Journal of Applied Mathematics and Physics, 2022, 10, 1477-1494
https://www.scirp.org/journal/jamp
ISSN Online: 2327-4379
ISSN Print: 2327-4352

Widened R^n General Relativity

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How to cite this paper: Strumia, A. (2022) Widened R^n General Relativity. *Journal of Applied Mathematics and Physics*, **10**, 1477-1494.
https://doi.org/10.4236/jamp.2022.105104

Received: April 5, 2022
Accepted: May 8, 2022
Published: May 11, 2022

Abstract

An extension of General Relativity is presented based on a scalar Lagrangian density which is a general function of all the independent invariant scalars that one is able to build by the powers of the Ricci tensor. It is shown how the new terms arising in the generalized Einstein filed equations may be interpreted as *dark matter* and *dark energy* contributions. Metricity condition fulfilled by a new tensor different than the usual metric tensor is also obtained. Moreover, it is shown that Schwarzschild-De Sitter, Robertson-Walker-De Sitter and Kerr-De Sitter metrics are exact solutions to the new field equations. Remarkably, the form of the equation of the geodesic trajectories of particle motions across space-time remains the same as in Einstein General Relativity unless the cosmological constant Lambda is no longer a constant becoming a function of the space-time co-ordinates.

Keywords

General Relativity, Cosmological Constant, Dark Energy, Dark Matter

1. Introduction

Several attempts have been proposed in order to modify General Relativity (GR) and its applications to cosmology [1] to explain the emergence of the so called *dark energy* (for a review see, *e.g.* [2] [3] [4]) related to the cosmological constant (see, *e.g.* [5] [6]), and of the great amount of *dark matter* in the universe (see, *e.g.* [7] [8]). Very many of such researches are based on the replacement of the curvature scalar R appearing in Einstein-Hilbert action integral with a generic function of R (see, *e.g.* [9]) or of more possible physically significant scalars as the trace T of the energy-momentum tensor [10] [11].

Instead of following the $f(R), f(R,T)$ approaches, in the present paper we will formulate a Widened General Relativity (WGR) involving a scalar Lagrangian density depending on the independent invariants that one is able to build starting from the independent powers of the Ricci tensor.

In Section 2, we define the independent powers and the independent invariant scalars that one is able to build starting from the Ricci tensor.

In Section 3, we deduce the Euler-Lagrange field equations arising from a Lagrangian density depending on such independent scalar invariants in a *n-dimensional* manifold V^n. In particular in subsection 3.1 we show that the scalar Lagrangian density governing the theory is required to be a homogenous function of the invariants of degree *n*/2. Moreover, we observe that the generalized Einstein equations for the gravitational field, when read from the viewpoint of usual GR, exhibit an energy-momentum tensor involving new terms candidate to be interpreted as *dark matter* and respectively *dark energy*.

While in subsection 3.2, we examine the relation between connection coefficients, the metric tensor and the new tensor $\mathsf{G} \equiv \left(G_{\mu\nu}\right)$ fulfilling the metricity condition in place of the usual $\mathsf{g} \equiv \left(g_{\mu\nu}\right)$.

Section 4 applies the results previously obtained to a four-dimensional physical space-time. It is emphasized that when a solution, involving a parameter Λ, fulfills the physically relevant condition $R_{\mu\nu} = -\Lambda g_{\mu\nu}$, the results are greatly simplified, since all the unknown constant parameters mutually cancel. In subsection 4.1, we show how the static Schwarzchild-De Sitter metric is a solution to the WGR field equations, allowing *dark energy* (with Λ as cosmological constant) and a *dark matter* to appear in the energy-momentum tensor, according to the GR view point, notwithstanding they are not present in the WGR Lagrangian density. In a similar way in subsection 4.2, the Robertson-Walker-De Sitter solution is obtained involving *dark matter* and *dark energy* contributions. Eventually, in subsection 4.3, also the Kerr-De Sitter solution is obtained in *untwisted co-ordinates* involving the presence of *dark matter* and *dark energy* terms.

Section 5 examines the geodesic equations describing motion of a test particle or a point representative of some cluster of masses, evaluating the extra tidal forces arising from the non-linear dependence of the Lagrangian density on the invariants of the powers of the Ricci tensor. Remarkably such *geodesic deviation* respect usual GR is null since the cosmological is really a constant. Otherwise, it involves some of the new unknown constant parameters appearing in the Lagrangian density, which are to be tuned with the experimental astrophysical observations. So the form of the equations of the geodesic trajectories of particle motions across space-time is not affected by dark matter and dark energy unless the cosmological contribution Λ is no longer a constant becoming a function of the space-time co-ordinates. In the latter case, it is shown that only the powers of Λ up to the seventh order are involved in the geodesic equation.

Section 6 concludes the paper.

2. The Independent Powers and Invariant Scalars Characterizing the Ricci Tensor

Let R be the Ricci tensor of components:

$$R_{\mu\nu} = \Gamma^{\alpha}_{\mu\nu,\alpha} - \Gamma^{\alpha}_{\mu\alpha,\nu} - \Gamma^{\alpha}_{\mu\beta}\Gamma^{\beta}_{\nu\alpha} + \Gamma^{\alpha}_{\mu\nu}\Gamma^{\beta}_{\alpha\beta}, \tag{1}$$

the co-ordinates, in an *n-dimensional* manifold V^n, being $x^{\mu}, (\mu = 0, k; k = 1, 2, \cdots, n-1)$, the components of the metric tensor g being $g_{\mu\nu}$, its signature being $(+,-,\cdots,-)$, and the connection Γ being assumed to be torsionless. A similar but simpler approach was already exploited, but considering only the scalar curvature R plus the squared Ricci tensor in [12] [13].

Let us define the powers of the Ricci tensor:

$$\begin{aligned} &R^{(0)}_{\mu\nu} = g_{\mu\nu}, \quad R^{(1)}_{\mu\nu} = R_{\mu\nu}, \quad R^{(2)}_{\mu\nu} = g^{\alpha\beta} R_{\mu\alpha} R_{\beta\nu}, \cdots \\ &R^{(n)}_{\mu\nu} = g^{\alpha_1\beta_1} \cdots g^{\alpha_{n-1}\beta_{n-1}} R_{\mu\alpha_1} R_{\beta_1\alpha_2} \cdots R_{\beta_{n-2}\alpha_{n-1}} R_{\beta_{n-1}\nu}, \end{aligned} \tag{2}$$

only n of which are independent (*i.e.*, as many as the number of dimensions of V^n). In fact, thanks to the Hamilton-Cayley's theorem, it results:

$$(-1)^n R^{(n)}_{\mu\nu} + (-1)^{n-1} a_{n-1} R^{(n-1)}_{\mu\nu} + \cdots + a_0 g_{\mu\nu} = 0, \tag{3}$$

higher powers being linearly dependent on the previous n ones, resulting:

$$(-1)^n R^{(n)}_{\mu\nu} = -(-1)^{n-1} a_{n-1} R^{(n-1)}_{\mu\nu} - \cdots - a_0 g_{\mu\nu}. \tag{4}$$

Each coefficient $a_{n-l}, l = 1, \cdots, n$ is the sum of the determinants of all the distinct minors of dimension $(n-l)\times(n-l)$ related to the elements of the main diagonal of $R_{\mu\nu}$.

The traces of the tensors of components $R^{(l)}_{\mu\nu}\,(l = 1, \cdots, n)$ provide a set of n independent invariants:

$$I^{(1)} = g^{\mu\nu} R_{\mu\nu} = R, \quad \cdots, \quad I^{(n)} = g^{\mu\nu} R^{(n)}_{\mu\nu} = R^{(n)}, \tag{5}$$

which are enough to build a *Widened General Relativity* theory (WGR).

3. Lagrangian Density and Field Equations

Let us now consider the most general scalar Lagrangian density one is able to obtain from the Ricci tensor independent powers, which is provided by any generic function of the independent invariants $I^{(l)}$:

$$\mathcal{L} \equiv \mathcal{L}\left(I^{(l)}\right), \tag{6}$$

where the previous notation is to be intended as a shortening for:

$$\mathcal{L}\left(I^{(1)}, I^{(2)}, \cdots, I^{(n)}\right). \tag{7}$$

Then the corresponding action integral governing WGR theory we are examining results to be:

$$S = \int_{\Omega} \sqrt{|g|}\, \mathcal{L}\left(I^{(l)}\right) \mathrm{d}^n x, \tag{8}$$

where g is as usual, the determinant of the metric tensor $\mathrm{g} \equiv (g_{\mu\nu})$ and Ω is the integration region in space-time. The Palatini variation [14] of S respect to the field variables $g^{\mu\nu}, \Gamma^{\alpha}_{\mu\nu}$, treated as independent, leads to the equations governing the WGR theory.

We have:

$$\delta S=\delta\int_{\Omega}\sqrt{|g|}\mathcal{L}\left(I^{(l)}\right)\mathrm{d}^{n}x=0. \tag{9}$$

Let us evaluate:

$$\delta\left[\sqrt{|g|}\mathcal{L}\left(I^{(l)}\right)\right]=\sqrt{|g|}\delta\mathcal{L}\left(I^{(l)}\right)+\mathcal{L}\left(I^{(l)}\right)\delta\sqrt{|g|}. \tag{10}$$

Developing calculations we obtain:

$$\delta\left[\sqrt{|g|}\mathcal{L}\left(I^{(l)}\right)\right]=\sqrt{|g|}\sum_{l=1}^{n}\mathcal{L}_{I^{(l)}}\delta I^{(l)}-\frac{1}{2}\mathcal{L}\sqrt{|g|}g_{\mu\nu}\delta g^{\mu\nu}, \tag{11}$$

where:

$$\mathcal{L}_{I^{(l)}}=\frac{\partial\mathcal{L}}{\partial I^{(l)}}, \tag{12}$$

and we have omitted the functional dependence of $\mathcal{L}$. Now:

$$\delta I^{(l)}=lR^{(l-1)\mu\nu}\delta R_{\mu\nu}+R^{(l)}_{\mu\nu}\delta g^{\mu\nu},\quad R^{(l-1)\mu\nu}=g^{\mu\rho}g^{\nu\sigma}R^{(l-1)}_{\rho\sigma}. \tag{13}$$

Then it follows:

$$\begin{aligned}\delta\left[\sqrt{|g|}\mathcal{L}\left(I^{(l)}\right)\right]=&\sqrt{|g|}\sum_{l=1}^{n}l\mathcal{L}_{I^{(l)}}R^{(l-1)\mu\nu}\delta R_{\mu\nu}\\&+\sqrt{|g|}\left[\sum_{l=1}^{n}\mathcal{L}_{I^{(l)}}R^{(l)}_{\mu\nu}-\frac{1}{2}\mathcal{L}g_{\mu\nu}\right]\delta g^{\mu\nu}.\end{aligned} \tag{14}$$

More, evaluating:

$$\begin{aligned}\delta R_{\beta\gamma}=&\delta^{\mu}_{\beta}\left(\delta^{\alpha}_{\lambda}\delta^{\nu}_{\gamma}-\delta^{\nu}_{\lambda}\delta^{\alpha}_{\gamma}\right)\delta\Gamma^{\lambda}_{\mu\nu,\alpha}\\&+\left[\delta^{\mu}_{\beta}\delta^{\nu}_{\gamma}\Gamma^{\rho}_{\lambda\rho}-\delta^{\nu}_{\gamma}\Gamma^{\mu}_{\beta\lambda}-\delta^{\mu}_{\beta}\Gamma^{\nu}_{\lambda\gamma}+\delta^{\nu}_{\lambda}\Gamma^{\mu}_{\beta\gamma}\right]\delta\Gamma^{\lambda}_{\mu\nu},\end{aligned} \tag{15}$$

we have:

$$\begin{aligned}&\delta\left[\sqrt{|g|}\mathcal{L}\left(I^{(l)}\right)\right]\\&=\sqrt{|g|}\sum_{l=1}^{n}l\mathcal{L}_{I^{(l)}}R^{(l-1)\beta\gamma}\delta^{\mu}_{\beta}\left(\delta^{\alpha}_{\lambda}\delta^{\nu}_{\gamma}-\delta^{\nu}_{\lambda}\delta^{\alpha}_{\gamma}\right)\delta\Gamma^{\lambda}_{\mu\nu,\alpha}\\&+\sqrt{|g|}\sum_{l=1}^{n}l\mathcal{L}_{I^{(l)}}R^{(l-1)\beta\gamma}\left[\delta^{\mu}_{\beta}\delta^{\nu}_{\gamma}\Gamma^{\rho}_{\lambda\rho}-\delta^{\nu}_{\gamma}\Gamma^{\mu}_{\beta\lambda}-\delta^{\mu}_{\beta}\Gamma^{\nu}_{\lambda\gamma}+\delta^{\nu}_{\lambda}\Gamma^{\mu}_{\beta\gamma}\right]\delta\Gamma^{\lambda}_{\mu\nu}\\&+\sqrt{|g|}\left[\sum_{l=1}^{n}\mathcal{L}_{I^{(l)}}R^{(l)}_{\mu\nu}-\frac{1}{2}\mathcal{L}g_{\mu\nu}\right]\delta g^{\mu\nu}.\end{aligned} \tag{16}$$

Therefore, the Euler-Lagrange field equations follow:

$$\sum_{l=1}^{n}\mathcal{L}_{I^{(l)}}R^{(l)}_{\mu\nu}-\frac{1}{2}\mathcal{L}g_{\mu\nu}=0, \tag{17}$$

$$\begin{aligned}&\sum_{l=1}^{n}l\left[\sqrt{|g|}\mathcal{L}_{I^{(l)}}R^{(l-1)\beta\gamma}\right]_{,\alpha}\delta^{\mu}_{\beta}\left(\delta^{\alpha}_{\lambda}\delta^{\nu}_{\gamma}-\delta^{\nu}_{\lambda}\delta^{\alpha}_{\gamma}\right)\\&-\sum_{l=1}^{n}l\sqrt{|g|}\mathcal{L}_{I^{(l)}}R^{(l-1)\beta\gamma}\left[\delta^{\mu}_{\beta}\left(\delta^{\nu}_{\gamma}\Gamma^{\rho}_{\lambda\rho}-\Gamma^{\nu}_{\lambda\gamma}\right)-\delta^{\nu}_{\gamma}\Gamma^{\mu}_{\beta\lambda}+\delta^{\nu}_{\lambda}\Gamma^{\mu}_{\beta\gamma}\right]=0.\end{aligned} \tag{18}$$

Equation (17) generalizes the Einstein equations of free gravitational field, while Equation (18) replaces the usual metricity condition, defining the connection coefficients.

3.1. WGR Filed Equations (Generalized Einstein Equations)

Let us examine the Euler-Lagrange field Equations (17):

$$\sum_{l=1}^{n} \mathcal{L}_{I^{(l)}} R_{\mu\nu}^{(l)} - \frac{1}{2} \mathcal{L} g_{\mu\nu} = 0.$$

The trace provides a necessary condition to which the Lagrangian density is required to obey, *i.e.*:

$$\sum_{l=1}^{n} \mathcal{L}_{I^{(l)}} I^{(l)} - \frac{n}{2} \mathcal{L} = 0. \tag{19}$$

Developing we have:

$$\mathcal{L}_{I^{(1)}} I^{(1)} + \mathcal{L}_{I^{(2)}} I^{(2)} + \cdots + \mathcal{L}_{I^{(n)}} I^{(n)} = \frac{n}{2} \mathcal{L}. \tag{20}$$

According to Euler formula it follows that $\mathcal{L}$ is required to be a homogenous function in the variables $I^{(l)}$ of degree $n/2$.

Besides we observe that Equation (17) can be usefully written as:

$$\mathcal{L}_{I^{(1)}} R_{\mu\nu}^{(1)} + \sum_{l=2}^{n} \mathcal{L}_{I^{(l)}} R_{\mu\nu}^{(l)} - \frac{1}{2} \mathcal{L} g_{\mu\nu} = 0. \tag{21}$$

More, recalling the definitions (2), and being necessarily:

$$\mathcal{L}_{I^{(1)}} \neq 0, \tag{22}$$

so that at the first order the theory approaches to usual Einstein Relativity, we have also:

$$R_{\mu\nu} - \frac{1}{2} \frac{\mathcal{L}}{\mathcal{L}_{I^{(1)}}} g_{\mu\nu} = -\sum_{l=2}^{n} \frac{\mathcal{L}_{I^{(l)}}}{\mathcal{L}_{I^{(1)}}} R_{\mu\nu}^{(l)}. \tag{23}$$

And also:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = -\sum_{l=2}^{n} \frac{\mathcal{L}_{I^{(l)}}}{\mathcal{L}_{I^{(1)}}} R_{\mu\nu}^{(l)} + \frac{1}{2} \left(\frac{\mathcal{L}}{\mathcal{L}_{I^{(1)}}} - R \right) g_{\mu\nu}. \tag{24}$$

On interpreting the latter equation from the viewpoint of usual GR one is led to consider the r.h.s. term as own to matter, governed by an equivalent energy-momentum tensor defined as:

$$\kappa T_{\mu\nu} = -\sum_{l=2}^{n} \frac{\mathcal{L}_{I^{(l)}}}{\mathcal{L}_{I^{(1)}}} R_{\mu\nu}^{(l)} + \frac{1}{2} \left(\frac{\mathcal{L}}{\mathcal{L}_{I^{(1)}}} - R \right) g_{\mu\nu}, \tag{25}$$

so that we may write Equation (23) in a more usual form as:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \kappa T_{\mu\nu}. \tag{26}$$

Appearing of an energy-momentum tensor in the last equation, in absence of observable matter and fields, may be interpreted as *dark matter* and *vacuum* or *dark energy* contributions. In fact on decomposing $T_{\mu\nu}$ onto the proper platform of the particles of a fluid traveling with velocity $\mathbf{u} \equiv (u_\mu)$, we may write:

$$T_{\mu\nu} = \rho^{(dm)} u_\mu u_\nu - p^{(de)} \gamma_{\mu\nu}, \quad \gamma_{\mu\nu} = g_{\mu\nu} - u_\mu u_\nu. \tag{27}$$

More, in presence of observable matter and fields a contribution $T_{\mu\nu}^{(obs)}$ is to be added. In principle, one could guess even that all (*dark* and *non-dark*) matter might arise from the energy-momentum tensor (25), especially when it is supposed that usual matter and non-gravitational fields are hidden in higher dimensions ($n > 4$) of space-time [15] [16].

3.2. Connection Equations in WGR

Now Equation (18) can be more conveniently written as:

$$\begin{aligned}&\sum_{l=1}^{n} l\left[\sqrt{|g|}\mathcal{L}_{I^{(l)}}\left(\delta_{\lambda}^{\alpha}R^{(l-1)\mu\nu}-\delta_{\lambda}^{\nu}R^{(l-1)\mu\alpha}\right)\right]_{,\alpha}\\&-\sum_{l=1}^{n} l\sqrt{|g|}\mathcal{L}_{I^{(l)}}\left[R^{(l-1)\mu\nu}\Gamma_{\lambda\rho}^{\rho}-R^{(l-1)\beta\mu}\Gamma_{\beta\lambda}^{\nu}-R^{(l-1)\nu\gamma}\Gamma_{\gamma\lambda}^{\mu}+\delta_{\lambda}^{\nu}R^{(l-1)\beta\gamma}\Gamma_{\beta\gamma}^{\mu}\right]=0,\end{aligned} \tag{28}$$

On developing the last relation, dividing by $\sqrt{|g|}$ and reordering we arrive at:

$$\begin{aligned}&\sum_{l=1}^{n} l\left[\left(\mathcal{L}_{I^{(l)}}R^{(l-1)\mu\nu}\right)_{,\lambda}+\mathcal{L}_{I^{(l)}}R^{(l-1)\beta\mu}\Gamma_{\beta\lambda}^{\nu}+\mathcal{L}_{I^{(l)}}R^{(l-1)\nu\gamma}\Gamma_{\gamma\lambda}^{\mu}\right]\\&-\delta_{\lambda}^{\nu}\sum_{l=1}^{n} l\left[\left(\mathcal{L}_{I^{(l)}}R^{(l-1)\mu\alpha}\right)_{,\alpha}+\mathcal{L}_{I^{(l)}}R^{(l-1)\beta\gamma}\Gamma_{\beta\gamma}^{\mu}\right]+\left(\log\sqrt{|g|}\right)_{,\lambda}\sum_{l=1}^{n} l\mathcal{L}_{I^{(l)}}R^{(l-1)\mu\nu}\\&-\delta_{\lambda}^{\nu}\left(\log\sqrt{|g|}\right)_{,\alpha}\sum_{l=1}^{n} l\mathcal{L}_{I^{(l)}}R^{(l-1)\mu\alpha}-\sum_{l=1}^{n} l\mathcal{L}_{I^{(l)}}R^{(l-1)\mu\nu}\Gamma_{\lambda\rho}^{\rho}=0.\end{aligned} \tag{29}$$

After some simple but tedious calculations we finally obtain:

$$\begin{aligned}&\left(\sum_{l=1}^{n} l\mathcal{L}_{I^{(l)}}R^{(l-1)\mu\nu}\right)_{,\lambda}+\sum_{l=1}^{n} l\mathcal{L}_{I^{(l)}}R^{(l-1)\beta\mu}\Gamma_{\beta\lambda}^{\nu}+\sum_{l=1}^{n} l\mathcal{L}_{I^{(l)}}R^{(l-1)\nu\gamma}\Gamma_{\gamma\lambda}^{\mu}\\&+\left[\left(\log\sqrt{|g|}\right)_{,\lambda}-\Gamma_{\lambda\rho}^{\rho}\right]\sum_{l=1}^{n} l\mathcal{L}_{I^{(l)}}R^{(l-1)\mu\nu}=0.\end{aligned} \tag{30}$$

The relation (30) defines implicitly the connection coefficients $\Gamma_{\mu\nu}^{\alpha}$ as an extension of the usual metricity condition. Let us now set:

$$G^{\mu\nu}=\frac{1}{\sqrt{\alpha}}\sum_{l=1}^{n} l\mathcal{L}_{I^{(l)}}R^{(l-1)\mu\nu}, \tag{31}$$

where α is a positive function which will be determined soon. So Equation (30) may be written as:

$$\left(\sqrt{\alpha}G^{\mu\nu}\right)_{,\lambda}+\sqrt{\alpha}G^{\beta\mu}\Gamma_{\beta\lambda}^{\nu}+\sqrt{\alpha}G^{\nu\gamma}\Gamma_{\gamma\lambda}^{\mu}+\left[\left(\log\sqrt{|g|}\right)_{,\lambda}-\Gamma_{\lambda\rho}^{\rho}\right]\sqrt{\alpha}G^{\mu\nu}=0. \tag{32}$$

Developing we obtain:

$$\begin{aligned}&\sqrt{\alpha}G_{,\lambda}^{\mu\nu}+G^{\mu\nu}\left(\sqrt{\alpha}\right)_{,\lambda}+\sqrt{\alpha}G^{\beta\mu}\Gamma_{\beta\lambda}^{\nu}+\sqrt{\alpha}G^{\nu\gamma}\Gamma_{\gamma\lambda}^{\mu}\\&+\left[\left(\log\sqrt{|g|}\right)_{,\lambda}-\Gamma_{\lambda\rho}^{\rho}\right]\sqrt{\alpha}G^{\mu\nu}=0.\end{aligned} \tag{33}$$

And also:

$$G_{,\lambda}^{\mu\nu}+G^{\mu\nu}\left(\log\sqrt{\alpha}\right)_{,\lambda}+G^{\beta\mu}\Gamma_{\beta\lambda}^{\nu}+G^{\nu\gamma}\Gamma_{\gamma\lambda}^{\mu}+\left[\left(\log\sqrt{|g|}\right)_{,\lambda}-\Gamma_{\lambda\rho}^{\rho}\right]G^{\mu\nu}=0. \tag{34}$$

Introducing $G_{\mu\nu}$, defined as the inverse of $G^{\mu\nu}$:

$$G_{\mu\rho}G^{\rho\nu} = \delta^{\nu}_{\mu}. \tag{35}$$

and multiplying (34) by $G_{\mu\nu}$ and taking account that:

$$\frac{1}{2}G_{\mu\nu}G^{\mu\nu}_{,\lambda} = -\left(\log\sqrt{|G|}\right)_{,\lambda}, \tag{36}$$

it follows:

$$-2\left(\log\sqrt{|G|}\right)_{,\lambda} + 2\Gamma^{\beta}_{\lambda\beta} + \left[\left(\log\sqrt{\alpha|g|}\right)_{,\lambda} - \Gamma^{\rho}_{\lambda\rho}\right]n = 0. \tag{37}$$

On choosing:

$$\alpha = \frac{|G|}{|g|}, \quad G = \det\left\|G_{\mu\nu}\right\|, \tag{38}$$

it remains:

$$(n-2)\left[\left(\log\sqrt{\alpha|g|}\right)_{,\lambda} - \Gamma^{\rho}_{\lambda\rho}\right] = 0. \tag{39}$$

And being $n > 2$ in physical space-times:

$$\left(\log\sqrt{\alpha|g|}\right)_{,\lambda} - \Gamma^{\rho}_{\lambda\rho} = 0, \tag{40}$$

Then (34) becomes the metricity condition:

$$G^{\mu\nu}_{,\lambda} + G^{\beta\mu}\Gamma^{\nu}_{\beta\lambda} + G^{\nu\gamma}\Gamma^{\mu}_{\gamma\lambda} = 0, \tag{41}$$

for the tensor $G^{\mu\nu}$, which is to be used to raise indices in place of $g^{\mu\nu}$, while the usual $g_{\mu\nu}$ still defines the metric of space-time, the interval being still:

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^{\mu}\mathrm{d}x^{\nu}. \tag{42}$$

In order to avoid confusion one could adopt the notation:

$$A^{\bar{\mu}} = G^{\mu\nu}A_{\nu}, \tag{43}$$

for any index raised by $G^{\mu\nu}$. From (41) one easily solves also:

$$\Gamma^{\alpha}_{\mu\nu} = \frac{1}{2}G^{\alpha\beta}\left(G_{\mu\beta,\nu} + G_{\nu\beta,\mu} - G_{\mu\nu,\beta}\right). \tag{44}$$

4. Equations in 4-D Space-Time

In a four-dimensional space-time V^4, from the definitions (2) we have simply:

$$\begin{gathered} R^{(0)}_{\mu\nu} = g_{\mu\nu}, \quad R^{(1)}_{\mu\nu} = R_{\mu\nu}, \quad R^{(2)}_{\mu\nu} = g^{\lambda\rho}R_{\mu\lambda}R_{\rho\nu}, \\ R^{(3)}_{\mu\nu} = g^{\lambda\rho}g^{\sigma\tau}R_{\mu\lambda}R_{\rho\sigma}R_{\tau\nu}, \quad R^{(4)}_{\mu\nu} = g^{\alpha\beta}g^{\lambda\rho}g^{\sigma\tau}R_{\mu\alpha}R_{\beta\lambda}R_{\rho\sigma}R_{\tau\nu}. \end{gathered} \tag{45}$$

And:

$$\begin{gathered} I^{(1)} = g^{\mu\nu}R^{(1)}_{\mu\nu} = R, \quad I^{(2)} = g^{\mu\nu}R^{(2)}_{\mu\nu}, \\ I^{(3)} = g^{\mu\nu}R^{(3)}_{\mu\nu}, \quad I^{(4)} = g^{\mu\nu}R^{(4)}_{\mu\nu}. \end{gathered} \tag{46}$$

We remember that here the usual inverse metric tensor $g^{\mu\nu}$ still appears into the invariant scalars, according to the definitions (5). Then into the WGR field equations:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \kappa \left(T_{\mu\nu}^{(dm)} + T_{\mu\nu}^{(de)} \right), \tag{47}$$

the *dark matter* and *dark energy* contributions become:

$$\kappa \left(T_{\mu\nu}^{(dm)} + T_{\mu\nu}^{(de)} \right) = -\frac{\mathcal{L}_{I^{(2)}}}{\mathcal{L}_{I^{(1)}}} R_{\mu\nu}^{(2)} - \frac{\mathcal{L}_{I^{(3)}}}{\mathcal{L}_{I^{(1)}}} R_{\mu\nu}^{(3)} - \frac{\mathcal{L}_{I^{(4)}}}{\mathcal{L}_{I^{(1)}}} R_{\mu\nu}^{(4)} + \frac{1}{2} \left(\frac{\mathcal{L}}{\mathcal{L}_{I^{(1)}}} - R \right) g_{\mu\nu}, \tag{48}$$

In particular, thanks to (20) we know that the Lagrangian density is a homogenous function of $I^{(1)}, I^{(2)}, I^{(3)}, I^{(4)}$ of degree 2:

$$\begin{aligned} \mathcal{L} = {} & a_{11} I^{(1)2} + a_{22} I^{(2)2} + a_{33} I^{(3)2} + a_{44} I^{(4)2} + 2a_{12} I^{(1)} I^{(2)} + 2a_{13} I^{(1)} I^{(3)} \\ & + 2a_{14} I^{(1)} I^{(4)} + 2a_{23} I^{(2)} I^{(3)} + 2a_{24} I^{(2)} I^{(4)} + 2a_{34} I^{(3)} I^{(4)}, \end{aligned} \tag{49}$$

the a_{jl} being constant coefficients to be determined. According to a matrix formalism we have in a more compact formula:

$$L = \mathbf{I} \cdot \mathsf{A} \cdot \mathbf{I}, \quad \mathbf{I} \equiv \left(I^{(l)} \right), \quad \mathsf{A} \equiv \left(a_{lk} \right). \tag{50}$$

The derivatives respect to the invariants $I^{(l)}$ become now:

$$\begin{aligned} \mathcal{L}_{I^{(1)}} &= 2a_{11} I^{(1)} + 2a_{12} I^{(2)} + 2a_{13} I^{(3)} + 2a_{14} I^{(4)}, \\ \mathcal{L}_{I^{(2)}} &= 2a_{12} I^{(1)} + 2a_{22} I^{(2)} + 2a_{23} I^{(3)} + 2a_{24} I^{(4)}, \\ \mathcal{L}_{I^{(3)}} &= 2a_{13} I^{(1)} + 2a_{23} I^{(2)} + 2a_{33} I^{(3)} + 2a_{34} I^{(4)}, \\ \mathcal{L}_{I^{(4)}} &= 2a_{14} I^{(1)} + 2a_{24} I^{(2)} + 2a_{34} I^{(3)} + 2a_{44} I^{(4)}, \end{aligned} \tag{51}$$

or in matrix formalism:

$$\nabla_{\mathbf{I}} \mathcal{L} \equiv 2\mathsf{A}\mathbf{I}. \tag{52}$$

The matrix elements a_{lk} are in general undetermined within the theory and, in principle, should be determined by adding special symmetry criteria and experimental results. So one could require, *e.g.*, that the matrix A is diagonal, so reducing to 4 the number of its elements. But it seems more relevant to observe that in correspondence to the known physically meaningful metrics (*i.e.*, Schwarzschild-De Sitter, Robertson-Walker-De Sitter, Kerr-De Sitter) the condition:

$$R_{\mu\nu} = -\Lambda g_{\mu\nu}, \tag{53}$$

holds, from which we obtain:

$$\begin{aligned} R_{\mu\nu}^{(1)} &= -\Lambda g_{\mu\nu}, \quad R_{\mu\nu}^{(2)} = \Lambda^2 g_{\mu\nu}, \\ R_{\mu\nu}^{(3)} &= -\Lambda^3 g_{\mu\nu}, \quad R_{\mu\nu}^{(4)} = \Lambda^4 g_{\mu\nu}, \end{aligned} \tag{54}$$

$$I^{(1)} = -4\Lambda, \quad I^{(2)} = 4\Lambda^2, \quad I^{(3)} = -4\Lambda^3, \quad I^{(4)} = 4\Lambda^4. \tag{55}$$

Therefore the Lagrangian density becomes simply:

$$\begin{aligned} \mathcal{L} = 16 \big(& a_{11}\Lambda^2 + a_{22}\Lambda^4 + a_{33}\Lambda^6 + a_{44}\Lambda^8 + a_{12}\Lambda^3 \\ & + a_{13}\Lambda^4 + a_{14}\Lambda^5 + a_{23}\Lambda^5 + a_{24}\Lambda^6 + a_{34}\Lambda^7 \big), \end{aligned} \tag{56}$$

And the derivatives respect to the invariants are:

$$
\begin{aligned}
\mathcal{L}_{I^{(1)}} &= -8a_{11}\Lambda + 8a_{12}\Lambda^2 - 8a_{13}\Lambda^3 + 8a_{14}\Lambda^4, \\
\mathcal{L}_{I^{(2)}} &= -8a_{12}\Lambda + 8a_{22}\Lambda^2 - 8a_{23}\Lambda^3 + 8a_{24}\Lambda^4, \\
\mathcal{L}_{I^{(3)}} &= -8a_{13}\Lambda + 8a_{23}\Lambda^2 - 8a_{33}\Lambda^3 + 8a_{34}\Lambda^4, \\
\mathcal{L}_{I^{(4)}} &= -8a_{14}\Lambda + 8a_{24}\Lambda^2 - 8a_{34}\Lambda^3 + 8a_{44}\Lambda^4,
\end{aligned} \tag{57}
$$

Then in Equation (48) it remains:

$$
\begin{aligned}
\kappa\left(T_{\mu\nu}^{(dm)} + T_{\mu\nu}^{(de)}\right) = &-\frac{\mathcal{L}_{I^{(2)}}}{\mathcal{L}_{I^{(1)}}}\Lambda^2 g_{\mu\nu} + \frac{\mathcal{L}_{I^{(3)}}}{\mathcal{L}_{I^{(1)}}}\Lambda^3 g_{\mu\nu} - \frac{\mathcal{L}_{I^{(4)}}}{\mathcal{L}_{I^{(1)}}}\Lambda^4 g_{\mu\nu} \\
&+ \frac{1}{2}\left(\frac{\mathcal{L}}{\mathcal{L}_{I^{(1)}}} - 4\Lambda\right) g_{\mu\nu},
\end{aligned} \tag{58}
$$

And also:

$$
\begin{aligned}
\kappa\left(T_{\mu\nu}^{(dm)} + T_{\mu\nu}^{(de)}\right) = &-\frac{1}{4\mathcal{L}_{I^{(1)}}}\left(\mathcal{L}_{I^{(2)}} I^{(2)} + \mathcal{L}_{I^{(3)}} I^{(3)} + \mathcal{L}_{I^{(4)}} I^{(4)}\right) g_{\mu\nu} \\
&+ \frac{1}{2}\left(\frac{\mathcal{L}}{\mathcal{L}_{I^{(1)}}} - 4\Lambda\right) g_{\mu\nu},
\end{aligned} \tag{59}
$$

Since:

$$
\mathcal{L}_{I^{(2)}} I^{(2)} + \mathcal{L}_{I^{(3)}} I^{(3)} + \mathcal{L}_{I^{(4)}} I^{(4)} = 2\mathcal{L} - \mathcal{L}_{I^{(1)}} I^{(1)}
$$

finally it results:

$$
\kappa\left(T_{\mu\nu}^{(dm)} + T_{\mu\nu}^{(de)}\right) = \Lambda g_{\mu\nu}. \tag{60}
$$

It is remarkable that, thanks to (53) the coefficients a_{jl} appearing in the Lagrangian density do not contribute to the solution, and their determination results to be irrelevant respect to the metric, even if they contribute to the definition of $G_{\mu\nu}$. Different solutions of lower symmetry, if any, might involve at least some of the coefficients in terms of physically significant parameters to be determined experimentally. What is relevant is that in WGR approach, interpreted from the viewpoint of GR, the cosmological constant Λ appears naturally in the known physically relevant solutions for the metric even if it does not appear in the Lagrangian density itself.

4.1. Spherical Symmetry: Schwarzschild-De Sitter Solution

Let us now test the most simple solution we can conjecture starting from the Schwarzschild-De Sitter metric modified by a time-time factor $(1+\varepsilon)$.

We have:

$$
\begin{aligned}
&g_{00} = c^2\left(1 - \frac{r_s}{r} - \frac{\Lambda}{3}r^2\right)(1+\varepsilon), \quad g_{11} = -\frac{1}{1 - \frac{r_s}{r} - \frac{\Lambda}{3}r^2}, \\
&g_{22} = -r^2, \quad g_{33} = -r^2\sin^2\theta, \quad r_s = \frac{2GM}{c^2}.
\end{aligned} \tag{61}
$$

We remark that in our WGR field Equations (17) no cosmological constant appears, but a cosmological contribution, which, in principle, might be even non-constant, arises in the decomposition (47), when we interpret the same field equations from the stand point of usual GR.

The spherical co-ordinates are, as usual:

$$x^0 = ct, \quad x^1 = r, \quad x^2 = \theta, \quad x^3 = \phi. \tag{62}$$

The Ricci tensor evaluated starting from (61) becomes:

$$\begin{aligned} R_{00} &= -\Lambda g_{00}, \quad R_{11} = -\Lambda g_{11}, \\ R_{22} &= -\Lambda g_{22}, \quad R_{33} = -\Lambda g_{44}. \end{aligned} \tag{63}$$

Or in explicit form:

$$\begin{aligned} R_{00} &= -\Lambda c^2 \left(1 - \frac{r_s}{r} - \frac{\Lambda}{3} r^2\right)(1+\varepsilon), \\ R_{11} &= \frac{\Lambda}{1 - \frac{r_s}{r} - \frac{\Lambda}{3} r^2}, \quad R_{22} = \Lambda r^2, \quad R_{33} = \Lambda r^2 \sin^2\theta. \end{aligned} \tag{64}$$

The powers of the Ricci tensor are consequently:

$$\begin{aligned} R_{00}^{(k)} &= c^2 (-\Lambda)^k \left(1 - \frac{r_s}{r} - \frac{\Lambda}{3} r^2\right)(1+\varepsilon), \quad R_{11}^{(k)} = -\frac{(-\Lambda)^k}{1 - \frac{r_s}{r} - \frac{\Lambda}{3} r^2}, \\ R_{22}^{(k)} &= -(-\Lambda)^k r^2, \quad R_{33}^{(k)} = -(-\Lambda)^k r^2 \sin^2\theta. \end{aligned} \tag{65}$$

Therefore the WGR field equations can be written in the form:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = -\frac{\mathcal{L}_{I^{(2)}}}{\mathcal{L}_{I^{(1)}}} R_{\mu\nu}^{(2)} - \frac{\mathcal{L}_{I^{(3)}}}{\mathcal{L}_{I^{(1)}}} R_{\mu\nu}^{(3)} - \frac{\mathcal{L}_{I^{(4)}}}{\mathcal{L}_{I^{(1)}}} R_{\mu\nu}^{(4)} + \frac{1}{2}\left(\frac{\mathcal{L}}{\mathcal{L}_{I^{(1)}}} - R\right) g_{\mu\nu}. \tag{66}$$

And then:

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = &-\frac{\mathcal{L}_{I^{(2)}}}{\mathcal{L}_{I^{(1)}}} \Lambda^2 g_{\mu\nu} + \frac{\mathcal{L}_{I^{(3)}}}{\mathcal{L}_{I^{(1)}}} \Lambda^3 g_{\mu\nu} - \frac{\mathcal{L}_{I^{(4)}}}{\mathcal{L}_{I^{(1)}}} \Lambda^4 g_{\mu\nu} \\ &+ \frac{1}{2}\left(\frac{\mathcal{L}}{\mathcal{L}_{I^{(1)}}} - R\right) g_{\mu\nu}. \end{aligned} \tag{67}$$

The $\mathcal{L}_{I^{(l)}}$ still result to be:

$$\begin{aligned} \mathcal{L}_{I^{(1)}} &= -8a_{11}\Lambda + 8a_{12}\Lambda^2 - 8a_{13}\Lambda^3 + 8a_{14}\Lambda^4, \\ \mathcal{L}_{I^{(2)}} &= -8a_{12}\Lambda + 8a_{22}\Lambda^2 - 8a_{23}\Lambda^3 + 8a_{24}\Lambda^4, \\ \mathcal{L}_{I^{(3)}} &= -8a_{13}\Lambda + 8a_{23}\Lambda^2 - 8a_{33}\Lambda^3 + 8a_{34}\Lambda^4, \\ \mathcal{L}_{I^{(4)}} &= -8a_{14}\Lambda + 8a_{24}\Lambda^2 - 8a_{34}\Lambda^3 + 8a_{44}\Lambda^4, \end{aligned}$$

Direct calculation leads, even now, after elision of several terms, to:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \Lambda g_{\mu\nu}, \tag{68}$$

making a cosmological constant to appear in the WGR equations, even if it is not explicitly included into the action integral (8).

In correspondence to that solution both a *dark matter* and a *dark energy* contribution appear, own to the emergent cosmological constant Λ.

In fact, interpreting the r.h.s. in (68) as an energy-momentum tensor, we have:

$$\kappa T_{00} = \Lambda \varepsilon c^2 \left(1 - \frac{r_s}{r} - \frac{\Lambda}{3}\right) + \Lambda c^2 \left(1 - \frac{r_s}{r} - \frac{\Lambda}{3}\right), \tag{69}$$

$$\kappa T_{11} = -\frac{\Lambda}{1 - \frac{r_s}{r} - \frac{\Lambda}{3}}, \tag{70}$$

$$\kappa T_{22} = -\Lambda r^2, \tag{71}$$

$$\kappa T_{33} = -\Lambda r^2 \sin^2\theta, \tag{72}$$

which corresponds to the Schwarzschild solutions in presence of a cosmological constant Λ and a time scale factor $(1+\varepsilon)$.

Now we can interpret the contributions:

$$\kappa \rho^{(dm)} = \Lambda \varepsilon, \quad \kappa p^{(de)} = -\kappa \rho^{(de)} = -\Lambda, \tag{73}$$

respectively as *dark matter* and *dark energy* or *vacuum energy* of a matter fluid in its local rest frame.

We remember that the state equation $p^{(de)} = -\rho^{(de)}$, holding for the Λ vacuum pressure state equation is a special case of the more general condition $p < \frac{1}{3}\rho$ which is generally allowed now to characterize *dark energy*, in order to give rise to the observed acceleration of the universe expansion [17] [18].

We observe that in correspondence to such static solution the coefficients a_{jk} appearing in the Lagrangian density do not play any role in the solution to the field equations, since they mutually cancel.

We emphasize that a non-vanishing constant ε is required to ensure a non-null dark matter density contribution.

4.2. Spherical Symmetry: Robertson-Walker-De Sitter Solution

We consider now the Robertson-Walker metric with g_{00}:

$$\begin{aligned} g_{00} &= c^2(1+\varepsilon), \quad g_{11} = -\frac{a(t)^2}{1-Kr^2}, \\ g_{22} &= -a(t)^2 r^2, \quad g_{33} = -a(t)^2 r^2 \sin^2\theta, \end{aligned} \tag{74}$$

$(1+\varepsilon)$ being still a constant time scale factor.

We test how *dark matter* and *dark energy* can appear in WGR cosmological solutions.

The Ricci tensor components are given by:

$$\begin{aligned} R_{00} &= -\frac{3\ddot{a}(t)}{a(t)}, \quad R_{11} = \frac{a(t)\ddot{a}(t) + 2\dot{a}(t)^2 + 2Kc^2}{c^2(1-Kr^2)}, \\ R_{22} &= R_{33} = \frac{r^2}{c^2}\left[a(t)\ddot{a}(t) + 2\dot{a}(t)^2 + 2Kc^2\right]. \end{aligned} \tag{75}$$

And then:

$$R = -\frac{6\left[a(t)\ddot{a}(t) + 2\dot{a}(t)^2 + 2Kc^2\right]}{c^2 a(t)^2} \tag{76}$$

While its powers are:

$$\begin{aligned} R_{00}^{(k)} &= -\frac{3^k \ddot{a}(t)^k}{a(t)^k}, \\ R_{11}^{(k)} &= \frac{\left[a(t)\ddot{a}(t) + 2\dot{a}(t)^2 + 2Kc^2\right]^k}{c^{2k}\left(1 - Kr^2\right)^k}, \\ R_{22}^{(k)} &= R_{33}^k = \frac{r^{2k}}{c^{2k}}\left[a(t)^k \ddot{a}(t) + 2\dot{a}(t)^2 + 2Kc^2\right]^k. \end{aligned} \tag{77}$$

Empty space

It is remarkable that in correspondence to the choice:

$$K = 0, \quad a(t) = a_0 \exp\left(\sqrt{\frac{\Lambda}{3}}ct\right), \tag{78}$$

it results just:

$$R_{\mu\nu} = -\Lambda g_{\mu\nu}, \tag{79}$$

as it happened for the metric (61).

The WGR field equations can be written in the form:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = -\frac{\mathcal{L}_{I^{(2)}}}{\mathcal{L}_{I^{(1)}}}R_{\mu\nu}^{(2)} - \frac{\mathcal{L}_{I^{(3)}}}{\mathcal{L}_{I^{(1)}}}R_{\mu\nu}^{(3)} - \frac{\mathcal{L}_{I^{(4)}}}{\mathcal{L}_{I^{(1)}}}R_{\mu\nu}^{(4)} + \frac{1}{2}\left(\frac{\mathcal{L}}{\mathcal{L}_{I^{(1)}}} - R\right)g_{\mu\nu}, \tag{80}$$

being:

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = &-\frac{\mathcal{L}_{I^{(2)}}}{\mathcal{L}_{I^{(1)}}}\Lambda^2 g_{\mu\nu} + \frac{\mathcal{L}_{I^{(3)}}}{\mathcal{L}_{I^{(1)}}}\Lambda^3 g_{\mu\nu} - \frac{\mathcal{L}_{I^{(4)}}}{\mathcal{L}_{I^{(1)}}}\Lambda^4 g_{\mu\nu} \\ &+ \frac{1}{2}\left(\frac{\mathcal{L}}{\mathcal{L}_{I^{(1)}}} - R\right)g_{\mu\nu}. \end{aligned} \tag{81}$$

Direct calculation leads, after elision of several terms, to:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \Lambda g_{\mu\nu}. \tag{82}$$

In correspondence to that solution both a *dark matter* and a *dark energy* contribution appear, own to the emergent cosmological constant Λ.

In fact, interpreting the r.h.s. in (68) as an energy-momentum tensor, we have:

$$\kappa T_{00} = \Lambda c^2 = \left(\frac{\Lambda\varepsilon}{1+\varepsilon} + \frac{\Lambda}{1+\varepsilon}\right)c^2, \tag{83}$$

$$\kappa T_{11} = -\frac{\Lambda}{1+\varepsilon}a(t)^2, \tag{84}$$

$$\kappa T_{22} = -\frac{\Lambda}{1+\varepsilon}a(t)^2 r^2, \tag{85}$$

$$\kappa T_{33} = -\frac{\Lambda}{1+\varepsilon} a(t)^2 r^2 \sin^2\theta, \tag{86}$$

which correspond to the De Sitter solution in presence of a cosmological constant Λ and a time scale factor $(1+\varepsilon)$. Now we can interpret the contributions:

$$\kappa \rho^{(dm)} = \frac{\Lambda\varepsilon}{1+\varepsilon}, \quad \kappa p^{(de)} = -\frac{\Lambda}{1+\varepsilon}, \tag{87}$$

respectively as *dark energy* and *dark energy* or *vacuum energy* of a matter fluid in its local rest frame. We emphasize that also in the present case a non-vanishing constant ε is required to ensure a non-null dark matter density contribution.

4.3. Cylindrical Symmetry: Kerr-De Sitter Solution

Let us now examine what happens when considering the Kerr metric with cosmological constant (Kerr-De Sitter metric). Usually it is represented in Boyer-Lunquist like co-ordinates, as the metric which defines the interval:

$$\begin{aligned} ds^2 = & -\rho^2\left[\frac{dr^2}{\Delta_r} + \frac{d\theta^2}{\Delta_\theta}\right] - \frac{\Delta_\theta \sin^2\theta}{\rho^2}\left[a\frac{dt}{\Xi} - \left(r^2+a^2\right)\frac{d\phi}{d\Xi}\right]^2 \\ & + \frac{\Delta_r}{\rho^2}\left[\frac{dt}{d\Xi} - a\sin^2\theta\frac{d\phi}{\Xi}\right]^2, \end{aligned} \tag{88}$$

where:

$$\begin{aligned} & \Delta_\theta = 1 + \frac{\Lambda}{3}a^2\cos^2\theta, \quad \Delta_r = \left(1 - \frac{\Lambda r^2}{3}\right)\left(r^2+a^2\right) - r_s r, \\ & \rho^2 = r^2 + a^2\cos^2\theta, \quad \Xi = 1 + \frac{\Lambda}{3}a^2, \\ & r_s = \frac{2GM}{c^2}, \quad a = \frac{2c^2 J}{r_s}. \end{aligned} \tag{89}$$

J is the angular momentum of the rotating mass M.

Even if when represented in such co-ordinates the Kerr-De Sitter metric is not diagonal, because of the cylindrical symmetry, and cannot fulfill the condition (79), it can be diagonalized thanks to the following *untwisting* co-ordinate transformation [19]:

$$\begin{aligned} & T = \frac{t}{\Xi}, \quad \bar{\phi} = \phi - \frac{a\Lambda t}{3\Xi}, \quad y\cos\Theta = r\cos\theta, \\ & y^2 = \frac{r^2\Delta_\theta + a^2\sin^2\theta}{\Xi}. \end{aligned} \tag{90}$$

The interval assuming the empty space-time De Sitter form:

$$ds^2 = \left(1 - \frac{\Lambda y^2}{3}\right)dT^2 - \frac{1}{1-\frac{\Lambda y^2}{3}}dy^2 - y^2\left(d\Theta^2 + \sin^2\Theta d\bar{\phi}^2\right). \tag{91}$$

It has been shown that the previous metric may be written as:

$$g_{\mu\nu} = \eta_{\mu\nu} + \frac{2r_s r}{\rho^2} l_\mu l_\nu, \tag{92}$$

$\mathbf{l} \equiv (l_\mu)$ being a light-like vector and $(\eta_{\mu\nu})$ the flat metric. Then, according to Taub's results [20] [21], condition (79) is satisfied.

From the previous results provided for the Schwarzschild-De Sitter metric we deduce, after rescaling time by the factor $(1+\varepsilon)^{1/2}$, for the widened Kerr-De Sitter metric:

$$\begin{aligned} &g_{00} = c^2\left(1-\frac{\Lambda}{3}y^2\right)(1+\varepsilon), \quad g_{11} = -\frac{1}{1-\frac{\Lambda}{3}y^2}, \\ &g_{22} = -y^2, \quad g_{33} = -y^2 \sin^2\theta. \end{aligned} \tag{93}$$

The Ricci tensor follows:

$$\begin{aligned} &R_{00} = -\Lambda c^2\left(1-\frac{\Lambda}{3}r^2\right)(1+\varepsilon), \\ &R_{11} = \frac{\Lambda}{1-\frac{\Lambda}{3}r^2}, \quad R_{22} = \Lambda r^2, \quad R_{33} = \Lambda r^2 \sin^2\theta. \end{aligned} \tag{94}$$

The powers of the Ricci tensor are consequently:

$$\begin{aligned} &R_{00}^{(k)} = c^2(-\Lambda)^k\left(1-\frac{\Lambda}{3}r^2\right)(1+\varepsilon), \quad R_{11}^{(k)} = -\frac{(-\Lambda)^k}{1-\frac{\Lambda}{3}r^2}, \\ &R_{22}^{(k)} = -(-\Lambda)^k r^2, \quad R_{33}^{(k)} = -(-\Lambda)^k r^2 \sin^2\theta. \end{aligned} \tag{95}$$

Then the field equations become again:

$$R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} = \Lambda g_{\mu\nu},$$

And then:

$$\begin{aligned} &\kappa T_{00} = \Lambda\varepsilon c^2\left(1-\frac{\Lambda}{3}y^2\right) + \Lambda c^2\left(1-\frac{\Lambda}{3}y^2\right), \\ &\kappa T_{11} = -\frac{\Lambda}{1-\frac{\Lambda}{3}y^2}, \\ &\kappa T_{22} = -\Lambda r^2, \\ &\kappa T_{33} = -\Lambda r^2 \sin^2\theta. \end{aligned} \tag{96}$$

So we may interpret:

$$\kappa\rho^{(dm)} = \Lambda\varepsilon, \quad \kappa p^{(de)} = -\kappa\rho^{(de)} = -\Lambda, \tag{97}$$

as *dark matter* and *dark energy* contributions.

5. Geodesics

In Section 3.2 we provided relation (31), which now, being $n=4$, specializes as:

$$G^{\mu\nu} = \frac{1}{\sqrt{\alpha}}\sum_{l=1}^{4} l\mathcal{L}_{I^{(k)}} R^{(l-1)\mu\nu}, \tag{98}$$

tensor $\mathsf{G} \equiv \left(G^{\mu\nu}\right)$ satisfying the metricity condition (41), which defines the connection coefficients (44):

$$\Gamma^{\alpha}_{\mu\nu} = \frac{1}{2} G^{\alpha\beta} \left(G_{\mu\beta,\nu} + G_{\nu\beta,\mu} - G_{\mu\nu,\beta} \right),$$

from which one is able to obtain the equation of the geodesic trajectory of a free particle traveling across space-time with velocity $\mathbf{u} \equiv \left(u^{\alpha}\right)$:

$$\frac{\mathrm{d}u^{\alpha}}{\mathrm{d}s} = -\Gamma^{\alpha}_{\mu\nu} u^{\mu} u^{\nu}, \tag{99}$$

where $\mathrm{d}s$ is the interval defined by Equation (42). When $R_{\mu\nu}$ is an isotropic tensor, as it happens for the physically relevant solutions examined, so that condition (53) holds, we easily obtain:

$$R^{(l-1)\mu\nu} = (-1)^{l-1} \Lambda^{l-1} g^{\mu\nu}, \tag{100}$$

from which it follows:

$$G^{\mu\nu} = \frac{f}{\sqrt{\alpha}} g^{\mu\nu}, \tag{101}$$

where we have introduced:

$$G^{\mu\nu} = \frac{1}{\sqrt{\alpha}} \left(\mathcal{L}_{I^{(1)}} R^{(0)} + 2\mathcal{L}_{I^{(2)}} R^{(1)} + 3\mathcal{L}_{I^{(3)}} R^{(2)} + 4\mathcal{L}_{I^{(4)}} R^{(3)} \right) g^{\mu\nu}$$

$$f = \mathcal{L}_{I^{(1)}} - 2\Lambda \mathcal{L}_{I^{(2)}} + 3\Lambda^2 \mathcal{L}_{I^{(3)}} - 4\Lambda^3 \mathcal{L}_{I^{(4)}}. \tag{102}$$

From (57) we obtain in explicit form:

$$\begin{aligned} f = {} & 8\Lambda \left(-a_{11} + a_{12}\Lambda - a_{13}\Lambda^2 + a_{14}\Lambda^3 \right) \\ & -16\Lambda^2 \left(-a_{12} + a_{22}\Lambda - a_{23}\Lambda^2 + a_{24}\Lambda^3 \right) \\ & +24\Lambda^3 \left(-a_{13} + a_{23}\Lambda - a_{33}\Lambda^2 + a_{34}\Lambda^3 \right) \\ & -32\Lambda^4 \left(-a_{14} + a_{24}\Lambda - a_{34}\Lambda^2 + a_{44}\Lambda^3 \right). \end{aligned} \tag{103}$$

The constant parameter f results to be proportional at most to the first seven power of Λ. And on the special case that $\mathsf{A} \equiv \left(a_{lk}\right)$ is diagonal we have simply:

$$f = -8\Lambda \left(a_{11} + 2a_{22}\Lambda^2 + 3a_{33}\Lambda^4 + 4a_{44}\Lambda^6 \right). \tag{104}$$

In principle experimental observations should give information to verify such predictions and determine the coefficients a_{jk}, but measurements need to be very refined because of the small values of the cosmological constant and the trajectory deviations.

The ratio of the determinants α, defined by (38) can be evaluated taking the determinants of both sides in the relation (101). We have:

$$\frac{1}{G} = \frac{f^4}{\alpha^2 g}. \tag{105}$$

From which:

$$\alpha = f^4. \tag{106}$$

It follows into (101):

$$G^{\mu\nu} = \frac{1}{f} g^{\mu\nu}. \tag{107}$$

And also:

$$G_{\mu\nu} = f g_{\mu\nu}. \tag{108}$$

so that:

$$G^{\mu\lambda} G_{\lambda\nu} = g^{\mu\lambda} g_{\lambda\nu} = \delta^{\mu}_{\nu}.$$

The equation of a geodesic trajectory is given by:

$$\frac{\mathrm{d}u^{\alpha}}{\mathrm{d}s} = -\frac{1}{2} G^{\alpha\beta} \left(G_{\mu\beta,\nu} + G_{\nu\beta,\mu} - G_{\mu\nu,\beta} \right) u^{\mu} u^{\nu}, \tag{109}$$

And developing calculations:

$$\frac{\mathrm{d}u^{\alpha}}{\mathrm{d}s} = -\gamma^{\alpha}_{\mu\nu} u^{\mu} u^{\nu} - \left(u^{\alpha} u^{\nu} - \frac{1}{2} g^{\alpha\nu} \right) (\log f)_{,\nu}, \tag{110}$$

where:

$$\gamma^{\alpha}_{\mu\nu} = \frac{1}{2} g^{\alpha\beta} \left(g_{\mu\beta,\nu} + g_{\nu\beta,\mu} - g_{\mu\nu,\beta} \right). \tag{111}$$

The last term in (110) represents the strength which deviates trajectory from usual GR prevision. Significantly that term is vanishing since Λ is a constant respect to the space-time co-ordinates. So the form of the equation of the geodesic trajectories is the same as in usual GR while the universe is expanding. Possible observed deviations will prove that Λ is not a constant being a function of the co-ordinates x^{μ}, and at most through the first seven powers of Λ.

6. Conclusions

We have tested a way to generalize Einstein General Relativity, which is an alternative to the $f(R), f(R,T)$ family of theories, starting from a scalar Lagrangian density which is a general function of all the independent invariants that one is able to obtain from the powers of the Ricci tensor. We have derived both the field equations and interpreted them from the point of view of usual General Relativity, showing how *dark matter* and *dark energy* contributions emerge. Most physically relevant exact solutions endowed either with spherical or cylindrical symmetries have been found. The geodesic equation has been obtained for particle trajectory which includes a geodesic deviation tidal force arising from the non-linear dependence of the Lagrangian density on the Ricci tensor powers, responsible for the *dark matter* and *dark energy* contributions. In principle, the theory involves too many constant undetermined parameters a_{jk}, but they cancel in correspondence to the most physically relevant solutions. Meanwhile, the parameters appear within the geodesic equations if and only if the cosmological constant is not a constant but depends on the co-ordinates. In the latter case, such parameters a_{jk} are to be determined in order to fit experimental data.

The WGR theory, like the usual GR, being a *macroscopic* theory, is still unable to provide a *microscopic* model to explain how the *dark matter* and *dark energy* contributions, involving parameters like Λ and ε, can arise from fundamental fields and elementary particles interaction. Possibly a quantum field theory including gravity could offer a more exhaustive understanding of how the universe intimately behaves. A similar situation appeared in the past, when thermodynamics provided a *macroscopic* theory of heat behavior and the *microscopic* model, which later kinetic theory of gases and mechanical statistics provided, was not yet set up.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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