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On the Transition from Newtonian Gravity to General Relativity

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Abstract

Using an alternative representation of the Ricci tensor, we argue that the theory of gravitation can be easily developed in such a way that the formal description of gravity in the transition from classical Newtonian physics to general relativity remains essentially unchanged. That is to say, we show how arguments concerning the plausible conceptual compatibility of Newtonian and general-relativistic models of gravity can be replaced by a demonstration of their actual formal identity. More specifically, we find that both the classical Newtonian and the general relativistic field equations are equivalent to a velocity-field divergence equation of the form $\boldsymbol{v}\left[\operatorname{div}(\boldsymbol{v})\right]+\operatorname{div}(\boldsymbol{v},\boldsymbol{v})=-4\pi\rho$ where the term $\operatorname{div}(\boldsymbol{v},\boldsymbol{v})$ is defined to be the trace of the square of the Jacobian derivative matrix of $\boldsymbol{v}$ (or of its general-relativistic analogue).

Keywords

General Relativity, Newtonian Gravitation, Field Equations, Uniform Velocity-Field Representation

1. Introduction

Those who wish to master the special and general theories of relativity commonly have to pass certain milestones in questioning and re-thinking some of their most dearly held reality assumption. The first of these, without a doubt, is the realization that the apparent constancy and observer-independence of the speed of light in Maxwell's electromagnetic wave equation causes space and time in special relativity to lose their familiar, Newtonian absoluteness. Once this vital fact has been grasped, the students of relativity can move on to explore in more detail the inner workings of special-relativistic dynamics, but they are liable to again feel confounded when these dynamics are subsequently integrated into a general-relativistic theory of gravity that ultimately aims to achieve nothing less than the rigorous mathematical elucidation of the large-scale geometric structure of the entire physical universe.

In facing this latter conceptual challenge, it is important, first and foremost, to thoroughly appreciate the fundamental significance of the equivalence of inertial and gravitational masses. For it is precisely this equivalence that renders the spacetime environment in a constant gravitational field indistinguishable from the spacetime environment that is experienced by a gravity-free observer who undergoes a steadily accelerated motion. And it is this equivalence as well and in consequence, that makes it possible to re-conceive a seeming acceleration under the influence of a gravitational force as a force-free uniform motion relative to an accelerated observer and that, thereby, also makes it possible to re-conceive the universe as a whole as a patchwork of infinitesimal constant-gravity regions that are each equal in structure to the spacetime world within a constant-acceleration frame of reference.

However, when it comes to the problem of how such a patchwork ought to be organized so as to yield a theory of gravity that is compatible, in the classical limit, with Newton's absolute-spacetime conception of a gravitational force that acts at a distance instantaneously, the matter quickly gets confusing. For not only must those who endeavor to tackle this problem be familiar with the mathematical formalisms of differential geometry, but they also must learn to bridge in their minds the seemingly deep divide between these formalisms on the one hand and the more elementary mathematical tools employed by Newton on the other. And the purpose of the present paper, therefore, is to suggest a derivation of the field equations of general relativity that greatly narrows that mathematical divide or even closes it completely.

That said, we must hasten to add that our purpose is not to overrule or discredit common approaches to the bridging of the gap between the theories of gravity of Newton and Einstein but merely to offer an alternative point of view. The method of stratification (as explained in Chapter 12 of [1]) and the consideration of weak-field limits (as discussed in Section 8.1 of [2]), for example, are perfectly valid and well-known means of establishing the inherent compatibility of Newton's and Einstein's theories, but they do not demonstrate their actual formal identity—as we hope to do in Sections 3 and 4 below. Moreover, for further discussions of the relation between Newtonian gravitation and general relativity the reader is referred to [3] [4] [5] and [6].

2. Prerequisites and Result Summary

Since general relativity cannot be divorced from differential geometry, it behooves us to recall to begin with some pertinent mathematical facts and constructions: denoting by TM the tangent space bundle of a C^∞-manifold M, it can be shown (see for instance [7], pp. 70-73) that there exists an open set $V \subset TM$ and a differentiable map $\exp : V \to M$ with the following properties:

1) For all $p \in M$ and all $\boldsymbol{w} \in T_p M$ the set $I_{\boldsymbol{w}} := \{ t \in R \mid t\boldsymbol{w} \in V \cap T_p M \}$ is an open interval containing zero, and $c_{\boldsymbol{w}} : I_{\boldsymbol{w}} \to M$, $t \mapsto \exp(t\boldsymbol{w})$ is a geodesic that satisfies the initial conditions $c_{\boldsymbol{w}}(0) = p$ and $c'_{\boldsymbol{w}}(0) = \boldsymbol{w}$. Furthermore, $c_{\boldsymbol{w}}$ is maximal and unique in the sense that any geodesic that satisfies the same initial conditions is a restriction of $c_{\boldsymbol{w}}$ to a subinterval of $I_{\boldsymbol{w}}$.

2) For all $p \in M$ there exists an open neighborhood V_p of $\boldsymbol{0}$ in $V \cap T_p M$ such that the restriction $\exp_p := \exp\restriction_{V_p}$ is a diffeomorphism from V_p onto an open neighborhood U_p of p in M. (Note: w.l.o.g. we may assume that for all $p \in M$ and all $\boldsymbol{w} \in T_p M$ the set $\{ t \in R \mid t\boldsymbol{w} \in V_p \}$ is an open interval.)

Assuming further that M is equipped with a Lorentz inner product $g(.,.)$ (with the sign convention (+−−−)) and that p is a given point in M, we denote by TL_p the set of timelike unit vectors in $T_p M$, *i.e.*,

$$TL_p := \{ \boldsymbol{v} \in T_p M \mid g(\boldsymbol{v}, \boldsymbol{v}) = 1 \}.$$

Moreover, by ML_p we denote the set of all points $q \in U_p \setminus \{p\}$ (with U_p as defined above) for which there exists a $\boldsymbol{v} \in TL_p$ and a $t \in I_{\boldsymbol{v}}$ such that $\exp_p(t\boldsymbol{v}) = q$, and by $\boldsymbol{u}$ we denote the geodesic velocity field that the exponential map induces on ML_p, *i.e.*, for $q = \exp_p(t\boldsymbol{v}) = c_{\boldsymbol{v}}(t) \in ML_p$ we set

$$\boldsymbol{u}(q) := c'_{\boldsymbol{v}}(t). \tag{1}$$

Since $c_{\boldsymbol{v}}$ is a geodesic, it follows that

$$g(\boldsymbol{u}(q), \boldsymbol{u}(q)) = g(c'_{\boldsymbol{v}}(t), c'_{\boldsymbol{v}}(t)) = g(c'_{\boldsymbol{v}}(0), c'_{\boldsymbol{v}}(0)) = g(\boldsymbol{v}, \boldsymbol{v}) = 1.$$

To complete our setup, we pick an arbitrary Lorentz frame $\{ \boldsymbol{e}_\mu(p) \}_{\mu=0}^{3}$ at p and create a frame field $\{ \boldsymbol{e}_\mu \}_{\mu=0}^{3}$ on U_p by parallel shifting the vectors $\boldsymbol{e}_\mu(p)$ along the geodesics that originate at p (and are described by $\exp_p$). This construction readily implies that

$$\nabla_{\boldsymbol{u}(q)} \boldsymbol{e}_\mu = \boldsymbol{0},$$

for all $q \in ML_p$ and that

$$\nabla_{\boldsymbol{w}} \boldsymbol{e}_\mu = \boldsymbol{0} \tag{2}$$

for all $\boldsymbol{w} \in T_p M$ and all $\mu \in \{0,1,2,3\}$. Setting further

$$\boldsymbol{e}^0 := \boldsymbol{e}_0, \boldsymbol{e}^1 := -\boldsymbol{e}_1, \boldsymbol{e}^2 := -\boldsymbol{e}_2, \boldsymbol{e}^3 := -\boldsymbol{e}_3,$$

a dual basis field on U_p is $\{ g(., \boldsymbol{e}^\mu) \}_{\mu=0}^{3}$, and the general-relativistic analogue of the Jacobian derivative matrix of $\boldsymbol{u}$ is

$$D\boldsymbol{u} := \left(g\left(\nabla_{\boldsymbol{e}_\mu} \boldsymbol{u}, \boldsymbol{e}^\nu \right) \right)_{\mu,\nu=0}^{3},$$

where ν is the row index and μ the column index.

Using brackets, as usual, to indicate the taking of the directional derivative (that is, $\boldsymbol{u}[f] = \partial_{\boldsymbol{u}} f$), we will show in Sections 3 and 4, that the matrix $D\boldsymbol{u}$ satisfies the equation

$$\boxed{\boldsymbol{u}[\operatorname{div}(\boldsymbol{u})] = \boldsymbol{u}[\operatorname{tr}(D\boldsymbol{u})] = -\operatorname{tr}(D\boldsymbol{u})^2} \tag{3}$$

in the classical Newtonian case, in which particle motion is governed by the Laplace equation $\Delta V = 0$, as well as in the free-particle special-relativistic case. Furthermore, since the validity of (3) in the classical Newtonian case turns out to be *equivalent* to the validity of the classical vacuum field equation $\Delta V = 0$, it is perfectly reasonable to require, in light of these results, that Equation (3) be satisfied as well in general-relativistic vacuum spacetime. And it is precisely this latter requirement that turns out to be equivalent to the validity of the vacuum field equation

$$R_{\mu\nu} = 0,$$

or equivalently,

$$\mathrm{Ric}(\boldsymbol{v}, \boldsymbol{w}) = 0.$$

Moreover, in Section 5 we will show that a similar unifying description—analogous to (3)—can be given for the classical and general-relativistic *matter* field equations as well. That is to say, in using velocity-field divergences, we will be able to reveal that the classical Newtonian and general-relativistic theories of gravity are formally essentially identical.

Introducing the natural, generalizing notation

$$\boxed{\mathrm{div}(\boldsymbol{v}, \boldsymbol{w}) := \mathrm{tr}(D\boldsymbol{v}D\boldsymbol{w})},$$

we summarize our findings in the following table so as to illustrate thereby how seemingly disparate classical and relativistic field equations become perfectly unified when represented by velocity-field divergence equations:

Equation	Standard Description	Velocity Field Description
cl. vac. feq.	$\Delta V = 0$	$\boldsymbol{v}[\mathrm{div}(\boldsymbol{v})] + \mathrm{div}(\boldsymbol{v}, \boldsymbol{v}) = 0$
rel. vac. feq.	$\mathrm{Ric}(\boldsymbol{v}, \boldsymbol{w}) = 0$	$\boldsymbol{u}[\mathrm{div}(\boldsymbol{u})] + \mathrm{div}(\boldsymbol{u}, \boldsymbol{u}) = 0$
cl. mat. feq.	$\Delta V = 4\pi\rho$	$\boldsymbol{v}[\mathrm{div}(\boldsymbol{v})] + \mathrm{div}(\boldsymbol{v}, \boldsymbol{v}) = -4\pi\rho$
rel. mat. feq. (free part.)	$\mathrm{Ric} - (R/2)g = 8\pi T$	$\boldsymbol{w}[\mathrm{div}(\boldsymbol{w})] + \mathrm{div}(\boldsymbol{w}, \boldsymbol{w}) = -4\pi\rho$ for $\boldsymbol{w} = \boldsymbol{u}$, $\boldsymbol{u}_\perp$, $(\boldsymbol{u} + \boldsymbol{u}_\perp)/\sqrt{2}$

As a note of caution, we wish to add that all the results derived in the present paper appear to be so elementary in character that it is difficult to imagine that they have not been previously established. However, since the present author is not aware of any pertinent reference, the results in question are here being offered—with considerable hesitation—as provisional novelties.

3. The Classical Vacuum Field Equation

To begin with, we consider the free-particle Newtonian case in which the possible particle trajectories are straight lines in four-dimensional absolute spacetime. Thus, we define the analogue of $\boldsymbol{u}$ in (1) (with the base point p at the origin $(0,0,0,0)$) via the equation

$$\boldsymbol{v}\left(x^0,x^1,x^2,x^3\right):=\begin{pmatrix}1\\ x^1/x^0\\ x^2/x^0\\ x^3/x^0\end{pmatrix},$$

and observe that

$$D\boldsymbol{v}=\begin{pmatrix}0 & 0 & 0 & 0\\ -x^1/\left(x^0\right)^2 & 1/x^0 & 0 & 0\\ -x^2/\left(x^0\right)^2 & 0 & 1/x^0 & 0\\ -x^3/\left(x^0\right)^2 & 0 & 0 & 1/x^0\end{pmatrix}.$$

This yields

$$\begin{aligned}\boldsymbol{v}\left[\operatorname{div}(\boldsymbol{v})\right]&=-\frac{3}{\left(x^0\right)^2}\\ &=-\operatorname{tr}\begin{pmatrix}0 & 0 & 0 & 0\\ 0 & 1/\left(x^0\right)^2 & 0 & 0\\ 0 & 0 & 1/\left(x^0\right)^2 & 0\\ 0 & 0 & 0 & 1/\left(x^0\right)^2\end{pmatrix}\\ &=-\operatorname{tr}\left(D\boldsymbol{v}\right)^2\end{aligned}$$

and Equation (3) has therefore been shown to be valid.

To proceed, we assume that a Newtonian test particle of mass m moves in a gradient force field

$$\boldsymbol{F}\left(x^0,x^1,x^2,x^2\right)=-m\begin{pmatrix}0\\ \partial V/\partial x^1\\ \partial V/\partial x^2\\ \partial V/\partial x^3\end{pmatrix}=-m\nabla V\left(x^0,x^1,x^2,x^3\right)$$

that satisfies the field equation

$$\Delta V=\operatorname{div}(\nabla V)=\sum_{\mu=1}^{3}\frac{\partial^2 V}{\left(\partial x^\mu\right)^2}=0.$$

Furthermore, setting $q:=\left(x^0,x^1,x^2,x^3\right)$, we denote by $c_q(s)=\left(z_q^\mu(s)\right)_{\mu=0}^3$ the spacetime curve, traced out by the test particle, that satisfies the boundary conditions

$$c_q(0)=(0,0,0,0)\ \text{and}\ c_q\left(x^0\right)=q$$

as well as the time-component condition $z_q^0(s)=s$. Given this definition, it is natural and plausible to assume that there exists a constant $T>0$ such that the boundary and time-component conditions above determine c_q uniquely for all $q\in(0,T)\times R^3$ (or perhaps $(0,T)\times N$ for some large' set $N\subset R^3$ that contains $(0,0,0)$). That is to say, we will in essence assume that

$$v(q) := c_q'\left(x^0\right) = \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=x^0}\begin{pmatrix} z_q^0(s) \\ z_q^1(s) \\ z_q^2(s) \\ z_q^3(s) \end{pmatrix} = \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=x^0}\begin{pmatrix} s \\ z_q^1(s) \\ z_q^2(s) \\ z_q^3(s) \end{pmatrix} = \begin{pmatrix} 1 \\ \left(z_q^1\right)'\left(x^0\right) \\ \left(z_q^2\right)'\left(x^0\right) \\ \left(z_q^3\right)'\left(x^0\right) \end{pmatrix}$$

is a well-defined vector field on a suitable subset of R^4 that contains $(0,0,0,0)$. Given this assumption, it follows that

$$v\left(c_q(s)\right) = c_q'(s)$$

for all $s \in (0,T)$, and that, by implication,

$$-\nabla V\big|_{c_q(s)} = \frac{1}{m}F\left(c_q(s)\right) = c_q''(s) = \frac{\mathrm{d}}{\mathrm{d}s}v\left(c_q(s)\right) = Dv\left(c_q(s)\right)c_q'(s).$$

Hence

$$\begin{aligned} v(q)\left[\operatorname{div}(v)\right] &= c_q'\left(x^0\right)\left[\operatorname{div}(v)\right] \\ &= \sum_{\mu,\nu=0}^{3}\left(z_q^\nu\right)'\left(x^0\right)\frac{\partial^2\left(z_q^\mu\right)'\left(x^0\right)}{\partial x^\nu \partial x^\mu} \\ &= \sum_{\mu,\nu=0}^{3}\frac{\partial}{\partial x^\mu}\left(\left(z_q^\nu\right)'\left(x^0\right)\frac{\partial\left(z_q^\mu\right)'\left(x^0\right)}{\partial x^\nu}\right) - \sum_{\mu,\nu=0}^{3}\frac{\partial\left(z_q^\nu\right)'\left(x^0\right)}{\partial x^\nu}\frac{\partial\left(z_q^\mu\right)'\left(x^0\right)}{\partial x^\nu} \\ &= \sum_{\mu=1}^{3}\frac{\partial}{\partial x^\mu}\left(-\left.\frac{\partial V}{\partial x^\mu}\right|_{c_q\left(x^0\right)=q}\right) - \operatorname{tr}\left(\left(Dv\right)^2\right)\Big|_{c_q\left(x^0\right)=q} \\ &= -\Delta V\big|_q - \operatorname{tr}\left(\left(Dv\right)^2\right)\Big|_q, \end{aligned}$$

and, by implication,

$$\boxed{v\left[\operatorname{div}(v)\right] + \operatorname{div}(v,v) = -\Delta V.} \tag{4}$$

Consequently, the classical vacuum field equation $\Delta V = 0$ is indeed satisfied if and only if

$$\boxed{v\left[\operatorname{div}(v)\right] + \operatorname{div}(v,v) = 0.}$$

4. The Relativistic Vacuum Field Equation

In order to prove that Equation (3) is satisfied as well in the free-particle special relativistic case, we set

$$g := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

and

$$u := \frac{1}{\sqrt{(x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2}} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}.$$

Given these definitions, we find (by way of a trivial computation) that

$$D\boldsymbol{u} = \frac{\boldsymbol{Id} - \boldsymbol{u}(\boldsymbol{gu})^t}{\sqrt{(x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2}} = \frac{\operatorname{div}(\boldsymbol{u})}{3}\left(\boldsymbol{Id} - \boldsymbol{u}(\boldsymbol{gu})^t\right),$$

and since

$$\boldsymbol{Id} - \boldsymbol{u}(\boldsymbol{gu})^t = \left(\boldsymbol{Id} - \boldsymbol{u}(\boldsymbol{gu})^t\right)^2,$$

it follows (again by way of a trivial computation) that

$$\boldsymbol{u}\left[\operatorname{div}(\boldsymbol{u})\right] = -\frac{(\operatorname{div}(\boldsymbol{u}))^2}{3} = -\frac{(\operatorname{div}(\boldsymbol{u}))^2 \operatorname{tr}\left(\left(\boldsymbol{Id} - \boldsymbol{u}(\boldsymbol{gu})^t\right)^2\right)}{9}$$
$$= -\operatorname{tr}\left((D\boldsymbol{u})^2\right) = -\operatorname{div}(\boldsymbol{u}, \boldsymbol{u}),$$

as desired.

In the light of this result and in the light as well of the preceding result concerning classical free particles, it is perfectly reasonable to expect that general-relativistic vacuum spacetime should be structured in such a way that

$$\boxed{\boldsymbol{u}\left[\operatorname{div}(\boldsymbol{u})\right] + \operatorname{div}(\boldsymbol{u}, \boldsymbol{u}) = 0.} \tag{5}$$

In order to show that this equation is indeed equivalent to the vacuum field equation $R_{\mu\nu} = 0$, we will now proceed to prove the following theorem:

Theorem 4.1. *For all spacetime vector fields* $\boldsymbol{v}$ *and* $\boldsymbol{w}$ *it is the case that*

$$\operatorname{Ric}(\boldsymbol{v}, \boldsymbol{w}) = \operatorname{div}(\nabla_{\boldsymbol{v}} \boldsymbol{w}) - \boldsymbol{v}\left[\operatorname{div}(\boldsymbol{w})\right] - \operatorname{div}(\boldsymbol{v}, \boldsymbol{w}).$$

Note: the term on the right is indeed a tensor because it is easily seen to be C^∞*-bilinear.*

Proof. Given a spacetime event *p*, and given the basis fields $\boldsymbol{e}_\mu$, defined in Section 2, we may employ Einstein's summation convention in conjunction with (4) to infer that

$$\begin{aligned}
\operatorname{Ric}(\boldsymbol{v}, \boldsymbol{w})\big|_p &= g\left(R(\boldsymbol{e}_\mu, \boldsymbol{v})\boldsymbol{w}, \boldsymbol{e}^\mu\right)\big|_p \\
&= g\left(\nabla_{\boldsymbol{e}_\mu}\nabla_{\boldsymbol{v}}\boldsymbol{w} - \nabla_{\boldsymbol{v}}\nabla_{\boldsymbol{e}_\mu}\boldsymbol{w} - \nabla_{[\boldsymbol{e}_\mu, \boldsymbol{v}]}\boldsymbol{w}, \boldsymbol{e}^\mu\right)\Big|_p \\
&= \operatorname{div}(\nabla_{\boldsymbol{v}}\boldsymbol{w})\big|_p - \boldsymbol{v}(p)\left[g\left(\nabla_{\boldsymbol{e}_\mu}\boldsymbol{w}, \boldsymbol{e}^\mu\right)\right] + g\left(\nabla_{\boldsymbol{e}_\mu}\boldsymbol{w}, \nabla_{\boldsymbol{v}}\boldsymbol{e}^\mu\right)\Big|_p \\
&\quad - g\left(\nabla_{\nabla_{\boldsymbol{e}_\mu}\boldsymbol{v} - \nabla_{\boldsymbol{v}}\boldsymbol{e}_\mu}\boldsymbol{w}, \boldsymbol{e}^\mu\right)\Big|_p \\
&= \operatorname{div}(\nabla_{\boldsymbol{v}}\boldsymbol{w})\big|_p - \boldsymbol{v}(p)\left[\operatorname{div}(\boldsymbol{w})\right] - g\left(\nabla_{\boldsymbol{e}_\mu}\boldsymbol{v}, \boldsymbol{e}^\nu\right) g\left(\nabla_{\boldsymbol{e}_\nu}\boldsymbol{w}, \boldsymbol{e}^\mu\right)\Big|_p \\
&= \operatorname{div}(\nabla_{\boldsymbol{v}}\boldsymbol{w})\big|_p - \boldsymbol{v}(p)\left[\operatorname{div}(\boldsymbol{w})\right] - \operatorname{div}(\boldsymbol{v}, \boldsymbol{w})\big|_p,
\end{aligned}$$

as desired. □

For later reference and in order to further explore this result, we introduce the following definition:

Definition 4.2. Given a spacetime event $q \in M$ and given a tangent vector $\boldsymbol{v} \in T_q M$, we say that a vector field $\boldsymbol{w}$ is a *geodesic extension* of $\boldsymbol{v}$ if $\boldsymbol{w}(q) = \boldsymbol{v}$ and if the domain of $\boldsymbol{w}$ contains an open neighborhood U of q such that $\nabla_{\boldsymbol{w}} \boldsymbol{w}|_p = \boldsymbol{0}$ for all $p \in U$.

Lemma 4.3. *Every spacetime tangent vector* $\boldsymbol{v} \in T_q M$ *admits a geodesic extension.*

Proof. This fact is essentially well known and hardly requires a proof, but one way to establish it is to use a local coordinate system at q to translate the geodesic equation $\nabla_{\boldsymbol{w}} \boldsymbol{w} = \boldsymbol{0}$ into a first-order system of differential equations and to find a solution of this system that satisfies the condition $\boldsymbol{w}(q) = \boldsymbol{v}$. Alternatively, a proof using the exponential map is feasible as well. □

Corollary 4.4. *If* $\boldsymbol{w}$ *is a geodesic extension of a vector* $\boldsymbol{v} \in T_q M$, *then*

$$\mathrm{Ric}(\boldsymbol{v}, \boldsymbol{v}) = -\boldsymbol{w}(q)\left[\mathrm{div}(\boldsymbol{w})\right] - \mathrm{div}(\boldsymbol{w}, \boldsymbol{w})\big|_q .$$

Proof. This is a trivial consequence of Theorem 4.1 and the fact that $\nabla_{\boldsymbol{w}} \boldsymbol{w} = \boldsymbol{0}$ on an open neighborhood of q. □

Since the geodesic velocity field $\boldsymbol{u}$, defined in (1), satisfies the equation $\nabla_{\boldsymbol{u}} \boldsymbol{u} = \boldsymbol{0}$, we may apply Theorem 4.1 to infer that

$$\mathrm{Ric}(\boldsymbol{u}, \boldsymbol{u}) = 0$$

if and only if (5) is satisfied. Using polarization in conjunction with the familiar symmetry of the Ricci-tensor (as well as the fact that the time-like unit vectors $\boldsymbol{u}$ span $T_q M$ as the base point p in the definition of $\boldsymbol{u}$ is properly varied), it follows that indeed Equation (5) is satisfied if and only if $R_{\mu\nu} = 0$.

To conclude our discussion in this section, it is worth noting that both terms in Equation (5) are frame-independent. That is to say, if $\{\boldsymbol{f}_\mu\}_{\mu=0}^3$ is some other arbitrary frame field, then

$$\mathrm{div}(\boldsymbol{u}) = g\left(\nabla_{\boldsymbol{e}_\mu} \boldsymbol{u}, \boldsymbol{e}^\mu\right) = g\left(\nabla_{\boldsymbol{f}_\mu} \boldsymbol{u}, \boldsymbol{f}^\mu\right)$$

and

$$\mathrm{div}(\boldsymbol{u}, \boldsymbol{u}) = g\left(\nabla_{\boldsymbol{e}_\mu} \boldsymbol{u}, \boldsymbol{e}^\nu\right) g\left(\nabla_{\boldsymbol{e}_\nu} \boldsymbol{u}, \boldsymbol{e}^\mu\right) = g\left(\nabla_{\boldsymbol{f}_\mu} \boldsymbol{u}, \boldsymbol{f}^\nu\right) g\left(\nabla_{\boldsymbol{f}_\nu} \boldsymbol{u}, \boldsymbol{f}^\mu\right).$$

The first of these two equations is well known, and the second can be derived as follows:

$$\begin{aligned}
& g\left(\nabla_{\boldsymbol{f}_\mu} \boldsymbol{u}, \boldsymbol{f}^\nu\right) g\left(\nabla_{\boldsymbol{f}_\nu} \boldsymbol{u}, \boldsymbol{f}^\mu\right) \\
& = g\left(\boldsymbol{f}_\mu, \boldsymbol{e}^\alpha\right) g\left(\boldsymbol{f}^\nu, \boldsymbol{e}_\beta\right) g\left(\boldsymbol{f}_\nu, \boldsymbol{e}^\gamma\right) g\left(\boldsymbol{f}^\mu, \boldsymbol{e}_\delta\right) g\left(\nabla_{\boldsymbol{e}_\alpha} \boldsymbol{u}, \boldsymbol{e}^\beta\right) g\left(\nabla_{\boldsymbol{e}_\gamma} \boldsymbol{u}, \boldsymbol{e}^\delta\right) \\
& = g\left(\boldsymbol{e}_\delta, \boldsymbol{e}^\alpha\right) g\left(\boldsymbol{e}_\beta, \boldsymbol{e}^\gamma\right) g\left(\nabla_{\boldsymbol{e}_\alpha} \boldsymbol{u}, \boldsymbol{e}^\beta\right) g\left(\nabla_{\boldsymbol{e}_\gamma} \boldsymbol{u}, \boldsymbol{e}^\delta\right) \\
& = g\left(\nabla_{\boldsymbol{e}_\alpha} \boldsymbol{u}, \boldsymbol{e}^\beta\right) g\left(\nabla_{\boldsymbol{e}_\beta} \boldsymbol{u}, \boldsymbol{e}^\alpha\right).
\end{aligned}$$

5. The Matter Field Equations

According to (4), the classical matter field equation

$$\Delta V = 4\pi\rho$$

is equivalent to

$$\boxed{\boldsymbol{v}\left[\operatorname{div}(\boldsymbol{v})\right] + \operatorname{div}(\boldsymbol{v},\boldsymbol{v}) = -4\pi\rho.} \tag{6}$$

Thus, we only need to show that an analogous representation is valid as well for the matter field equation of general relativity. To this end we will consider to begin with the very simple special case where the curvature of spacetime is induced by the gravitational interactions of a swarm of free particles whose rest-frame mass density is ρ and whose unit-length geodesic velocity field is $\boldsymbol{u}$ (*i.e.*, $\nabla_{\boldsymbol{u}}\boldsymbol{u} = \boldsymbol{0}$ and $g(\boldsymbol{u},\boldsymbol{u}) = 1$). In other words, we will assume that the stress-energy tensor is

$$\rho\boldsymbol{u}\otimes\boldsymbol{u},$$

or equivalently, that

$$\boldsymbol{T}(\boldsymbol{v}) = \rho g(\boldsymbol{v},\boldsymbol{u})\boldsymbol{u}.$$

To proceed, we require that (6) be valid as well in the general-relativistic case in which $\boldsymbol{v}$ is replaced by $\boldsymbol{u}$. That is to say, we demand that

$$\boldsymbol{u}\left[\operatorname{div}(\boldsymbol{u})\right] + \operatorname{div}(\boldsymbol{u},\boldsymbol{u}) = -4\pi\rho.$$

Inspired by this natural requirement, we establish the following general theorem:

Theorem 5.1. *Let q be a fixed spacetime event in the domain of* $\boldsymbol{u}$. *Then the matter field equation*

$$\operatorname{Ric}(\boldsymbol{x},\boldsymbol{y}) - \frac{R}{2}g(\boldsymbol{x},\boldsymbol{y}) = 8\pi\rho g(\boldsymbol{x},\boldsymbol{u}(q))g(\boldsymbol{u}(q),\boldsymbol{y}) = 8\pi T(\boldsymbol{x},\boldsymbol{y}) \tag{7}$$

is satisfied for all $\boldsymbol{x},\boldsymbol{y}\in T_qM$ *if and only if*

$$\boxed{\boldsymbol{w}(q)\left[\operatorname{div}(\boldsymbol{w})\right] + \operatorname{div}(\boldsymbol{w},\boldsymbol{w})\big|_q = -4\pi\rho(q)} \tag{8}$$

for any geodesic vector field $\boldsymbol{w}$ (*i.e.*, $\nabla_{\boldsymbol{w}}\boldsymbol{w} = \boldsymbol{0}$), *defined on an open neighborhood of q, that satisfies one of the following conditions*:

a) $\boldsymbol{w}(q) = \boldsymbol{u}(q)$,

b) $\boldsymbol{w}(q) = \boldsymbol{u}_\perp$ for some spacelike unit vector $\boldsymbol{u}_\perp \in T_qM$ that is Lorentz-perpendicular to $\boldsymbol{u}(q)$, that is, $g(\boldsymbol{u}_\perp,\boldsymbol{u}(q)) = 0$ and $g(\boldsymbol{u}_\perp,\boldsymbol{u}_\perp) = -1$,

c) $\boldsymbol{w}(q) = (\boldsymbol{u}(q)+\boldsymbol{u}_\perp)/\sqrt{2}$ for some vector $\boldsymbol{u}_\perp$ as described in (b).

Proof. Throughout the proof below we will assume that $\{\boldsymbol{e}_\mu\}_{\mu=0}^{3}$ is a Lorentz frame based at q such that $\boldsymbol{e}_0 = \boldsymbol{u}(q)$. (Note: this latter assumption will be needed only in the second part of the proof, not in the first.)

"$\Rightarrow$" If (7) is valid, then the geodesic equation $\nabla_{\boldsymbol{w}}\boldsymbol{w} = \boldsymbol{0}$ in conjunction with Theorem 4.1 implies that

$$\begin{aligned}\boldsymbol{w}(q)\left[\operatorname{div}(\boldsymbol{w})\right] + \operatorname{div}(\boldsymbol{w},\boldsymbol{w})\big|_q &= -\operatorname{Ric}(\boldsymbol{w}(q),\boldsymbol{w}(q))\\ &= -\frac{R(q)}{2}g(\boldsymbol{w}(q),\boldsymbol{w}(q)) - 8\pi\rho(q)g(\boldsymbol{w}(q),\boldsymbol{u}(q))^2.\end{aligned}$$

Since (7) also implies that

$$\begin{aligned} R(q) &= \mathrm{Ric}\left(\boldsymbol{e}_\mu, \boldsymbol{e}^\mu\right) \\ &= \frac{R(q)}{2} g\left(\boldsymbol{e}_\mu, \boldsymbol{e}^\mu\right) + 8\pi\rho(q) g\left(\boldsymbol{e}_\mu, \boldsymbol{u}(q)\right) g\left(\boldsymbol{u}(q), \boldsymbol{e}^\mu\right) \\ &= 2R(q) + 8\pi\rho(q) g\left(\boldsymbol{u}(q), \boldsymbol{u}(q)\right) \\ &= 2R(q) + 8\pi\rho(q), \end{aligned}$$

it follows that

$$R(q) = -8\pi\rho(q),$$

and therefore,

$$\boldsymbol{w}(q)\left[\operatorname{div}(\boldsymbol{w})\right] + \operatorname{div}(\boldsymbol{w}, \boldsymbol{w})\big|_q = 4\pi\rho(q) g\left(\boldsymbol{w}(q), \boldsymbol{w}(q)\right) - 2g\left(\boldsymbol{w}(q), \boldsymbol{u}(q)\right)^2 .$$

Given this equation, it is easy to verify that

$$\boldsymbol{w}(q)\left[\operatorname{div}(\boldsymbol{w})\right] + \operatorname{div}(\boldsymbol{w}, \boldsymbol{w})\big|_q = -4\pi\rho(q)$$

whenever one of the conditions (a), (b), or (c) above is satisfied.

"$\Leftarrow$" Since all the tensors in Equation (7) are symmetric, it is sufficient—by polarization—to show that

$$\mathrm{Ric}(\boldsymbol{x}, \boldsymbol{x}) - \frac{R}{2} g(\boldsymbol{x}, \boldsymbol{x}) = 8\pi\rho(q) g\left(\boldsymbol{x}, \boldsymbol{u}(q)\right)^2$$

for all $\boldsymbol{x} \in T_q M$. To do so, we will show to begin with that

$$\mathrm{Ric}\left(\boldsymbol{e}_\mu, \boldsymbol{e}_\nu\right) = 4\pi\rho(q)\delta_{\mu\nu} \tag{9}$$

for all $\mu, \nu \in \{0,1,2,3\}$. If $\mu = \nu$, then we pick a geodesic extension $\boldsymbol{w}$ of $\boldsymbol{e}_\mu$ (see Lemma 4.3) and apply Corollary 4.4 in conjunction with (8) and either (a) or (b) to infer that

$$\mathrm{Ric}\left(\boldsymbol{e}_\mu, \boldsymbol{e}_\mu\right) = -\boldsymbol{w}(q)\left[\operatorname{div}(\boldsymbol{w})\right] - \operatorname{div}(\boldsymbol{w}, \boldsymbol{w})\big|_q = 4\pi\rho(q), \tag{10}$$

as desired. If $\mu \neq \nu$ and $\mu \neq 0 \neq \nu$, then

$$g\left(\boldsymbol{e}_\mu + \boldsymbol{e}_\nu, \boldsymbol{e}_\mu + \boldsymbol{e}_\nu\right) = -2.$$

Consequently, if $\boldsymbol{w}$ is a geodesic extension of $\boldsymbol{u}_\perp := \left(\boldsymbol{e}_\mu + \boldsymbol{e}_\nu\right)/\sqrt{2}$, then Corollary 4.4 in conjunction with (8), (10), and (b) implies that

$$\begin{aligned} 4\pi\rho(q) &= -\boldsymbol{w}(q)\left[\operatorname{div}(\boldsymbol{w})\right] - \operatorname{div}(\boldsymbol{w}, \boldsymbol{w})\big|_q = \mathrm{Ric}\left(\boldsymbol{u}_\perp, \boldsymbol{u}_\perp\right) \\ &= \frac{1}{2}\left(\mathrm{Ric}\left(\boldsymbol{e}_\mu, \boldsymbol{e}_\mu\right) + 2\mathrm{Ric}\left(\boldsymbol{e}_\mu, \boldsymbol{e}_\nu\right) + \mathrm{Ric}\left(\boldsymbol{e}_\nu, \boldsymbol{e}_\nu\right)\right) \\ &= \frac{1}{2}\left(8\pi\rho(q) + 2\mathrm{Ric}\left(\boldsymbol{e}_\mu, \boldsymbol{e}_\nu\right)\right), \end{aligned}$$

and therefore,

$$\mathrm{Ric}\left(\boldsymbol{e}_\mu, \boldsymbol{e}_\nu\right) = 0,$$

as desired. Finally, if $\mu = 0 \neq \nu$ and if $\boldsymbol{w}$ is a geodesic extension of

$$\frac{\boldsymbol{u}(q) + \boldsymbol{u}_\perp}{\sqrt{2}} = \frac{\boldsymbol{e}_\mu + \boldsymbol{e}_\nu}{\sqrt{2}},$$

then Corollary 4.4 in conjunction with (8), (10), and (c) implies that

$$\begin{aligned} 4\pi\rho(q) &= -\boldsymbol{w}(q)\left[\operatorname{div}(\boldsymbol{w})\right] - \operatorname{div}(\boldsymbol{w},\boldsymbol{w})\big|_q \\ &= \operatorname{Ric}\left(\frac{\boldsymbol{u}(q)+\boldsymbol{u}_\perp}{\sqrt{2}}, \frac{\boldsymbol{u}(q)+\boldsymbol{u}_\perp}{\sqrt{2}}\right) \\ &= \frac{1}{2}\left(8\pi\rho(q) + 2\operatorname{Ric}\left(\boldsymbol{e}_\mu,\boldsymbol{e}_\nu\right)\right), \end{aligned}$$

and again we find that

$$\operatorname{Ric}\left(\boldsymbol{e}_\mu,\boldsymbol{e}_\nu\right) = 0,$$

as desired. Having thus established Equation (9), it follows that

$$\operatorname{Ric}(\boldsymbol{x},\boldsymbol{x}) = g\left(\boldsymbol{x},\boldsymbol{e}^\mu\right) g\left(\boldsymbol{x},\boldsymbol{e}^\nu\right) \operatorname{Ric}\left(\boldsymbol{e}_\mu,\boldsymbol{e}_\nu\right) = 4\pi\rho(q)\sum_{\mu=0}^{3} g\left(\boldsymbol{x},\boldsymbol{e}^\mu\right)^2$$

and

$$R(q) = \operatorname{Ric}\left(\boldsymbol{e}_\mu,\boldsymbol{e}^\mu\right) = 4\pi\rho(q) - 12\pi\rho(q) = -8\pi\rho(q).$$

Hence

$$\begin{aligned} &\operatorname{Ric}(\boldsymbol{x},\boldsymbol{x}) - \frac{R}{2} g(\boldsymbol{x},\boldsymbol{x}) \\ &= 4\pi\rho(q)\sum_{\mu=0}^{3} g\left(\boldsymbol{x},\boldsymbol{e}^\mu\right)^2 + 4\pi\rho(q) g\left(\boldsymbol{x},\boldsymbol{e}_\mu\right) g\left(\boldsymbol{x},\boldsymbol{e}^\mu\right) \\ &= 8\pi\rho(q) g\left(\boldsymbol{x},\boldsymbol{u}(q)\right)^2, \end{aligned}$$

as desired. □

Concerning the conditions (a), (b), and (c) in Theorem 5.1, we wish to remark that the validity of (6) in the Newtonian case naturally suggests that Equation (8) ought to be valid if (a) and (b) are satisfied because $\rho(q)$ is the rest-frame density and $\boldsymbol{u}(q)$ and $\boldsymbol{u}_\perp$ are timelike and spacelike rest-frame vectors, respectively. Moreover, regarding the validity of (8) in the remaining case, where $\boldsymbol{w}$ satisfies (c), it is helpful to notice that Newtonian gravity can be regarded as a classical limit that emerges from general relativity as the speed of light diverges to infinity. For in adopting this point of view, the Newtonian spatial rest frame at q merges with the relativistic lightcone at q, and the lightlike vector $\boldsymbol{u}(q)+\boldsymbol{u}_\perp$, multiplied with the Euclidean scaling factor $1/\sqrt{2}$, may therefore be considered to be a spatial rest-frame vector in the Newtonian limit. However, regardless of whether we consider this latter interpretation to be convincing or not, Theorem 5.1 remains perfectly valid as a mathematical fact. So ultimately (c) is simply a condition that needs to be added in order to guarantee that (7) and (8) are mathematically equivalent and that, by implication, Newtonian gravity and general relativity may justifiably be viewed to be *formally identical*.

Furthermore, the somewhat unsatisfactory restriction to a swarm of particles moving on geodesics can easily be lifted by considering an entire family of swarms in dependence on the geodesic unit vector fields $\boldsymbol{u}$—that is, by considering a perfect fluid in which particles of common rest mass m move along geodesics so that at each point q the mass density of particles moving with velocity

$$\boldsymbol{u}_q(v,\theta,\phi)=\left.\frac{\boldsymbol{e}_0+v\cos(\theta)\sin(\phi)\boldsymbol{e}_1+v\sin(\theta)\sin(\phi)\boldsymbol{e}_2+v\cos(\phi)\boldsymbol{e}_3}{\sqrt{1-v^2}}\right|_q$$

is equal to $\rho_q(v)$ (in the particles' rest frames) for all spherical coordinate triples $(v,\theta,\phi)\in[0,1)\times[0,2\pi)\times[0,\pi)$. (Note: the suggested dependence of ρ_q on v alone—rather than on v, θ, and ϕ—is due to the fact that in a perfect fluid the distribution of particle velocities may be assumed to be isotropic relative to the fluid's rest frame $\{\boldsymbol{e}_\mu(q)\}_{\mu=0}^3$.) Given this setup, it follows that the mass density corresponding to $\rho_q(v)$ relative to the fluid's rest frame $\{\boldsymbol{e}_\mu(q)\}_{\mu=0}^3$ is

$$\frac{\rho_q(v)}{\sqrt{1-v^2}\sqrt{1-v^2}}=\frac{\rho_q(v)}{1-v^2} \tag{11}$$

where the first factor $1/\sqrt{1-v^2}$ is due to relativistic length contraction (in the direction of motion) and the second accounts for the relativistic increase in mass. Furthermore, the stress energy tensor $T_{q,v,\theta,\phi}$ corresponding to $\boldsymbol{u}_q(v,\theta,\phi)$ is given by the equation

$$T_{q,v,\theta,\phi}(\boldsymbol{x})=\rho_q(v)g\left(\boldsymbol{x},\boldsymbol{u}_q(v,\theta,\phi)\right)\boldsymbol{u}_q(v,\theta,\phi),$$

and the matrix representing $T_{q,v,\theta,\phi}$ at q with respect to the frame $\{\boldsymbol{e}_\mu(q)\}_{\mu=0}^3$ is

$$\left(T_\mu^\nu(q,v,\theta,\phi)\right)_{\mu,\nu=0}^3=\rho_q(v)\left(g\left(\boldsymbol{u}_q(v,\theta,\phi),\boldsymbol{e}^\nu(q)\right)g\left(\boldsymbol{u}_q(v,\theta,\phi),\boldsymbol{e}_\mu(q)\right)\right)_{\mu,\nu=0}^3.$$

That said, we now are justified in asserting that Equation (8) plausibly suggests—by way of its proven equivalence to Equation (7)—that the spacetime environment of an ideal gas or perfect fluid is described by the equation

$$\mathrm{Ric}(\boldsymbol{x},\boldsymbol{y})-\frac{R}{2}g(\boldsymbol{x},\boldsymbol{y})=8\pi T(\boldsymbol{x},\boldsymbol{y}),$$

where the matrix representing $T(\boldsymbol{x})$ (as defined by the equation $T(\boldsymbol{x},\boldsymbol{y})=g(T(\boldsymbol{x}),\boldsymbol{y})$) is

$$\left(T_\mu^\nu(q)\right)_{\mu,\nu=0}^3=\int_0^{2\pi}\int_0^\pi\int_0^1\left(T_\mu^\nu(q,v,\theta,\phi)\right)_{\mu,\nu=0}^3 v^2\sin(\phi)\mathrm{d}v\mathrm{d}\phi\mathrm{d}\theta.$$

Since the definition of $\boldsymbol{u}_q(v,\theta,\phi)$ readily implies (by way of elementary integration) that

$$\int_0^{2\pi}\int_0^\pi\int_0^1 T_\mu^\nu(q,v,\theta,\phi)v^2\sin(\phi)\mathrm{d}v\mathrm{d}\phi\mathrm{d}\theta=0$$

whenever $\mu\neq\nu$, it follows that $\left(T_\mu^\nu(q)\right)_{\mu,\nu=0}^3$ is a diagonal matrix with diagonal elements

$$\rho(q):=T_0^0(q)=\int_0^{2\pi}\int_0^\pi\int_0^1\frac{\rho_q(v)v^2\sin(\phi)}{1-v^2}\mathrm{d}v\mathrm{d}\phi\mathrm{d}\theta=4\pi\int_0^1\frac{\rho_q(v)v^2}{1-v^2}\mathrm{d}v$$

and

$$T_1^1(q)=-\int_0^{2\pi}\int_0^\pi\int_0^1\frac{\rho_q(v)\cos^2(\theta)\sin^3(\phi)v^4}{1-v^2}\mathrm{d}v\mathrm{d}\phi\mathrm{d}\theta=-\frac{4\pi}{3}\int_0^1\frac{\rho_q(v)v^4}{1-v^2}\mathrm{d}v,$$

$$T_2^2(q) = -\int_0^{2\pi}\int_0^{\pi}\int_0^{1}\frac{\rho_q(v)\sin^2(\theta)\sin^3(\phi)v^4}{1-v^2}\mathrm{d}v\mathrm{d}\phi\mathrm{d}\theta = -\frac{4\pi}{3}\int_0^1\frac{\rho_q(v)v^4}{1-v^2}\mathrm{d}v,$$

$$T_3^3(q) = -\int_0^{2\pi}\int_0^{\pi}\int_0^{1}\frac{\rho_q(v)\cos^2(\phi)\sin(\phi)v^4}{1-v^2}\mathrm{d}v\mathrm{d}\phi\mathrm{d}\theta = -\frac{4\pi}{3}\int_0^1\frac{\rho_q(v)v^4}{1-v^2}\mathrm{d}v.$$

Setting $P(q) := -T_1^1(q) = -T_2^2(q) = -T_3^3(q)$, we may infer that the matrix representing T at q with reference to the frame $\{\boldsymbol{e}_\mu(q)\}_{\mu=0}^3$ is

$$\begin{pmatrix} \rho(q) & 0 & 0 & 0 \\ 0 & -P(q) & 0 & 0 \\ 0 & 0 & -P(q) & 0 \\ 0 & 0 & 0 & -P(q) \end{pmatrix}$$

and that, by implication,

$$\boxed{T(q) = (\rho(q)+P(q))\boldsymbol{e}_0(q)\otimes\boldsymbol{e}_0(q) - P(q)g.}$$

Thus we have arrived at the well-known representation of the stress-energy tensor of a perfect fluid (with the metric sign convention (+−−−)) because, according to (11), $\rho(q)$ is the total mass density as measured in $\{\boldsymbol{e}_\mu(q)\}_{\mu=0}^3$, and $P(q)$ is easily seen to be and also well-known to be the fluid's pressure.

Finally, to round up our discussion of the relativistic matter field equation, we wish to point out that the equation

$$\mathrm{Ric}(\boldsymbol{w},\boldsymbol{w}) = -\boldsymbol{w}[\mathrm{div}(\boldsymbol{w})] - \mathrm{div}(\boldsymbol{w},\boldsymbol{w})$$

always allows us to compute all the components of the stress-energy tensor T by properly choosing $\boldsymbol{w}$ in the term on the right-hand side of this equation, but recovering the components of T from a single equation of the form

$$\boldsymbol{w}[\mathrm{div}(\boldsymbol{w})] - \mathrm{div}(\boldsymbol{w},\boldsymbol{w}) = -4\pi\rho$$

is not always possible. There are special cases in which it is possible because the components of T are all equal to a constant factor multiplied by the energy density ρ, but in general, of course, the components of T may contain a wide variety of quantities other than ρ. Two prominent special cases in which ρ completely characterizes T are encountered when the curvature-generating gravitational energy is produced by an electric field $\boldsymbol{E}$ or by a plane electromagnetic wave. For in the former case, the electromagnetic field tensor $F(\boldsymbol{x})$ is represented by the matrix

$$\left(F_\mu^\nu\right)_{\mu,\nu=0}^3 = \begin{pmatrix} 0 & \|\boldsymbol{E}\| & 0 & 0 \\ \|\boldsymbol{E}\| & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(relative to the basis $\{\boldsymbol{e}_0, \boldsymbol{E}/\|\boldsymbol{E}\|, \boldsymbol{e}_2, \boldsymbol{e}_3\}$), and the electromagnetic stress-energy tensor

$$T = \frac{1}{4\pi}\left(F^2 - \frac{1}{4}\mathrm{tr}(F^2)g\right) \tag{12}$$

is easily seen to be represented by the matrix

$$\left(F_{\mu}^{\nu}\right)_{\mu,\nu=0}^{3}=\frac{1}{4\pi}\begin{pmatrix}\|\boldsymbol{E}\|^{2}/2 & 0 & 0 & 0\\ 0 & \|\boldsymbol{E}\|^{2}/2 & 0 & 0\\ 0 & 0 & -\|\boldsymbol{E}\|^{2}/2 & 0\\ 0 & 0 & 0 & -\|\boldsymbol{E}\|^{2}/2\end{pmatrix}$$
$$=\begin{pmatrix}\rho & 0 & 0 & 0\\ 0 & \rho & 0 & 0\\ 0 & 0 & -\rho & 0\\ 0 & 0 & 0 & -\rho\end{pmatrix}; \tag{13}$$

and in the latter case, of a plane wave, the electromagnetic field tensor is represented by the matrix

$$\left(F_{\mu}^{\nu}\right)_{\mu,\nu=0}^{3}=\begin{pmatrix}0 & \|\boldsymbol{E}\| & 0 & 0\\ \|\boldsymbol{E}\| & 0 & 0 & -\|\boldsymbol{B}\|\\ 0 & 0 & 0 & 0\\ 0 & \|\boldsymbol{B}\| & 0 & 0\end{pmatrix}=\begin{pmatrix}0 & \|\boldsymbol{E}\| & 0 & 0\\ \|\boldsymbol{E}\| & 0 & 0 & -\|\boldsymbol{E}\|\\ 0 & 0 & 0 & 0\\ 0 & \|\boldsymbol{E}\| & 0 & 0\end{pmatrix}$$

(relative to the basis $\{\boldsymbol{e}_0, \boldsymbol{E}/\|\boldsymbol{E}\|, \boldsymbol{B}/\|\boldsymbol{B}\|, \boldsymbol{E}\times\boldsymbol{B}/\|\boldsymbol{E}\times\boldsymbol{B}\|\}$ because for a plane electromagnetic wave it is the case that $\boldsymbol{E}\perp\boldsymbol{B}$ and $\|\boldsymbol{E}\|=\|\boldsymbol{B}\|$) and the matrix representing the corresponding stress-energy tensor turns out to be

$$\left(F_{\mu}^{\nu}\right)_{\mu,\nu=0}^{3}=\frac{1}{4\pi}\begin{pmatrix}\|\boldsymbol{E}\|^{2} & 0 & 0 & -\|\boldsymbol{E}\|^{2}\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ \|\boldsymbol{E}\|^{2} & 0 & 0 & -\|\boldsymbol{E}\|^{2}\end{pmatrix}=\begin{pmatrix}\rho & 0 & 0 & -\rho\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ \rho & 0 & 0 & -\rho\end{pmatrix} \tag{14}$$

(because $\rho=\left(\|\boldsymbol{E}\|^{2}+\|\boldsymbol{B}\|^{2}\right)/(8\pi)=\|\boldsymbol{E}\|^{2}/(4\pi)$). So in either case, the components of the matrix representing T are either 0 or $\pm\rho$, and theorems analogous to Theorem 5.1 can therefore be formulated.

In order to see this more clearly, we may want to take another look at Theorem 5.1: the central step in proving this theorem was to demonstrate that the equation

$$\mathrm{Ric}(\boldsymbol{w},\boldsymbol{w})=-\boldsymbol{w}\left[\mathrm{div}(\boldsymbol{w})\right]-\mathrm{div}(\boldsymbol{w},\boldsymbol{w})$$

in conjunction with the assumed validity of the equation

$$\boldsymbol{w}\left[\mathrm{div}(\boldsymbol{w})\right]+\mathrm{div}(\boldsymbol{w},\boldsymbol{w})=-4\pi\rho$$

for all geodesic vector fields $\boldsymbol{w}$, as specified in (a), (b), and (c), completely characterizes the Ricci tensor, that is, it implies that for all tangent vectors $\boldsymbol{x}=x^{\mu}\boldsymbol{e}_{\mu}$ it is the case that

$$\mathrm{Ric}(\boldsymbol{x},\boldsymbol{x})=\frac{R}{2}\boldsymbol{g}(\boldsymbol{x},\boldsymbol{x})+8\pi T(\boldsymbol{x},\boldsymbol{x})=-4\pi\rho\left(\boldsymbol{g}(\boldsymbol{x},\boldsymbol{x})-2\boldsymbol{g}(\boldsymbol{x},\boldsymbol{u})^{2}\right)$$
$$=-4\pi\rho\left(\boldsymbol{g}(\boldsymbol{x},\boldsymbol{x})-2\boldsymbol{g}(\boldsymbol{x},\boldsymbol{e}_0)^{2}\right)=4\pi\rho\sum_{\mu=0}^{3}\left(x^{\mu}\right)^{2}.$$

In essence, therefore, the statement of Theorem 5.1 is a somewhat stronger version of the assertion that the equation

$$\mathrm{Ric}(\boldsymbol{x},\boldsymbol{x}) = -4\pi\rho\left(\boldsymbol{g}(\boldsymbol{x},\boldsymbol{x}) - 2\boldsymbol{g}(\boldsymbol{x},\boldsymbol{e}_0)^2\right)$$

must be satisfied for all $\boldsymbol{x}$ if the equation

$$\mathrm{Ric}(\boldsymbol{x},\boldsymbol{x}) = 4\pi\rho \tag{15}$$

is valid for all vectors $\boldsymbol{x} = x^\mu \boldsymbol{e}_\mu$ for which

$$\sum_{\mu=0}^{3}\left(x^\mu\right)^2 = 1. \tag{16}$$

(Note: Equation (16) is easily seen to be satisfied for all vectors specified in (a), (b), and (c), and therefore, the assertion above is slightly weaker than the statement of Theorem 5.1 because the vectors specified in (a), (b), and (c) form a strict subset of the vectors that satisfy (16)).

That said, we will now proceed to consider the electromagnetic stress-energy tensors given in (13) and (14): for the former of these we readily find that

$$T(\boldsymbol{x},\boldsymbol{x}) = \rho\left(\left(x^0\right)^2 - \left(x^1\right)^2 + \left(x^2\right)^2 + \left(x^3\right)^2\right)$$

and for the latter, we find that

$$T(\boldsymbol{x},\boldsymbol{x}) = \rho\left(\left(x^0\right)^2 - 2x^0x^3 + \left(x^3\right)^2\right).$$

Consequently, since (12) implies that the trace of T is equal to zero and that therefore R is equal to zero as well, the assertion above, concerning the sufficiency of Equations (15) and (16) for the generation of the Ricci tensor, leads us to assert, by analogy, that

$$\mathrm{Ric}(\boldsymbol{x},\boldsymbol{x}) = 8\pi T(\boldsymbol{x},\boldsymbol{x})$$

for all $\boldsymbol{x}$ if the equation

$$\mathrm{Ric}(\boldsymbol{x},\boldsymbol{x}) = 4\pi\rho$$

is valid for all vectors $\boldsymbol{x} = x^\mu \boldsymbol{e}_\mu$ for which either

$$\left(x^0\right)^2 - \left(x^1\right)^2 + \left(x^2\right)^2 + \left(x^3\right)^2 = \frac{1}{2}$$

(in the case where T is given as in (13)) or

$$\left(x^0\right)^2 - 2x^0x^3 + \left(x^3\right)^2 = \frac{1}{2}$$

(in the case where T is given as in (13)). Not surprisingly, both of these respective assertions can be readily established by using essentially the same methods as in the proof of Theorem 5.1. But since the pertinent details would add little value to the results derived in this paper, we will not attempt to include them.

6. Conclusion

In reinterpreting the classical and relativistic gravitational field equations as velocity-field divergence equations, we were able to show that classical and relativistic descriptions of gravitation may be considered to be formally strictly analogous. This observation in itself does not elucidate many of the deeper structural questions that limit-like transitions from general relativity to Newtonian gravity are commonly thought to raise, but it does appear to bring to light a surprisingly simple and straightforward mathematical kinship—and hence it does appear to be worth mentioning.

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

References

[1] Misner, C.W., Thorne, K.S. and Wheeler, J.A. (1973) Gravitation. W. H. Freeman and Company, San Francisco.

[2] Callahan, J.J. (2000) The Geometry of Spacetime. Springer-Verlag, New York. https://doi.org/10.1007/978-1-4757-6736-0

[3] Ellman, R. (2017) The Experimental Data Validation of Modern Newtonian Gravitation over General Relativity Gravitation. *International Journal of Geosciences*, **8**, 444-461. https://doi.org/10.4236/ijg.2017.84023

[4] Ellman, R. (2016) Comprehensive Gravitation. Research Gate. https://www.researchgate.net/profile/Roger-Ellman

[5] Fletcher, S. (2019) On the Reduction of General Relativity to Newtonian Gravitation. *Studies in History and Philosophy of Modern Physics*, **68**, 1-15. https://doi.org/10.1016/j.shpsb.2019.04.005

[6] Fritelli, S. and Reula, O. (1994) On the Newtonian Limit of General Relativity. *Communications in Mathematical Physics*, **166**, 221-235. https://doi.org/10.1007/BF02112314

[7] O'Neill, B. (1983) Semi-Riemannian Geometry. Academic Press, New York.