\right\|\left\|\nabla^{2m}\overline{u_t}\right\|\\&\leq M'(\chi)\left|\left\|\nabla^m u_1\right\|_p^p-\left\|\nabla^m u_2\right\|_p^p\right|\left\|\nabla^{2m}u_2\right\|\left\|\nabla^{2m}\overline{u_t}\right\|\\&\quad+M'(\chi)\left|\left\|\nabla^m v_1\right\|^2-\left\|\nabla^m v_2\right\|^2\right|\left\|\nabla^{2m}u_2\right\|\left\|\nabla^{2m}\overline{u_t}\right\|,\end{aligned} \tag{56}$$

where $\chi\in(s_1,s_2)$; by using Holder's inequality, Young's inequality and Poincare's inequality, (56) is treated as follows,

$$\begin{aligned}&M'(\chi)\left|\left\|\nabla^m u_1\right\|_p^p-\left\|\nabla^m u_2\right\|_p^p\right|\left\|\nabla^{2m}u_2\right\|\left\|\nabla^{2m}\overline{u_t}\right\|\\&\leq pM'(\chi)\left\|\chi_1\right\|_{p-1}^{p-1}\left\|\nabla^m\overline{u}\right\|_\infty\left\|\nabla^{2m}u_2\right\|\left\|\nabla^{2m}\overline{u_t}\right\|\\&\leq C_3\frac{1}{\lambda_1^{\frac{m}{2}}}\left\|\nabla^{2m}\overline{u}\right\|\left\|\nabla^{2m}\overline{u_t}\right\|\leq\frac{C_3}{2\varepsilon\lambda_1^m}\left\|\nabla^{2m}\overline{u}\right\|^2+\frac{\varepsilon C_3}{2}\left\|\nabla^{2m}\overline{u_t}\right\|^2,\end{aligned} \tag{57}$$

where $\chi_1\in\left(\nabla^m u_1,\nabla^m u_2\right)$,

$$\begin{aligned}&M'(\chi)\left|\left\|\nabla^m v_1\right\|^2-\left\|\nabla^m v_2\right\|^2\right|\left\|\nabla^{2m}u_2\right\|\left\|\nabla^{2m}\overline{u_t}\right\|\\&\leq M'(\chi)\left|\left\|\nabla^m v_1\right\|-\left\|\nabla^m v_2\right\|\right|\left\|\nabla^m\overline{v}\right\|\left\|\nabla^{2m}u_2\right\|\left\|\nabla^{2m}\overline{u_t}\right\|\\&\leq C_4\frac{1}{\lambda_1^{\frac{m}{2}}}\left\|\nabla^m\overline{v}\right\|\left\|\nabla^{2m}\overline{u_t}\right\|\leq\frac{C_4}{2\varepsilon\lambda_1^m}\left\|\nabla^{2m}\overline{v}\right\|^2+\frac{\varepsilon C_4}{2}\left\|\nabla^{2m}\overline{u_t}\right\|^2.\end{aligned} \tag{58}$$

Through the (56)-(58), finally (55) will become

$$\begin{aligned}&\left|\left(M(s_1)(-\Delta)^{2m}u_1-M(s_2)(-\Delta)^{2m}u_2,\overline{u_t}\right)\right|\\&\geq\frac{\mu}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\nabla^{2m}\overline{v}\right\|^2-\frac{C_3}{2\varepsilon\lambda_1^m}\left\|\nabla^{2m}\overline{u}\right\|^2-\frac{C_4}{2\varepsilon\lambda_1^m}\left\|\nabla^{2m}\overline{v}\right\|^2-\frac{\varepsilon\left(C_3+C_4\right)}{2}\left\|\nabla^{2m}\overline{u_t}\right\|^2.\end{aligned} \tag{59}$$

Then, according to the mean value theorem, the nonlinear source term is treated as follows,

$$\begin{aligned}&\left|\left(g_1(u_{2t},v_{2t})-g_1(u_{1t},v_{1t}),\overline{u_t}\right)\right|\\&\leq\left|\left(g_1(u_{2t},v_{2t})-g_1(u_{2t},v_{1t}),\overline{u_t}\right)\right|+\left|\left(g_1(u_{2t},v_{1t})-g_1(u_{1t},v_{1t}),\overline{u_t}\right)\right|\end{aligned}$$

$$
\begin{aligned}
&= \left| g_1'\left(u_{2_t}, \chi_2\right)\left(-\overline{v_t}, \overline{u_t}\right) \right| + \left| g_1'\left(\chi_3, v_{1_t}\right)\left(-\overline{u_t}, \overline{u_t}\right) \right| \\
&\le \left\| g_1'\left(u_{2_t}, \chi_2\right) \right\|_\infty \left\| \overline{v_t} \right\| \left\| \overline{u_t} \right\| + \left\| g_1'\left(\chi_3, v_{1_t}\right) \right\|_\infty \left\| \overline{u_t} \right\|^2 \\
&\le C_5 \left(\frac{\varepsilon}{2} \left\| \overline{v_t} \right\|^2 + \frac{1}{2\varepsilon} \left\| \overline{u_t} \right\|^2 \right) + C_6 \left\| \overline{u_t} \right\|^2 .
\end{aligned} \tag{60}
$$

Similarly, Let $\overline{v_t}$ inner product with the second equation of the above (51) and obtain

$$
\begin{aligned}
&\left(\overline{v_{tt}} + M\left(s_1\right)(-\Delta)^{2m} v_1 - M\left(s_2\right)(-\Delta)^{2m} v_2 - \beta(-\Delta)^{2m} \overline{v_t}, \overline{v_t} \right) \\
&= \left(g_2\left(u_{2t}, v_{2t}\right) - g_2\left(u_{1t}, v_{1t}\right), \overline{v_t} \right).
\end{aligned} \tag{61}
$$

The treatment of some items in (61) is similar to that (53)-(59), and the above results are sorted out that

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} \left(\mu \left(\left\| \nabla^{2m} \overline{u} \right\|^2 + \left\| \nabla^{2m} \overline{v} \right\|^2 \right) + \left\| \overline{u_t} \right\|^2 + \left\| \overline{v_t} \right\|^2 \right) \\
&+ \left(2\beta + \left(C_3 + C_4\right)\varepsilon \right) \left(\left\| \nabla^{2m} \overline{u_t} \right\|^2 + \left\| \nabla^{2m} \overline{v_t} \right\|^2 \right) \\
&\le \frac{2C_7}{\varepsilon \lambda_1^{2m}} \left(\left\| \nabla^m \overline{u} \right\|^2 + \left\| \nabla^m \overline{u} \right\|^2 \right) + \left(2C_8 + \frac{C_8 + \varepsilon^2 C_8}{\varepsilon} \right) \left(\left\| \overline{u_t} \right\|^2 + \left\| \overline{v_t} \right\|^2 \right),
\end{aligned} \tag{62}
$$

where $C_7 = \max\left\{C_4, C_3\right\}, C_8 = \max\left\{C_5, C_6\right\}$.

So

$$
\frac{\mathrm{d}}{\mathrm{d}t} \overline{y_2(t)} \le \alpha_2 \overline{y_2(t)} \tag{63}
$$

where $\overline{y_2(t)} = \mu \left(\left\| \nabla^{2m} \overline{u} \right\|^2 + \left\| \nabla^{2m} \overline{v} \right\|^2 \right) + \left\| \overline{u_t} \right\|^2 + \left\| \overline{v_t} \right\|^2$,

$\alpha_2 = \max\left\{ \frac{2C_7}{\varepsilon \lambda_1^{2m}}, \left(\frac{2C_8}{\mu} + \frac{C_8 + \varepsilon^2 C_8}{\varepsilon \mu} \right) \right\}$, according to Gronwall's inequality

$$
\overline{y_2(t)} \le \overline{y_2(0)} \mathrm{e}^{\alpha_2 t}, \tag{64}
$$

where

$$
\overline{y_2(0)} = 0, \tag{65}
$$

then $\mu \left(\left\| \nabla^{2m} \overline{u} \right\|^2 + \left\| \nabla^{2m} \overline{v} \right\|^2 \right) + \left\| \overline{u_t} \right\|^2 + \left\| \overline{v_t} \right\|^2 = 0$, so

$\left\| \nabla^{2m} \overline{u} \right\| = \left\| \nabla^{2m} \overline{v} \right\| = \left\| \overline{u_t} \right\| = \left\| \overline{v_t} \right\| = 0$.

Theorem 1 is proved.

3. The Family of Global Attractors and Dimension Estimation

Theorem 2. [7] Assume E is a Banach space, and $\left\{S(t)\right\}_{t\ge 0}$ is the operator semigroup on E,

$$
S(t): E \to E, \quad S(t+r) = S(t) + S(r) \; (\forall t, r > 0), \quad S(0) = I
$$

where I is the identity operator, if $S(t)$ satisfies

1) Semigroup $S(t)$ is uniformly bounded in *E*, *i.e.* $\forall R>0$, exists a constant $C(R)$ such that when $\|u\|_E \le R$, *T*, there is $\|S(t)u\|_E \le C(R)$ $(\forall t\in[0,\infty))$;

2) There exists a bounded absorbing set B_0 in *E*, that is, for any bounded set $B\subset E$, there exists a constant $t_0>0$, such that $S(t)B\subset B_0$ $(\forall t\ge t_0)$;

3) $\{S(t)\}_{t\ge0}$ is completely continuous operator;

then operator semigroup $S(t)$ has compact global attractor A.

In theorem 2, if $S(t)$ is a solution semigroup generated by the initial boundary value problem (1)-(5), $(u(t),w(t),v(t),q(t))=S(t)(u_0,w_0,v_0,q_0)$, and Banach space *E* is changed into Hilbert space E_k, there are the family of global attractors.

Theorem 3. Let $S(t)$ is a solution semigroup generated by the initial boundary value problems (1)-(5) under the hypothesis of lemma 1 and lemma 2, then the initial boundary value problems (1)-(5) have the family of global attractors. There are compact sets satisfying:

$$A_k\subset E_k\subset E_0\ \ (k=1,2,\cdots,2m),\ \text{and}\ \ A_k=\omega(B_k)=\bigcap_{s\ge0}\overline{\bigcup_{t\ge s}S(t)B_k},$$

where $B_k=\left\{(u,w,v,q)\in E_k:\|\nabla^{2m+k}u\|^2+\|\nabla^k w\|^2+\|\nabla^{2m+k}v\|^2+\|\nabla^k q\|^2\le C(R_k)\right\}$,

1) $S(t)A_k=A_k$;

2) A_k attracts all bounded sets of E_k, that is, any bounded set $B_k\subset E_k$, having

$$\lim_{t\to\infty}dist(S(t)B_k,A_k)=0,\ \text{where}\ \ dist(S(t)B_k,A_k)=\sup_{x\in B_K}\inf_{y\in A_k}\|S(t)x-y\|_{E_k};$$

then compact set A_k are called the family of global attractors of semigroup $S(t)$.

Proof: Verify the three conditions in theorem 2 to prove the existence of family of global attractors, under the condition of theorem 1, and the initial boundary value problems (1)-(5) generate solution semigroups $S(t):E_k\to E_k$

1) So for any bounded set $B_k\subset E_k$, having

$$\|S(t)(u_0,w_0,v_0,q_0)\|_{E_k}^2=\|\nabla^{2m+k}u\|^2+\|\nabla^k w\|^2+\|\nabla^{2m+k}v\|^2+\|\nabla^k q\|^2\le C(R_k),$$

where $t\ge0$ and $(u_0,w_0,v_0,q_0)\in B_k$, shows that $\{S(t)\}_{t\ge0}$ is uniformly bounded in E_k;

2) $\forall(u_0,w_0,v_0,q_0)\in E_k$, when $t\ge\max\{t_0,t_{0k}\}$, there is

$$\|S(t)(u_0,w_0,v_0,q_0)\|_{E_k}^2=\|\nabla^{2m+k}u\|^2+\|\nabla^k w\|^2+\|\nabla^{2m+k}v\|^2+\|\nabla^k q\|^2\le C(R_k),$$

thus B_k is a bounded absorption set of semigroup $S(t)$;

3) E_k is compactly embedded in E_0, *i.e.*, the bounded set in E_k is a compact set in E_0, so the operator semigroup $S(t)$ is completely continuous operator.

Theorem 3 is proved.

After the family of global attractors is obtained, in order to estimate the Hausdroff dimension and Fractal dimension of the family of global attractors, the initial boundary value problem (1)-(5) is linearized and obtain that

$$\begin{cases} U_{tt}+M'\left(\left\|\nabla^m u\right\|_p^p+\left\|\nabla^m v\right\|^2\right)\left(\left(\left\|\nabla^m u\right\|_p^p\right)'\nabla^m U+\left(\left\|\nabla^m v\right\|^2\right)'\nabla^m V\right)(-\Delta)^{2m}u \\ +M\left(\left\|\nabla^m u\right\|_p^p+\left\|\nabla^m v\right\|^2\right)(-\Delta)^{2m}U+\beta(-\Delta)^{2m}U_t+g_{1u_t}(u_t,v_t)U_t+g_{1v_t}(u_t,v_t)V_t=0, \\ V_{tt}+M'\left(\left\|D^m u\right\|_p^p+\left\|D^m v\right\|^2\right)\left(\left(\left\|\nabla^m u\right\|_p^p\right)'\nabla^m U+\left(\left\|\nabla^m v\right\|^2\right)'\nabla^m V\right)(-\Delta)^{2m}v \\ +M\left(\left\|\nabla^m u\right\|_p^p+\left\|\nabla^m v\right\|^2\right)(-\Delta)^{2m}V+\beta(-\Delta)^{2m}V_t+g_{2u_t}(u_t,v_t)U_t+g_{2v_t}(u_t,v_t)V_t=0, \\ \left.\frac{\partial U^i(x,t)}{\partial n^i}\right|_{\partial\Omega}=0,\left.\frac{\partial V^i(x,t)}{\partial n^i}\right|_{\partial\Omega}=0, i=0,1,2,\cdots,2m-1, t\ge 0, \\ U(x,0)=m_1, U_t(x,0)=m_2, \\ V(x,0)=n_1, V_t(x,0)=n_2, \end{cases} \tag{66}$$

where $U_t+\varepsilon U=W$, $V_t+\varepsilon V=Q$, $(m_1,m_2,n_1,n_2)\in E_k$, and $(u,w,v,q)=S(t)(u_0,w_0,v_0,q_0)$ is the solution of the initial boundary value problem (1)-(5). Given $(u_0,w_0,v_0,q_0)\in A_k$, $S(t):E_k\to E_k$, for any $(m_1,m_2,n_1,n_2)\in E_k$, there exists a unique solution $(U(t),W(t),V(t),Q(t))\in L^\infty([0,\infty);E_k)$ to the linear initial boundary value problem (66).

Lemma 3. For any $t>0$, $C(R_k)>0$, the mapping $S(t):E_k\to E_k$ is Frechet differentiable. The derivative on $\boldsymbol{\varphi}=(u,w,v,q)^{\mathrm{T}}$ is a linear operator on E_k,

$$L:(m_1,m_2,n_1,n_2)^{\mathrm{T}}\to(U(t),W(t),V(t),Q(t))^{\mathrm{T}},$$

where $(U(t),W(t),V(t),Q(t))$ is the solution of the problem (66).

Proof: suppose $\boldsymbol{\varphi}_0=(u_0,w_0,v_0,q_0)^{\mathrm{T}}\in E_k$, $\underline{\boldsymbol{\varphi}_0}=(u_0+m_1,w_0+m_2,v_0+n_1,q_0+n_2)^{\mathrm{T}}\in E_k$, where $\|\boldsymbol{\varphi}_0\|_{E_k}\le C(R_k)$, $\|\underline{\boldsymbol{\varphi}_0}\|_{E_k}\le C(R_k)$, definition $(u,w,v,q)^{\mathrm{T}}=S(t)\boldsymbol{\varphi}_0$, $(\underline{u},\underline{w},\underline{v},\underline{q})^{\mathrm{T}}=S(t)\underline{\boldsymbol{\varphi}_0}$. From this, we can obtain the Lipchitz property of $S(t)$ on the bounded set E_k, that is

$$\left\|S(t)\underline{\boldsymbol{\varphi}_0}-S(t)\boldsymbol{\varphi}_0\right\|_{E_k}^2\le \mathrm{e}^{\kappa t}\left\|(m_1,m_2,n_1,n_2)\right\|_{E_k}^2,$$

where κ is arbitrary constant.

Let $y=\underline{u}-u-U, z=\underline{v}-v-V$ is the solution of the following problem (67)-(68),

$$\begin{cases} y_{tt}+M\left(\left\|\nabla^m u\right\|_p^p+\left\|\nabla^m v\right\|^2\right)(-\Delta)^{2m}y+\beta(-\Delta)^{2m}y_t=h_1, & (67) \\ z_{tt}+M\left(\left\|\nabla^m u\right\|_p^p+\left\|\nabla^m v\right\|^2\right)(-\Delta)^{2m}z+\beta(-\Delta)^{2m}z_t=h_2, & (68) \\ y(x,0)=0, y(x,0)=0, x\in\Omega, \\ z(x,0)=0, z(x,0)=0, x\in\Omega, \\ \frac{\partial^i y}{\partial^i n}=0,\frac{\partial^i z}{\partial^i n}=0, i=0,1,2,\cdots,2m-1, \end{cases}$$

where letting $h_1 = h_{11} + h_{12}, h_2 = h_{21} + h_{22}$,

$$\begin{aligned} h_{11} &= M\left(\left\|\nabla^m u\right\|_p^p + \left\|\nabla^m v\right\|^2\right)(-\Delta)^{2m} \underline{u} - M\left(\left\|\nabla^m \underline{u}\right\|_p^p + \left\|\nabla^m \underline{v}\right\|^2\right)(-\Delta)^{2m} \underline{u} \\ &+ M'\left(\left\|\nabla^m u\right\|_p^p + \left\|\nabla^m v\right\|^2\right)\left(\left(\left\|\nabla^m u\right\|_p^p\right)' \nabla^m U + \left(\left\|\nabla^m v\right\|^2\right)' \nabla^m V\right)(-\Delta)^{2m} u, \end{aligned} \tag{69}$$

$$h_{12} = -g_1\left(\underline{u_t}, \underline{v_t}\right) + g_1\left(u_t, v_t\right) + g_{1u_t}\left(u_t, v_t\right)U_t + g_{1v_t}\left(u_t, v_t\right)V_t, \tag{70}$$

$$\begin{aligned} h_{21} &= M\left(\left\|\nabla^m u\right\|_p^p + \left\|\nabla^m v\right\|^2\right)(-\Delta)^{2m} \underline{v} - M\left(\left\|D^m \underline{u}\right\|_p^p + \left\|D^m \underline{v}\right\|^2\right)(-\Delta)^{2m} \underline{v} \\ &+ M'\left(\left\|D^m u\right\|_p^p + \left\|D^m v\right\|_p^p\right)\left(\left(\left\|\nabla^m u\right\|_p^p\right)' \nabla^m U + \left(\left\|\nabla^m v\right\|^2\right)' \nabla^m V\right)(-\Delta)^{2m} v, \end{aligned} \tag{71}$$

$$h_{22} = -g_2\left(\underline{u_t}, \underline{v_t}\right) + g_2\left(u_t, v_t\right) + g_{2u_t}\left(u_t, v_t\right)U_t + g_{2v_t}\left(u_t, v_t\right)V_t. \tag{72}$$

Let $(-\Delta)^k y_t$ inner product with Equation (69) and obtain,

$$\left(y_{tt} + M\left(\left\|\nabla^m u\right\|_p^p + \left\|\nabla^m v\right\|^2\right)(-\Delta)^{2m} y + \beta(-\Delta)^{2m} y_t, (-\Delta)^k y_t\right) = \left(h_1, (-\Delta)^k y_t\right). \tag{73}$$

Some items are treated as follows

$$\left(y_{tt}, (-\Delta)^k y_t\right) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\nabla^k y_t\right\|^2, \tag{74}$$

$$\left(M\left(\left\|\nabla^m u\right\|_p^p + \left\|\nabla^m v\right\|^2\right)(-\Delta)^{2m} y, (-\Delta)^k y_t\right) = \frac{\mu}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\nabla^{2m+k} y\right\|^2, \tag{75}$$

$$\left(\beta(-\Delta)^{2m} y_t, (-\Delta)^k y_t\right) = \beta\left\|\nabla^{2m+k} y_t\right\|^2. \tag{76}$$

First deal with Equation (69), for convenience, and the following symbols are introduced

$$\overline{s} = \nabla^m u, \underline{s} = \nabla^m \underline{u}, \overline{t} = D^m v, \underline{t} = D^m \underline{v}, \tilde{u} = u - \underline{u}, \tilde{v} = v - \underline{v},$$

$$N\left(\overline{s}\right) = M'\left(\left\|\overline{s}\right\|_p^p + \left\|\overline{t}\right\|^2\right)\left(\left\|\overline{s}\right\|_p^p\right)', \quad G\left(\overline{t}\right) = M'\left(\left\|\overline{s}\right\|_p^p + \left\|\overline{t}\right\|^2\right)\left(\left\|\overline{t}\right\|^2\right),$$

$$B\left(\underline{t}\right) = M'\left(\left\|\underline{s}\right\|_p^p + \left\|\underline{t}\right\|^2\right)\left(\left\|\underline{t}\right\|^2\right)',$$

$$\begin{aligned} h_{11} &= \left(M\left(\left\|\overline{s}\right\|_p^p + \left\|\overline{t}\right\|^2\right) - M\left(\left\|\underline{s}\right\|_p^p + \left\|\underline{t}\right\|^2\right)\right)(-\Delta)^{2m} \underline{u} \\ &\quad + M'\left(\left\|\overline{s}\right\|_p^p + \left\|\overline{t}\right\|^2\right)\left(\left(\left\|\overline{s}\right\|_p^p\right)' \nabla^m U + \left(\left\|\overline{t}\right\|^2\right)' \nabla^m V\right)(-\Delta)^{2m} u \\ &= h_{111} + h_{112}, \end{aligned} \tag{77}$$

where

$$\begin{aligned} h_{111} &= \left(M\left(\left\|\overline{s}\right\|_p^p + \left\|\overline{t}\right\|^2\right) - M\left(\left\|\underline{s}\right\|_p^p + \left\|\overline{t}\right\|^2\right)\right)(-\Delta)^{2m} \underline{u} \\ &\quad + M'\left(\left\|\overline{s}\right\|_p^p + \left\|\overline{t}\right\|^2\right)\left(\left\|\overline{s}\right\|_p^p\right)' \nabla^m U (-\Delta)^{2m} u, \end{aligned} \tag{78}$$

$$\begin{aligned} h_{112} &= \left(M\left(\left\|\underline{s}\right\|_p^p + \left\|\overline{t}\right\|^2\right) - M\left(\left\|\underline{s}\right\|_p^p + \left\|\underline{t}\right\|^2\right)\right)(-\Delta)^{2m} \underline{u} \\ &\quad + M'\left(\left\|\overline{s}\right\|_p^p + \left\|\overline{t}\right\|^2\right)\left(\left\|\overline{t}\right\|^2\right)' \nabla^m V (-\Delta)^{2m} u. \end{aligned} \tag{79}$$

Next, h_{111}, h_{112} is processed as follows

$$
\begin{aligned}
h_{111} &= M'\left(\|\chi_4\|_p^p + \|\overline{t}\|^2\right)\left(\|\chi_4\|_p^p\right)' \nabla^m \tilde{u}(-\Delta)^{2m} \underline{u} \\
&\quad + M'\left(\|\overline{s}\|_p^p + \|\overline{t}\|^2\right)\left(\|\overline{s}\|_p^p\right)' \nabla^m U(-\Delta)^{2m} u \\
&= N(\chi_4)\nabla^m \tilde{u}(-\Delta)^{2m} \underline{u} + N(\overline{s})\nabla^m U(-\Delta)^{2m} u \\
&= N(\chi_4)\nabla^m \tilde{u}(-\Delta)^{2m} u - N(\chi_4)\nabla^m \tilde{u}(-\Delta)^{2m} \tilde{u} \\
&\quad - N(\overline{s})\nabla^m \tilde{u}(-\Delta)^{2m} u - N(\overline{s})\nabla^m y(-\Delta)^{2m} u \\
&= N'(\chi_5)\left(\nabla^m \tilde{u}\right)^2 (-\Delta)^{2m} u(1-\theta_1) - N(\chi_4)\nabla^m \tilde{u}(-\Delta)^{2m} \tilde{u} \\
&\quad - N(\overline{s})\nabla^m y(-\Delta)^{2m} u,
\end{aligned}
\tag{80}
$$

$$
\begin{aligned}
h_{112} &= M'\left(\|\underline{s}\|_p^p + \|\chi_6\|^2\right)\left(\|\chi_6\|^2\right)' \nabla^m \tilde{v}(-\Delta)^{2m} \underline{u} \\
&\quad + M'\left(\|\overline{s}\|_p^p + \|\overline{t}\|^2\right)\left(\|\overline{t}\|^2\right)' \nabla^m V(-\Delta)^{2m} u \\
&= B(\chi_6)\nabla^m \tilde{v}(-\Delta)^{2m} \underline{u} + G(\overline{t})\nabla^m V(-\Delta)^{2m} u \\
&= -B(\chi_6)\nabla^m V(-\Delta)^{2m} \underline{u} - B(\chi_6)\nabla^m z(-\Delta)^{2m} \underline{u} + G(\overline{t})\nabla^m V(-\Delta)^{2m} u,
\end{aligned}
\tag{81}
$$

where $\chi_4 = \theta_1 \overline{s} + (1-\theta_1)\underline{s}, \chi_5 = \theta_2 \chi_4 + (1-\theta_2)\overline{s}, \chi_6 = \theta_3 \overline{t} + (1-\theta_3)\underline{t}$, $\theta_1, \theta_2, \theta_3 \in (0,1)$.

By using Holder's inequality, Young's inequality and Poincare's inequality, $\left((-\Delta)^k y_t, h_{11}\right)$ is treated as follows

$$
\begin{aligned}
\left((-\Delta)^k y_t, h_{11}\right) &= \left((-\Delta)^k y_t, h_{111} + h_{112}\right) \\
&\le C_9 \left\|\nabla^{2m+k}\tilde{u}\right\|^2 \frac{1}{\lambda_1^{\frac{k}{2}}}\left\|\nabla^k y_t\right\| + C_{10}\left\|\nabla^k y_t\right\| \frac{1}{\lambda_1^{\frac{k-m}{2}}}\left\|\nabla^{2m+k}\tilde{u}\right\|^2 \\
&\quad + C_{11}\frac{1}{\lambda_1^{\frac{m}{2}}}\left\|\nabla^{2m+k} y\right\|\left\|\nabla^k y_t\right\| + C_{12}\frac{1}{\lambda_1^{\frac{m}{2}}}\left\|\nabla^{2m+k} z\right\|\left\|\nabla^k y_t\right\| \\
&\le \left(\frac{C_9}{2\lambda_1^k} + \frac{(C_{10}+C_{11}+C_{12})\varepsilon}{2}\right)\left\|\nabla^k y_t\right\|^2 + \frac{C_{11}}{2\lambda_1^m \varepsilon}\left\|\nabla^{2m+k} y\right\|^2 \\
&\quad + \frac{C_{12}}{2\lambda_1^m \varepsilon}\left\|\nabla^{2m+k} z\right\|^2 + \left(\frac{C_9}{2} + \frac{C_{10}}{2\lambda_1^{k-m}\varepsilon}\right)\left\|\nabla^{2m+k}\tilde{u}\right\|^4.
\end{aligned}
\tag{82}
$$

The following processing is carried out through the mean value theorem

$$
\begin{aligned}
h_{12} &= g_1(u_t, v_t) - g_1(\underline{u_t}, \underline{v_t}) + g_{1u_t}(u_t, v_t) U_t + g_{1v_t}(u_t, v_t) V_t \\
&= g_1(u_t, v_t) - g_1(u_t, \underline{v_t}) + g_1(u_t, \underline{v_t}) - g_1(\underline{u_t}, \underline{v_t}) + \frac{\partial g_1(u_t, v_t)}{\partial u_t} U_t + \frac{\partial g_1(u_t, v_t)}{\partial v_t} V_t \\
&\le \frac{\partial g_1(u_t, \chi_7)}{\partial \chi_7}\tilde{v}_t + \frac{\partial g_1(\chi_8, \underline{v_t})}{\partial \chi_8}\tilde{u}_t + \frac{\partial g_1(u_t, v_t)}{\partial u_t} U_t + \frac{\partial g_1(u_t, v_t)}{\partial v_t} V_t \\
&\le \left(\frac{\partial g_1(u_t, \chi_7)}{\partial \chi_7} - \frac{\partial g_1(u_t, v_t)}{\partial v_t}\right)\tilde{v}_t + \frac{\partial g_1(u_t, v_t)}{\partial v_t}(\tilde{v}_t + V_t) + \left(\frac{\partial g_1(\chi_8, \underline{v_t})}{\partial \chi_8}\right.
\end{aligned}
$$

$$-\frac{\partial g_1(u_t,\underline{v_t})}{\partial u_t}\Bigg)\tilde{u}_t+\left(\frac{\partial g_1(u_t,\underline{v_t})}{\partial u_t}-\frac{\partial g_1(u_t,v_t)}{\partial u_t}\right)\tilde{u}_t+\frac{\partial g_1(u_t,v_t)}{\partial u_t}(\tilde{u}_t+U_t)$$

$$\le\frac{\partial^2 g_1(u_t,\chi_9)}{\partial\chi_9^2}(1-\theta_4)(\tilde{v}_t)^2+\frac{\partial g_1(u_t,v_t)}{\partial v_t}z_t+\frac{\partial^2 g_1(\chi_{10},\underline{v_t})}{\partial\chi_{10}^2}(1-\theta_5)(\tilde{u}_t)^2 \quad (83)$$

$$+\frac{\partial^2 g_1(u_t,\chi_{11})}{\partial\chi_{11}^2}\tilde{v}_t\tilde{u}_t+\frac{\partial g_1(u_t,v_t)}{\partial u_t}y_t,$$

where

$$\chi_7=\theta_4 v_t+(1-\theta_4)\underline{v_t},\chi_8=\theta_5 u_t+(1-\theta_5)\underline{u_t},\chi_9=\theta_6\chi_7+(1-\theta_6)v_t,$$

$$\chi_{10}=\theta_7\chi_8+(1-\theta_7)u_t,\chi_{11}=\theta_8\underline{v_t}+(1-\theta_8)v_t,\theta_4,\cdots,\theta_8\in(0,1).$$

By using Holder's inequality, Young's inequality and Poincare's inequality, $\left((-\Delta)^k y_t,h_{12}\right)$ is treated as follows

$$\begin{aligned}&\left((-\Delta)^k y_t,h_{12}\right)\\&\le\left\|\frac{\partial^2 g_1(u_t,\chi_9)}{\partial\chi_9^2}\right\|_\infty(1-\theta_4)\|\nabla^k\tilde{v}_t\|^2\frac{1}{\lambda_1^{\frac{k}{2}}}\|\nabla^k y_t\|+\left\|\frac{\partial g_1(u_t,v_t)}{\partial v_t}\right\|_\infty\|\nabla^k z_t\|\|\nabla^k y_t\|\\&+\left\|\frac{\partial^2 g_1(\chi_{10},\underline{v_t})}{\partial\chi_{10}^2}\right\|_\infty(1-\theta_5)\|\nabla^k\tilde{u}_t\|^2\frac{1}{\lambda_1^{\frac{k}{2}}}\|\nabla^k y_t\|+\left\|\frac{\partial g_1(u_t,v_t)}{\partial u_t}\right\|_\infty\|\nabla^k y_t\|^2\\&+\left\|\frac{\partial^2 g_1(u_t,\chi_{11})}{\partial\chi_{11}^2}\right\|_\infty\frac{1}{\lambda_1^k}\|\nabla^k\tilde{v}_t\|\|\nabla^k\tilde{u}_t\|\\&\le\frac{C_{14}\varepsilon}{2}\|\nabla^k\tilde{v}_t\|^4+\frac{C_{16}\varepsilon}{2}\|\nabla^k\tilde{u}_t\|^4+\left(\frac{C_{14}}{2\lambda_1^k\varepsilon}+\frac{C_{16}}{2\lambda_1^k\varepsilon}+C_{17}+\frac{C_{15}}{2}\right)\|\nabla^k y_t\|^2\\&+\frac{C_{15}}{2}\|\nabla^k z_t\|^2.\end{aligned} \quad (84)$$

Similarly, Let $(-\Delta)^k z_t$ inner product with Equation (68) and obtain,

$$\left(z_{tt}+M\left(\|\nabla^m u\|_p^p+\|\nabla^m v\|^2\right)(-\Delta)^{2m}z+\beta(-\Delta)^{2m}z_t,(-\Delta)^k z_t\right)=\left(h_2,(-\Delta)^k z_t\right). \quad (84)$$

Some items are treated as follows

$$\left(z_{tt},(-\Delta)^k z_t\right)=\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla^k z_t\|^2, \quad (85)$$

$$\left(M\left(\|\nabla^m u\|_p^p+\|\nabla^m v\|^2\right)(-\Delta)^{2m}z,(-\Delta)^k z_t\right)=\frac{\mu}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla^{2m+k}z\|^2, \quad (86)$$

$$\left(\beta(-\Delta)^{2m}z_t,(-\Delta)^k z_t\right)=\beta\|\nabla^{2m+k}z_t\|^2, \quad (87)$$

$$\begin{aligned}\left((-\Delta)^k z_t,h_{21}\right)\le&\left(\frac{C_9}{2\lambda_1^k}+\frac{(C_{10}+C_{11}+C_{12})\varepsilon}{2}\right)\|\nabla^k z_t\|^2+\frac{C_{11}}{2\lambda_1^m\varepsilon}\|\nabla^{2m+k}z\|^2\\&+\frac{C_{12}}{2\lambda_1^m\varepsilon}\|\nabla^{2m+k}y\|^2+\left(\frac{C_9}{2}+\frac{C_{10}}{2\lambda_1^{k-m}\varepsilon}\right)\|\nabla^{2m+k}\tilde{v}\|^4,\end{aligned} \quad (88)$$

$$\left((-\Delta)^k z_t, h_{22}\right) \le \left(\frac{C_{14}}{2\lambda_1^k \varepsilon} + \frac{C_{16}}{2\lambda_1^k \varepsilon} + C_{17} + \frac{C_{15}}{2}\right)\left\|\nabla^k z_t\right\| + \frac{C_{15}}{2}\left\|\nabla^k y_t\right\| + \frac{C_{14}\varepsilon}{2}\left\|\nabla^k \tilde{u}_t\right\|^4 + \frac{C_{16}\varepsilon}{2}\left\|\nabla^k \tilde{v}_t\right\|^4. \tag{89}$$

Based on the above Equations (74)-(89), it is sorted out that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\left\|\nabla^k y_t\right\|^2 + \left\|\nabla^k z_t\right\|^2 + \mu\left(\left\|\nabla^{2m+k} y\right\|^2 + \left\|\nabla^{2m+k} z\right\|^2\right)\right) \\
&+ 2\beta\left(\left\|\nabla^{2m+k} y_t\right\|^2 + \left\|\nabla^{2m+k} z_t\right\|^2\right) \\
&\le \left(\left(C_{10} + C_{11} + C_{13}\right)\varepsilon + \frac{C_{14} + C_{16} + 2\varepsilon C_9}{2\lambda_1^k} + 2\left(C_{15} + C_{17}\right)\right)\left(\left\|\nabla^k y_t\right\|^2 + \left\|\nabla^k z_t\right\|^2\right) \\
&+ \frac{C_{11} + C_{12}}{\varepsilon \lambda_1^m}\left(\left\|\nabla^{2m+k} y\right\|^2 + \left\|\nabla^{2m+k} z\right\|^2\right) + \left(C_9 + \frac{C_{10}}{\varepsilon \lambda_1^{k-m}}\right)\left(\left\|\nabla^{2m+k} \tilde{u}\right\|^4 + \left\|\nabla^{2m+k} \tilde{v}\right\|^4\right) \\
&+ \left(C_{14} + C_{16}\right)\varepsilon\left(\left\|\nabla^k \tilde{u}_t\right\|^4 + \left\|\nabla^k \tilde{v}_t\right\|^4\right),
\end{aligned} \tag{90}$$

finally simplified

$$\frac{\mathrm{d}}{\mathrm{d}t}\overline{y_3(t)} \le \alpha_3 \overline{y_3(t)} + \alpha_4\left(\left\|\nabla^{2m+k}\tilde{u}\right\|^4 + \left\|\nabla^{2m+k}\tilde{v}\right\|^4 + \left\|\nabla^k \tilde{u}_t\right\|^4 + \left\|\nabla^k \tilde{v}_t\right\|^4\right),$$

where $\alpha_3 = \max\left\{\left(C_{10} + C_{11} + C_{13}\right)\varepsilon + \frac{C_{14} + C_{16} + 2\varepsilon C_9}{2\lambda_1^k} + 2\left(C_{15} + C_{17}\right), \frac{C_{11} + C_{12}}{\varepsilon \lambda_1^m \mu}\right\}$,

$$\alpha_4 = \max\left\{C_9 + \frac{C_{10}}{\varepsilon \lambda_1^{k-m}}, \left(C_{14} + C_{16}\right)\varepsilon\right\},$$

$$\overline{y_3(t)} = \left\|\nabla^k y_t\right\|^2 + \left\|\nabla^k z_t\right\|^2 + \mu\left(\left\|\nabla^{2m+k} y\right\|^2 + \left\|\nabla^{2m+k} z\right\|^2\right),$$

according to Gronwall's inequality

$$\begin{aligned}
\overline{y_3(t)} &\le \overline{y_3(0)}\mathrm{e}^{\alpha_3 t} + \mathrm{e}^{\alpha_3}\int_0^t \left\|\nabla^{2m+k}\tilde{u}\right\|^4 + \left\|\nabla^{2m+k}\tilde{v}\right\|^4 + \left\|\nabla^k \tilde{u}_t\right\|^4 + \left\|\nabla^k \tilde{v}_t\right\|^4 \mathrm{d}t \\
&\le C_{18}\mathrm{e}^{\alpha_3 t}\left\|\left(m_1, m_2, n_1, n_2\right)\right\|_{E_k}^4.
\end{aligned}$$

when $\left\|\left(m_1, m_2, n_1, n_2\right)\right\|_{E_k}^2 \to 0$, there is

$$\frac{\left\|S(t)\underline{\varphi_0} - S(t)\varphi_0 - L\left(\left(m_1, m_2, n_1, n_2\right)^{\mathrm{T}}\right)\right\|_{E_k}^2}{\left\|\left(m_1, m_2, n_1, n_2\right)\right\|_{E_k}^2} \le C_{18}\mathrm{e}^{\alpha_3 t}\left\|\left(m_1, m_2, n_1, n_2\right)\right\|_{E_k}^2 \to 0.$$

Lemma 3 is proved.

Theorem 4. Under the condition of Theorem 3, the global attractors of initial boundary value problems (1)-(5) have finite dimensional Hausdroff dimension and fractal dimension, and then $d_H\left(A_k\right) < \frac{3}{7}N, d_F\left(A_k\right) < \frac{6}{7}N$.

Proof: In order to estimate the dimension of the global attractor, the initial boundary value problems (1)-(5) are rewritten as

$$\boldsymbol{\varphi}_t + \boldsymbol{H} * \boldsymbol{\varphi} = \boldsymbol{F}(\boldsymbol{\varphi}), \tag{91}$$

where $\boldsymbol{\varphi}=(u,w,v,q)^{\mathrm{T}}$, $w=u_t+\varepsilon u$, $q=v_t+\varepsilon v$,

$$\boldsymbol{H}=\begin{pmatrix} \varepsilon I & -I & 0 & 0 \\ (1-\beta\varepsilon)(-\Delta)^{2m}+\varepsilon^2 I & \beta(-\Delta)^{2m}-\varepsilon I & 0 & 0 \\ 0 & 0 & \varepsilon I & -I \\ 0 & 0 & (1-\beta\varepsilon)(-\Delta)^{2m}+\varepsilon^2 I & \beta(-\Delta)^{2m}-\varepsilon I \end{pmatrix},$$

$$\boldsymbol{F}(\boldsymbol{\varphi})=\begin{pmatrix} 0 \\ f_1(x)-g_1(u_t,v_t)+\left(1-M\left(\|\nabla^m u\|_p^p+\|\nabla^m v\|^2\right)\right)(-\Delta)^{2m}u \\ 0 \\ f_2(x)-g_2(u_t,v_t)+\left(1-M\left(\|\nabla^m u\|_p^p+\|\nabla^m v\|^2\right)\right)(-\Delta)^{2m}v \end{pmatrix}.$$

Let $L(\boldsymbol{\varphi})=\boldsymbol{\varphi}_t=\boldsymbol{F}(\boldsymbol{\varphi})-\boldsymbol{H}*\boldsymbol{\varphi}$, $\boldsymbol{\psi}_t=L_t(\boldsymbol{\varphi})$.

According to Lemma 3, $L:E_k\to E_k$ is Frechet differentiable, so (91) can be rewritten as

$$\boldsymbol{\psi}_t+\boldsymbol{P}(\boldsymbol{\varphi})*\boldsymbol{\psi}=\boldsymbol{\Gamma}_1(\boldsymbol{\varphi})*\boldsymbol{\psi}+\boldsymbol{\Gamma}_2(\boldsymbol{\varphi})*\boldsymbol{\psi}, \tag{92}$$

where $\boldsymbol{\psi}=(U,W,V,Q)^{\mathrm{T}}$, $W=U_t+\varepsilon U$, $Q=V_t+\varepsilon V$,

$$\boldsymbol{P}(\boldsymbol{\varphi})=\begin{pmatrix} \varepsilon I & -I & 0 & 0 \\ (1-\beta\varepsilon)(-\Delta)^{2m}+\varepsilon^2 I & \beta(-\Delta)^{2m}-\varepsilon I & 0 & 0 \\ 0 & 0 & \varepsilon I & -I \\ 0 & 0 & (1-\beta\varepsilon)(-\Delta)^{2m}+\varepsilon^2 I & \beta(-\Delta)^{2m}-\varepsilon I \end{pmatrix},$$

$$\boldsymbol{\Gamma}_1(\boldsymbol{\varphi})=\begin{pmatrix} 0 & 0 & 0 & 0 \\ \varepsilon g_{1u_t}(u_t,v_t) & -g_{1u_t}(u_t,v_t) & \varepsilon g_{1v_t}(u_t,v_t) & -g_{1v_t}(u_t,v_t) \\ 0 & 0 & 0 & 0 \\ \varepsilon g_{2u_t}(u_t,v_t) & -g_{2u_t}(u_t,v_t) & \varepsilon g_{2v_t}(u_t,v_t) & -g_{2v_t}(u_t,v_t) \end{pmatrix},$$

$$\boldsymbol{\Gamma}_2(\boldsymbol{\varphi})=\begin{pmatrix} 0 & 0 & 0 & 0 \\ (1-M(s))(-\Delta)^{2m}-D_u & 0 & -D_u & 0 \\ 0 & 0 & 0 & 0 \\ D_v & 0 & (1-M(s))(-\Delta)^{2m}-D_V & 0 \end{pmatrix},$$

$$D_uU=M'\left(\|\nabla^m u\|_p^p+\|\nabla^m v\|^2\right)\left(\|\nabla^m u\|_p^p\right)'\nabla^m U(-\Delta)^{2m}u,$$

$$D_uV=M'\left(\|\nabla^m u\|_p^p+\|\nabla^m v\|^2\right)\left(\|\nabla^m v\|^2\right)'\nabla^m V(-\Delta)^{2m}u,$$

$$D_vU=M'\left(\|\nabla^m u\|_p^p+\|\nabla^m v\|^2\right)\left(\|\nabla^m u\|_p^p\right)'\nabla^m U(-\Delta)^{2m}v,$$

$$D_vV=M'\left(\|\nabla^m u\|_p^p+\|\nabla^m v\|^2\right)\left(\|\nabla^m v\|^2\right)'\nabla^m V(-\Delta)^{2m}v,$$

and U,V is the solution of problem (66).

For every fixed $(u,w,v,q)\in E_k$, assume that $d_1,d_2,\cdots,d_N$ are N elements in

E_k, and $\boldsymbol{\psi}_1(t),\boldsymbol{\psi}_2(t),\cdots,\boldsymbol{\psi}_N(t)$ are N solutions of the linearized Equation (92) with an initial value $\boldsymbol{\psi}_1(0)=d_1,\boldsymbol{\psi}_2(0)=d_2,\cdots,\boldsymbol{\psi}_N(0)=d_N$, where N is a natural number.

It can be obtained by calculation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\|\boldsymbol{\psi}_1(t)\wedge\boldsymbol{\psi}_2(t)\wedge\cdots\wedge\boldsymbol{\psi}_N(t)\right\|_{\wedge E_k}^2-2tr\left(L_t\left(\boldsymbol{\varphi}(\tau)\right)Q_N(\tau)\right)\left\|d_1\wedge d_2\wedge\cdots\wedge d_N\right\|=0,$$

where $L_t\left(\boldsymbol{\varphi}(t)\right)=L_t\left(S(t)\boldsymbol{\varphi}_0\right)$ is a linear mapping, $\boldsymbol{\varphi}(t)=S(t)\boldsymbol{\varphi}_0$ is the solution of problems (1)-(5), $\wedge$ represents the outer product, tr represents the trace of the operator, Q_N represents the orthogonal projection from E_k to $span\{\boldsymbol{\psi}_1(t),\boldsymbol{\psi}_2(t),\cdots,\boldsymbol{\psi}_N(t)\}$, and the N dimensional volume $\wedge_{j=1}^N d_j$ is

$$\varpi_N(t)=\sup_{\varphi_0\in A_k}\sup_{d_j\in E_k}\left\|\boldsymbol{\psi}_1(t)\wedge\boldsymbol{\psi}_2(t)\wedge\cdots\wedge\boldsymbol{\psi}_N(t)\right\|_{\wedge^N E_k}^2.$$

Through Gronwall's inequality, can get

$$\begin{aligned}&\left\|\boldsymbol{\psi}_1(t)\wedge\cdots\wedge\boldsymbol{\psi}_N(t)\right\|_{\wedge^N E_k}^2\\&\le\left\|\boldsymbol{\psi}_1(0)\wedge\cdots\wedge\boldsymbol{\psi}_N(0)\right\|_{\wedge^N E_k}^2\exp\int_0^t tr\left(L_t\left(\boldsymbol{\varphi}(\tau)\right)Q_N(\tau)\right)\mathrm{d}\tau.\end{aligned}$$

For any given time τ, suppose $\boldsymbol{h}_j(\tau)=\left(\xi_j(\tau),\zeta_j(\tau),\eta_j(\tau),\sigma_j(\tau)\right)^{\mathrm{T}}$, $j=1,2,\cdots,N$ are the standard orthogonal basis of space $span\{\boldsymbol{\psi}_1(t),\boldsymbol{\psi}_2(t),\cdots,\boldsymbol{\psi}_N(t)\}$, then define the inner product on E_k

$$\begin{aligned}\left(\boldsymbol{h}_j,\overline{\boldsymbol{h}_j}\right)_{E_k}&=\left(\nabla^{2m+k}\xi_j,\nabla^{2m+k}\overline{\xi_j}\right)+\left(\nabla^k\zeta_j,\nabla^k\overline{\zeta_j}\right)\\&\quad+\left(\nabla^{2m+k}\eta_j,\nabla^{2m+k}\overline{\eta_j}\right)+\left(\nabla^k\sigma_j,\nabla^k\overline{\sigma_j}\right),\end{aligned}$$

and norm is

$$\left\|\boldsymbol{h}_j\right\|_{E_k}^2=\left(\boldsymbol{h}_j,\boldsymbol{h}_j\right)_{E_k}=\left\|\nabla^{2m+k}\xi_j\right\|^2+\left\|\nabla^k\zeta_j\right\|^2+\left\|\nabla^{2m+k}\eta_j\right\|^2+\left\|\nabla^k\sigma_j\right\|^2=1. \quad (93)$$

Through the above conditions, can get

$$\begin{aligned}tr\left(L_t\left(\boldsymbol{\varphi}(\tau)\right)Q_N(\tau)\right)&=\sum_{j=1}^N\left(L_t\left(\boldsymbol{\varphi}(\tau)\right)Q_N(\tau)\boldsymbol{h}_j(\tau),\boldsymbol{h}_j(\tau)\right)_{E_k}\\&=\sum_{j=1}^N\left(L_t\left(\boldsymbol{\varphi}(\tau)\right)\boldsymbol{h}_j(\tau),\boldsymbol{h}_j(\tau)\right)_{E_k},\end{aligned}\quad (94)$$

where

$$\begin{aligned}&\left(L_t\left(\boldsymbol{\varphi}(\tau)\right)\boldsymbol{h}_j(\tau),\boldsymbol{h}_j(\tau)\right)_{E_k}\\&=\left(\boldsymbol{\Gamma}_1(\boldsymbol{\varphi})\boldsymbol{h}_j(\tau),\boldsymbol{h}_j(\tau)\right)+\left(\boldsymbol{\Gamma}_2(\boldsymbol{\varphi})\boldsymbol{h}_j(\tau),\boldsymbol{h}_j(\tau)\right)-\left(\boldsymbol{P}(\boldsymbol{\varphi})\boldsymbol{h}_j(\tau),\boldsymbol{h}_j(\tau)\right).\end{aligned}$$

Some terms are treated as follows by using Holder's inequality, Young's inequality and Poincare's inequality

$$\begin{aligned}&\left(\boldsymbol{P}(\boldsymbol{\varphi})\boldsymbol{h}_j(\tau),\boldsymbol{h}_j(\tau)\right)\\&=\varepsilon\left(\nabla^{2m+k}\xi_j,\nabla^{2m+k}\xi_j\right)-\left(\nabla^{2m+k}\zeta_j,\nabla^{2m+k}\xi_j\right)+(1-\beta\varepsilon)\left(\nabla^{2m+k}\xi_j,\nabla^{2m+k}\zeta_j\right)\\&\quad+\varepsilon^2\left(\nabla^k\xi_j,\nabla^k\zeta_j\right)+\beta\left(\nabla^{2m+k}\zeta_j,\nabla^{2m+k}\zeta_j\right)-\varepsilon\left(\nabla^k\zeta_j,\nabla^k\zeta_j\right)\\&\quad+\varepsilon\left(\nabla^{2m+k}\eta_j,\nabla^{2m+k}\eta_j\right)-\left(\nabla^{2m+k}\sigma_j,\nabla^{2m+k}\eta_j\right)+(1-\beta\varepsilon)\left(\nabla^{2m+k}\eta_j,\nabla^{2m+k}\sigma_j\right)\\&\quad+\varepsilon^2\left(\nabla^k\eta_j,\nabla^k\sigma_j\right)+\beta\left(\nabla^{2m+k}\sigma_j,\nabla^{2m+k}\sigma_j\right)-\varepsilon\left(\nabla^k\sigma_j,\nabla^k\sigma_j\right)\end{aligned}$$

$$\begin{aligned}
&\geq \varepsilon\left\|\nabla^{2m+k}\xi_j\right\|^2-(2-\beta\varepsilon)\left\|\nabla^{2m+k}\zeta_j\right\|\left\|\nabla^{2m+k}\xi_j\right\|-\varepsilon^2\left\|\nabla^k\xi_j\right\|\left\|\nabla^k\zeta_j\right\| \\
&\quad+\beta\left\|\nabla^{2m+k}\zeta_j\right\|^2-\varepsilon\left\|\nabla^k\zeta_j\right\|^2+\varepsilon\left\|\nabla^{2m+k}\eta_j\right\|^2-(2-\beta\varepsilon)\left\|\nabla^{2m+k}\sigma_j\right\|\left\|\nabla^{2m+k}\eta_j\right\| \\
&\quad-\varepsilon^2\left\|\nabla^k\eta_j\right\|\left\|\nabla^k\sigma_j\right\|+\beta\left\|\nabla^{2m+k}\sigma_j\right\|^2-\varepsilon\left\|\nabla^k\sigma_j\right\|^2 \\
&\geq\left(\varepsilon-\frac{C_{19}(2-\beta\varepsilon)}{2}\right)\left\|\nabla^{2m+k}\xi_j\right\|^2 \\
&\quad+\left(\beta\lambda_1^{2m}-\frac{\varepsilon^2+2\varepsilon}{2}-\frac{C_{19}(2-\beta\varepsilon)\lambda_1^{2m}}{2}\right)\left\|\nabla^k\zeta_j\right\|^2-\frac{\varepsilon^2}{2}\left\|\nabla^k\xi_j\right\|^2 \\
&\quad+\left(\varepsilon-\frac{C_{20}(2-\beta\varepsilon)}{2}\right)\left\|\nabla^{2m+k}\eta_j\right\|^2 \\
&\quad+\left(\beta\lambda_1^{2m}-\frac{\varepsilon^2+2\varepsilon}{2}-\frac{C_{20}(2-\beta\varepsilon)\lambda_1^{2m}}{2}\right)\left\|\nabla^k\sigma_j\right\|^2-\frac{\varepsilon^2}{2}\left\|\nabla^k\eta_j\right\|^2,
\end{aligned}\tag{95}$$

$$\begin{aligned}
&\left(\Gamma_1(\boldsymbol{\varphi})\boldsymbol{h}_j(\tau),\boldsymbol{h}_j(\tau)\right) \\
&=\varepsilon\left(g'_{1u_t}\nabla^k\xi_j,\nabla^k\zeta_j\right)-\left(g'_{1u_t}\nabla^k\zeta_j,\nabla^k\zeta_j\right)+\varepsilon\left(g'_{1v_t}\nabla^k\eta_j,\nabla^k\zeta_j\right) \\
&\quad-\left(g'_{1v_t}\nabla^k\sigma_j,\nabla^k\zeta_j\right)+\varepsilon\left(g'_{2u_t}\nabla^k\xi_j,\nabla^k\sigma_j\right)-\left(g'_{2u_t}\nabla^k\zeta_j,\nabla^k\sigma_j\right) \\
&\quad+\varepsilon\left(g'_{2v_t}\nabla^k\eta_j,\nabla^k\sigma_j\right)-\left(g'_{2v_t}\nabla^k\sigma_j,\nabla^k\sigma_j\right) \\
&\leq\varepsilon\left\|g'_{1u_t}\right\|_\infty\left\|\nabla^k\xi_j\right\|\left\|\nabla^k\zeta_j\right\|-\left\|g'_{1u_t}\right\|_\infty\left\|\nabla^k\zeta_j\right\|^2+\varepsilon\left\|g'_{1v_t}\right\|_\infty\left\|\nabla^k\eta_j\right\|\left\|\nabla^k\zeta_j\right\| \\
&\quad+\left\|g'_{1v_t}\right\|_\infty\left\|\nabla^k\sigma_j\right\|\left\|\nabla^k\zeta_j\right\|+\varepsilon\left\|g'_{2u_t}\right\|_\infty\left\|\nabla^k\xi_j\right\|\left\|\nabla^k\sigma_j\right\| \\
&\quad+\left\|g'_{2u_t}\right\|_\infty\left\|\nabla^k\zeta_j\right\|\left\|\nabla^k\sigma_j\right\|+\varepsilon\left\|g'_{2v_t}\right\|_\infty\left\|\nabla^k\eta_j\right\|\left\|\nabla^k\sigma_j\right\|-\left\|g'_{2v_t}\right\|_\infty\left\|\nabla^k\sigma_j\right\|^2 \\
&\leq\frac{\varepsilon^2(C_{21}+C_{25})}{2}\left\|\nabla^k\xi_j\right\|^2+\frac{\varepsilon^2(C_{23}+C_{27})}{2}\left\|\nabla^k\eta_j\right\|^2 \\
&\quad+\left(\frac{\varepsilon(C_{21}+C_{23})+C_{24}+C_{26}}{2\varepsilon}-C_{22}\right)\left\|\nabla^k\zeta_j\right\|^2 \\
&\quad+\left(\frac{C_{27}+C_{25}+\varepsilon(C_{26}+C_{24})}{2}-C_{28}\right)\left\|\nabla^k\sigma_j\right\|^2 \\
&\leq\frac{\varepsilon^2(C_{21}+C_{25})}{2}\left\|\nabla^k\xi_j\right\|^2+\frac{\varepsilon^2(C_{23}+C_{27})}{2}\left\|\nabla^k\eta_j\right\|^2 \\
&\quad+(C_{29}-C_{21})\left\|\nabla^k\zeta_j\right\|^2+(C_{30}-C_{28})\left\|\nabla^k\sigma_j\right\|^2,
\end{aligned}\tag{96}$$

where $C_{29}=\frac{\varepsilon(C_{20}+C_{24})+C_{23}+C_{25}}{2\varepsilon},C_{30}=\frac{C_{20}+C_{24}+\varepsilon(C_{25}+C_{23})}{2}$,

$$\begin{aligned}
&\left(\Gamma_2(\boldsymbol{\varphi})\boldsymbol{h}_j(\tau),\boldsymbol{h}_j(\tau)\right) \\
&=(1-M(s))\left(\nabla^{2m+k}\xi_j,\nabla^{2m+k}\zeta_j\right)-M'(s)\left(\left\|\nabla^m u\right\|_p^p\right)'\left((-\Delta)^{2m}u\nabla^{m+k}\xi_j,\nabla^k\zeta_j\right) \\
&\quad-M'(s)\left(\left\|\nabla^m v\right\|^2\right)'\left((-\Delta)^{2m}u\nabla^{m+k}\eta_j,\nabla^k\zeta_j\right)+(1-M(s))\left(\nabla^{2m+k}\eta_j,\nabla^{2m+k}\sigma_j\right) \\
&\quad-M'(s)\left(\left\|\nabla^m u\right\|_p^p\right)'\left((-\Delta)^{2m}v\nabla^{m+k}\xi_j,\nabla^k\sigma_j\right)
\end{aligned}$$

$$
\begin{aligned}
&-M'(s)\left(\left\|\nabla^m v\right\|^2\right)'\left((-\Delta)^{2m} v\nabla^{m+k}\eta_j,\nabla^k\sigma_j\right)\\
&\le\left|\left(1-M(s)\right)\right|C_{27}\left\|\nabla^{2m+k}\xi_j\right\|\lambda_1^m\left\|\nabla^k\zeta_j\right\|+\left|N(\bar{s})\right|\left\|(-\Delta)^{2m}u\right\|\frac{1}{\lambda_1^{\frac{m}{2}}}\left\|\nabla^{2m+k}\xi_j\right\|\left\|\nabla^k\zeta_j\right\|\\
&+\left|G(\bar{t})\right|\left\|(-\Delta)^{2m}u\right\|\frac{1}{\lambda_1^{\frac{m}{2}}}\left\|\nabla^{2m+k}\eta_j\right\|\left\|\nabla^k\zeta_j\right\|+\left|\left(1-M(s)\right)\right|C_{30}\left\|\nabla^{2m+k}\eta_j\right\|\lambda_1^m\left\|\nabla^k\sigma_j\right\|\\
&+\left|N(\bar{s})\right|\left\|(-\Delta)^{2m}v\right\|\frac{1}{\lambda_1^{\frac{m}{2}}}\left\|\nabla^{2m+k}\xi_j\right\|\left\|\nabla^k\sigma_j\right\|\\
&+\left|G(\bar{t})\right|\left\|(-\Delta)^{2m}v\right\|\frac{1}{\lambda_1^{\frac{m}{2}}}\left\|\nabla^{2m+k}\eta_j\right\|\left\|\nabla^k\sigma_j\right\|\\
&\le\left(\frac{\left(m^*-1\right)C_{31}\varepsilon}{2}+\frac{\varepsilon^2C_{32}+C_{35}}{2\varepsilon\lambda_1^m}\right)\left\|\nabla^{2m+k}\xi_j\right\|^2\\
&+\left(\frac{\left(m^*-1\right)C_{34}\varepsilon}{2}+\frac{\varepsilon^2C_{36}+C_{33}}{2\varepsilon\lambda_1^m}\right)\left\|\nabla^{2m+k}\eta_j\right\|^2\\
&+\left(\frac{\left(m^*-1\right)C_{31}\lambda_1^{2m}}{2\varepsilon}+\frac{C_{32}+\varepsilon^2C_{33}}{2\varepsilon}\right)\left\|\nabla^k\zeta_j\right\|^2\\
&+\left(\frac{\left(m^*-1\right)C_{34}\lambda_1^{2m}}{2\varepsilon}+\frac{\varepsilon^2C_{35}+C_{36}}{2\varepsilon}\right)\left\|\nabla^k\sigma_j\right\|^2.
\end{aligned}\tag{97}
$$

Substitute (95)-(97) into (94) to obtain

$$
\begin{aligned}
&\left(L_t\left(\boldsymbol{\varphi}(\tau)\right)\boldsymbol{h}_j(\tau),\boldsymbol{h}_j(\tau)\right)_{E_k}\\
&=\left(\left(-\boldsymbol{P}(\boldsymbol{\varphi})+\boldsymbol{\Gamma}_1(\boldsymbol{\varphi})+\boldsymbol{\Gamma}_2(\boldsymbol{\varphi})\right)\boldsymbol{h}_j,\boldsymbol{h}_j\right)\\
&\le-\left(\varepsilon-\frac{C_{19}(2-\beta\varepsilon)}{2}-\frac{\left(m^*-1\right)C_{31}\varepsilon}{2}-\frac{\varepsilon^2C_{32}+C_{35}}{2\varepsilon\lambda_1^m}\right)\left\|\nabla^{2m+k}\xi_j\right\|^2\\
&-\left(\varepsilon-\frac{C_{20}(2-\beta\varepsilon)}{2}-\frac{\left(m^*-1\right)C_{34}\varepsilon}{2}-\frac{\varepsilon^2C_{36}+C_{33}}{2\varepsilon\lambda_1^m}\right)\left\|\nabla^{2m+k}\eta_j\right\|^2\\
&-\left(\beta\lambda_1^{2m}+C_{22}-C_{29}-\frac{\varepsilon^2+2\varepsilon}{2}-\frac{C_{19}(2-\beta\varepsilon)\lambda_1^{2m}}{2}-\frac{\left(m^*-1\right)C_{31}\lambda_1^{2m}}{2\varepsilon}\right.\\
&\left.-\frac{C_{32}+\varepsilon^2C_{33}}{2\varepsilon}\right)\left\|\nabla^k\zeta_j\right\|^2-\left(\beta\lambda_1^{2m}+C_{28}-C_{30}-\frac{\varepsilon^2+2\varepsilon}{2}\right.\\
&\left.-\frac{C_{20}(2-\beta\varepsilon)\lambda_1^{2m}}{2}-\frac{\left(m^*-1\right)C_{34}\lambda_1^{2m}}{2\varepsilon}-\frac{\varepsilon^2C_{35}+C_{36}}{2\varepsilon}\right)\left\|\nabla^k\sigma_j\right\|^2\\
&+\left(\frac{\varepsilon^2}{2}+\frac{\varepsilon^2\left(C_{21}+C_{25}\right)}{2}\right)\left\|\nabla^k\xi_j\right\|^2+\left(\frac{\varepsilon^2}{2}+\frac{\varepsilon^2\left(C_{23}+C_{27}\right)}{2}\right)\left\|\nabla^k\eta_j\right\|^2\\
&\le-b_1\left\|\nabla^{2m+k}\xi_j\right\|^2-b_2\left\|\nabla^{2m+k}\eta_j\right\|^2-b_3\left\|\nabla^k\zeta_j\right\|^2-b_4\left\|\nabla^k\sigma_j\right\|^2\\
&+a_1\left\|\nabla^k\xi_j\right\|^2+a_2\left\|\nabla^k\eta_j\right\|^2,
\end{aligned}\tag{98}
$$

where

$$b_1 = \varepsilon - \frac{C_{19}(2-\beta\varepsilon)}{2} - \frac{(m^*-1)C_{31}\varepsilon}{2} - \frac{\varepsilon^2 C_{32} + C_{35}}{2\varepsilon\lambda_1^m},$$

$$b_2 = \varepsilon - \frac{C_{20}(2-\beta\varepsilon)}{2} - \frac{(m^*-1)C_{34}\varepsilon}{2} - \frac{\varepsilon^2 C_{36} + C_{33}}{2\varepsilon\lambda_1^m},$$

$$b_3 = \beta\lambda_1^{2m} + C_{22} - C_{29} - \frac{\varepsilon^2 + 2\varepsilon}{2} - \frac{C_{19}(2-\beta\varepsilon)\lambda_1^{2m}}{2} - \frac{(m^*-1)C_{31}\lambda_1^{2m}}{2\varepsilon} - \frac{C_{32} + \varepsilon^2 C_{33}}{2\varepsilon},$$

$$b_4 = \beta\lambda_1^{2m} + C_{28} - C_{30} - \frac{\varepsilon^2 + 2\varepsilon}{2} - \frac{C_{20}(2-\beta\varepsilon)\lambda_1^{2m}}{2} - \frac{(m^*-1)C_{34}\lambda_1^{2m}}{2\varepsilon} - \frac{\varepsilon^2 C_{35} + C_{36}}{2\varepsilon},$$

$$a_1 = \frac{\varepsilon^2}{2} + \frac{\varepsilon^2(C_{21} + C_{25})}{2}, a_2 = \frac{\varepsilon^2}{2} + \frac{\varepsilon^2(C_{23} + C_{27})}{2}.$$

Supposing $0 < b = \min\{b_1, b_2, b_3, b_4\}$, $a = \max\{a_1, a_2\}$, combining the above Equations (94)-(98) can obtain

$$\sum_{j=1}^{N}\left(L_t(\boldsymbol{\varphi}(\tau))\boldsymbol{h}_j(\tau), \boldsymbol{h}_j(\tau)\right)_{E_k} \le -Nb + a\sum_{j=1}^{N}\left(\left\|\nabla^k \xi_j\right\|^2 + \left\|\nabla^k \eta_j\right\|^2\right). \quad (99)$$

For almost all times τ, existence $\overline{\overline{s}} = \frac{k}{2m+k}$ and $0 < \overline{\overline{s}} < 1$, there is

$$\sum_{j=1}^{N}\left\|\nabla^k \xi_j\right\|^2 \le \sum_{j=1}^{N}\lambda_j^{\overline{\overline{s}}-1}, \sum_{j=1}^{N}\left\|\nabla^k \eta_j\right\|^2 \le \sum_{j=1}^{N}\lambda_j^{\overline{\overline{s}}-1}.$$

So

$$\sum_{j=1}^{N}\left(L_t(\boldsymbol{\varphi}(\tau))\boldsymbol{h}_j(\tau), \boldsymbol{h}_j(\tau)\right)_{E_k} \le -Nb + 2a\sum_{j=1}^{N}\lambda_j^{\overline{\overline{s}}-1}. \quad (100)$$

Because $q_N = \lim_{t\to\infty} q_N(t)$, where $q_N(t) = \sup_{\boldsymbol{\varphi}_0 \in A}\sup_{d_j \in E_k}\frac{1}{t}\left(\int_0^t tr\left(L_t\left(S(\tau)\boldsymbol{\varphi}_0\right)Q_N(\tau)\right)\mathrm{d}\tau\right)$, can obtain $q_N \le -Nb + 2a\sum_{j=1}^{N}\lambda_j^{\overline{\overline{s}}-1}$.

Therefore, the Lyapunov exponent $g_1, g_2, \cdots, g_N\ (N > 1)$ on set B_{0k} is uniformly bounded, and $g_1 + g_2 + \cdots + g_N \le -Nb + 2a\sum_{j=1}^{N}\lambda_j^{\overline{\overline{s}}-1}$, where λ_j is the eigenvalue of $(-\Delta)$, then $0 < \lambda_1 \le \lambda_2 \le \lambda_3 \le \cdots \le \lambda_N$.

So $(q_j)_+ \le -Nb + 2a\sum_{j=1}^{N}\lambda_j^{\overline{\overline{s}}-1} \le 2a\sum_{j=1}^{N}\lambda_j^{\overline{\overline{s}}-1} \le \frac{3}{7}Nb$,

$q_N \le -Nb\left(1 - \frac{2a}{Nb}\sum_{j=1}^{N}\lambda_j^{\overline{\overline{s}}-1}\right) \le -\frac{4}{7}Nb$, further $\max_{1\le j\le N}\frac{(q_j)_+}{q_N} \le \frac{3}{4}$.

Thus, we can obtain $d_H(A_k) < \frac{3N}{7}$, $d_F(A_k) < \frac{6N}{7}$.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

References

[1] Kirchhoff, G. (1883) Vorlesungen Fiber Mechanic. Tenbner, Stuttgart.

[2] Hosoya, M. and Yamada, Y. (1991) On Some Nonlinear Wave Equations II: Global Existence and Energy Decay of Solutions. *Journal of the Faculty of Science*, *University of Tokyo*: *Mathematics. Section IA*, **38**, 239-250.

[3] Lin, G.G. and Zhou, C.M. (2021) The Dynamic Behavior of a Class of Kirchhoff Equations with High Order Strong Damping. *Journal of Applied Mathematics and Physics*, **9**, 1041-1055. https://doi.org/10.4236/jamp.2021.95071

[4] Lin, G.G. and Hu, L.J. (2018) Estimate on the Dimension of Global Attractor for Nonlinear Higher-Order Coupled Kirchhoff Type Equations. *Advances in Pure Mathematics*, **8**, 11-24.

[5] Cavalcanti, M.M., Domingos Cavalcanti, V.N., Prates Filho, J.S. and Soriano, J.A. (1998) Existence and Exponential Decay for a Kirchhoff-Carrier Model with Viscosity. *Journal of Mathematical Analysis and Applications*, **226**, 40-60. https://doi.org/10.1006/jmaa.1998.6057

[6] Ono, K. (1997) Global Existence, Decay, and Blowup of Solutions for Some Mildly Degenerate Nonlinear Kirchhoff Strings. *Journal of Differential Equations*, **137**, 273-301. https://doi.org/10.1006/jdeq.1997.3263

[7] Lin, G.G. (2011) Nonlinear Evolution Equations. Yunnan University Press, Kunming.

[8] Nakao, M. (2008) An Attractor for a Nonlinear Dissipative Wave Equation of Kirchhoff Type. *Journal of Mathematical Analysis and Applications*, **353**, 652-659. https://doi.org/10.1016/j.jmaa.2008.09.010

[9] Gao, Y.L., Sun, Y.T. and Lin, G.G. (2016) The Global Attractors and Their Hausdorff and Fractal Dimensions Estimation for the Higher-Order Nonlinear Kirchhoff-Type Equation with Strong Linear Damping. *International Journal of Modern Nonlinear Theory and Application*, **5**, 185-202.

[10] Lin, G.G. (2019) Dynamic Properties of Several Kinds of the Kirchhoff Equations. Chongqing University Press, Chongqing.

[11] Lin, G.G. and Yang, L.J. (2021) Global Attractors and Their Dimension Estimates for a Class of Generalized Kirchhoff Equations. *Advances in Pure Mathematics*, **11**, 317-333. https://doi.org/10.4236/apm.2021.114020

[12] Lin, G.G. and Guan, L.P. (2019) Random Attractor Family for a Class of Stochastic Higher-Order Kirchhoff Equations. *Journal of Mathematics Research*, **11**, 23-33. https://doi.org/10.5539/jmr.v11n3p23

[13] Wang, M.X., Tian, C.C. and Lin, G.G. (2014) Global Attractor and Dimension Estimation for a 2D Generalized Anisotropy Kuramoto-Sivashinsky Equation. *International Journal of Modern Nonlinear Theory and Application*, **3**, 163-172. https://doi.org/10.4236/ijmnta.2014.34018

[14] Lin, G.G. and Zhang, M. (2018) The Estimates of the Upper Bounds of Hausdorff Dimensions for the Global Attractor for a Class of Nonlinear Coupled Kirchhoff-Type Equations. *Advances in Pure Mathematics*, **8**, 1-10. https://doi.org/10.4236/apm.2018.81001

[15] Lv, P.H., Lin, G.G. and Sun, Y.T. (2022) The Family of Random Attractors for Nonautomous Stochastic Higher-Order Kirchhoff Equations with Variable Coefficients. *Open Mathematics*, **20**, 63-83. https://doi.org/10.1515/math-2022-0003

[16] Qin, C.L., Du, J.J. and Lin, G.G. (2017) Random Attractors for the Kirchhoff-Type Suspension Bridge Equations with Strong Damping and White Noises. *International Journal of Modern Nonlinear Theory and Application*, **6**, 134-147.

[17] Liao, Y.H., Lin, G.G. and Liu, J. (2022) A Family of Global Attractors for a Class of Generalized Kirchhoff-Beam Equations. *Journal of Applied Mathematics and Physics*, **10**, 930-951. https://doi.org/10.4236/jamp.2022.103064