Journal of Applied Mathematics and Physics, 2022, 10, 1362-1374
https://www.scirp.org/journal/jamp
ISSN Online: 2327-4379
ISSN Print: 2327-4352

Solving Navier-Stokes with Maclaurin Series

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How to cite this paper: Martino, G. (2022) Solving Navier-Stokes with Maclaurin Series. *Journal of Applied Mathematics and Physics*, **10**, 1362-1374.
https://doi.org/10.4236/jamp.2022.104096

Received: March 29, 2022
Accepted: April 26, 2022
Published: April 29, 2022

Abstract

In this paper I propose a method for founding solutions of Navier-Stokes equations. Purpose of the research is to solve equations giving form to relations between pressure, velocity and stream. Starting from the fact we do not know the form of functions we give a general representation in Maclaurin Series and prove that with reasonable values of parameters, representation holds and therefore has meaning in continuum. Then we solve the system of equations with respect to the pressure and match equations relation between parameters: matches of equations are possible because of the physical dimensions of equations. Then values of Continuity Equation are verified. The result is a polynomial finite and that coincides with the function in continuum, or is anyway one of its representation. The result under hydrostatic condition returns Stevino formula.

Keywords

Navier-Stokes, Momentum, Continuity, Differential Equation, Maclaurin, Polynomial

1. Introduction

The most of knowledge can be found in Reference [1] [2] and [3] with the expression of Navier-Stokes Equations, further are used some well known formulas that can be found almost everywhere but they are contained in the article and you can find links in others: References [4] (see method to solve differential equation with power series [5], for turbulence [6], Navier Stokes State [7], incompressible Navier Stokes [8], Multivariable Taylor [9], Cauchy product [10], Fluid Dynamics Knowledge [11] [12] [13] [14]). At the end of the article the results of the pressure match a well known formula when velocity vanishes. Further an Appendix is provided.

The fluid motion is governed by knowing the velocity vector for most. The velocity vector is one of the parameters of the Navier-Stokes equation. Together with continuity equation they constitute a system of differential equations in the unknowns $p(x,y,z,t)$ (pressure) and $v(x,y,z,t)$ (velocity) in the variables of space (x,y,z) and time *t*. The system for incompressible fluid can be written like

$$\nabla\cdot\mathbf{V}=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \qquad \text{(1. Continuity)}$$

$$\rho\cdot\frac{D\mathbf{V}}{Dt}=-\nabla p+\rho\cdot\mathbf{g}+\mu\nabla^{2}\mathbf{V} \qquad \text{(2. Navier-Stokes)}$$

for a detailed explanation of the system (see the Appendix and References).

2. Method Overview

Consider the Navier-Stokes equation

$$\rho\cdot\frac{D\mathbf{V}}{Dt}=-\nabla p+\rho\cdot\mathbf{g}+\mu\nabla^{2}\mathbf{V} \qquad \text{(3. NS)}$$

They are three equations. We give the V vector a form in Maclaurin series, then substitute the value of V in the three Navier-Stokes equations and then try to resolve each of this equation with respect to pressure with integration of each equation in the respective variable. We match the form of the pressure in the three equations and try to find a relation of parameters. After a partial comparison those parameters result in a relation. Applying Continuity Equation gives values. If the expansion for V is finite then the polynomial coincides with function of *V*.

We give an old style to composition but not less correct.

3. Maclaurin Expansion for Velocity

We don't know the form of the function but we know that the function has four variables *x*,*y*,*z*,*t*. If we want to find analytic solution we can represent functions *u*, *v*, *w* of $V=u\cdot\mathbf{i}+v\cdot\mathbf{j}+w\cdot\mathbf{k}$ as Taylor Series around point $\mathbf{a}=(a_1,a_2,a_3,a_4)$ (see we don't know this assumption is legal now). Any finite polynomial in Taylor series corresponds to the same polynomial: if we find terms in the polynomial expansion and those identify a finite polynomial we have found the function. The form of Taylor function in multidimensional variable is of the kind (we used a non compact form another could be compact with Hessian Matrix, see Wikipedia):

$$T(S)=\sum_{\alpha\ge 0}\frac{\prod_j (x_j-a_j)^{\alpha_j}}{\alpha!}\left(\frac{\partial^{\alpha} f}{\prod_j \partial x_j^{\alpha_j}}\right)(a) \qquad \text{(4. Taylor expansion)}$$

where $\alpha=\sum\alpha_j$, $\alpha!=\alpha_1!\cdots\alpha_n!$ and $\mathbf{x}=(x_1,x_2,x_3,x_4)=(x,y,z,t)$. If the function is defined in the derivative point than terms

$$\frac{1}{\alpha!}\frac{\partial^{\alpha} f}{\prod_j \partial x_j^{\alpha_j}}(a)=c_{(\alpha)}(a)$$

are constant at least one $\neq 0$.

Represent the function with respect to $a = (0,0,0,0)$ (Maclaurin Series). We must divide and reunion the spatial convergence with time convergence. A product of power does not have all same values as exponent. Radius of convergence is possible if all are convergent. Indeed if we call $\mathbf{x}_s$ variable of $\mathbf{x}$ of space then space radius of convergence $\|\mathbf{x}_s - \mathbf{b}\|_3 = \left(\sum_{k=1}^{3} x_k^3\right)^{\frac{1}{3}} < r$ where $\mathbf{b} = (0,0,0)$ for Mc Laurin Series and dimension of $[x] = [y] = [z] = [m]$ meters and $|t| < p$ with dimension $[t]$ time, measured according to International System of measure that see the convergence of total series S if all series are convergent [or all mutually exclusively subset are convergent].

Represent $x_1 = x$, $x_2 = y$, $x_3 = z$, $x_4 = t$, with a compact form for summation, with notation $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = S_j$. From now on we will call $\alpha_1 = \alpha_1; \alpha_2 = \beta; \alpha_3 = \gamma; \alpha_4 = \delta$ to be clearer this does not prevent previous consideration.

$$u = \sum_{S_u}^{\infty} c_u(0) \cdot x^{\alpha_u} \cdot y^{\beta_u} \cdot z^{\gamma_u} \cdot t^{\delta_u}$$

$$v = \sum_{S_v}^{\infty} c_v(0) \cdot x^{\alpha_v} \cdot y^{\beta_v} \cdot z^{\gamma_v} \cdot t^{\delta_v}$$

$$w = \sum_{S_w}^{\infty} c_w(0) \cdot x^{\alpha_w} \cdot y^{\beta_w} \cdot z^{\gamma_w} \cdot t^{\delta_w} \quad \text{(6—General expression for unknown velocity)}$$

this assumption can be thought legal if we think the fluid in continuum. See Figure 1.

4. 1st Equation of Navier-Stokes

Consider the formula of the Navier-Stokes equations

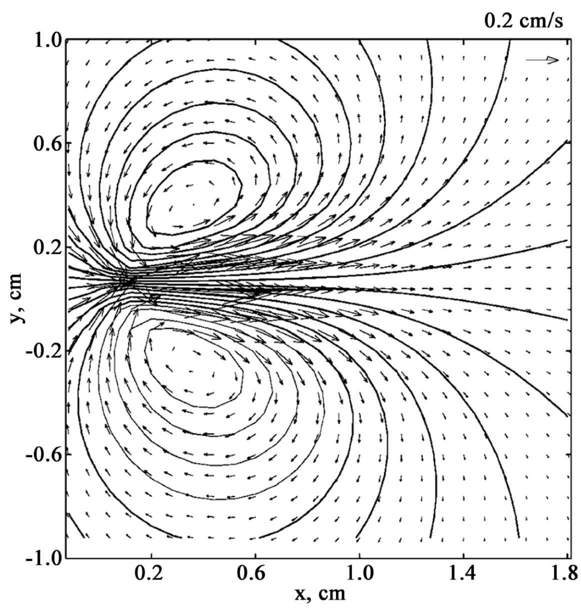

Figure 1. Example of velocity field vector.

$$\rho \cdot \frac{D\mathbf{V}}{Dt} = -\nabla p + \rho \cdot g + \mu \nabla^2 \mathbf{V} \tag{7.NS}$$

and his first equation

$$\rho \cdot \frac{\partial u}{\partial t} + \rho \cdot u \frac{\partial u}{\partial x} + \rho \cdot v \frac{\partial u}{\partial y} + \rho \cdot w \frac{\partial u}{\partial z} = -\frac{\partial p}{\partial x} + \rho \cdot g_x + \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right)$$

(1st Equation)

We have the form of the unknown u but we have not yet pressure. To solve we substitute the unknown in the equations of Navier stokes, respectively u, v, w. Then we solve the integral of the pressure in the three variables respectively x, y, z and confront physical equations (all elements multiplied by $\mu, \cdots$), found parameters values. We express each term separately because of the long expression and substitute the values of $V(u, v, w)$ in each term. We put a tag at the end of each expression as: 2nd addend right member, that refers at right member the second addend in the expression. Substituting the value of $V = (u, v, w)$ in the equation we have terms a at left member and b at right member.

$$a_{11} = \rho \cdot \frac{\partial u}{\partial t} = \rho \cdot \sum_{S_u=0}^{\infty} c_u(0) \cdot x^{\alpha_u} \cdot y^{\beta_u} \cdot z^{\gamma_u} \cdot \delta_u t^{\delta_u - 1}$$ (1st Addend Left Member—1st Equation)

$$\begin{aligned} a_{12} &= \rho \cdot u \frac{\partial u}{\partial x} \\ &= \rho \cdot \sum_{S_u=0}^{\infty} c_u(0) \cdot x^{\alpha_u} \cdot y^{\beta_u} \cdot z^{\gamma_u} \cdot t^{\delta_u} \cdot \sum_{S_u=0}^{\infty} c_u(0) \cdot \alpha_u \cdot x^{\alpha_u - 1} \cdot y^{\beta_u} \cdot z^{\gamma_u} \cdot t^{\delta_u} \end{aligned}$$

This series must be solved with Cauchy product

$$\sum_{k=0}^{\infty} s_k \cdot \sum_{k=0}^{\infty} t_k = \sum_{k=0}^{\infty} \sum_{n=0}^{k} s_n t_{k-n} \tag{8. Cauchy Product}$$

if $S_k = \alpha_k + \beta_k + \gamma_k + \delta_k$ and $S_n = \alpha_n + \beta_n + \gamma_n + \delta_n$ then (2) became

$$a_{12} = \rho \cdot \sum_{S_k=0}^{\infty} \sum_{S_n=0}^{S_k} c_{u(n)}(0) \cdot c_{u(k-n)}(0) \cdot (\alpha_k - \alpha_n) x^{\alpha_k - 1} \cdot y^{\beta_k} \cdot z^{\gamma_k} \cdot t^{\delta_k}$$

(2nd Addend Left Member—1st Equation)

Note that we set $\alpha_u = \alpha_{k-n} = (\alpha_k - \alpha_n)$ and if c equals one of $\alpha, \beta, \gamma, \delta$ and $1, 2s$ the first and second summatory in Cauchy product then $c_1(n) = c_1$ and $c_2(k-n) = c_2$ we reported anyway this notation to remember which one refers to the first summation and which one to second.

Other terms are then

$$\begin{aligned} a_{31} &= \rho \cdot v \frac{\partial u}{\partial y} \\ &= \rho \cdot \sum_{S_v=0}^{\infty} c_v(0) \cdot x^{\alpha_v} \cdot y^{\beta_v} \cdot z^{\gamma_v} \cdot t^{\delta_v} \cdot \sum_{S_u=0}^{\infty} c_u(0) \cdot x^{\alpha_u} \cdot \beta_u \cdot y^{\beta_u - 1} \cdot z^{\gamma_u} \cdot t^{\delta_u} \\ &= \rho \cdot \sum_{S_k=0}^{\infty} \sum_{S_n=0}^{S_k} c_{v(n)}(0) \cdot c_{u(k-n)}(0) \cdot x^{\alpha_k} \cdot (\beta_k - \beta_n) y^{\beta_k - 1} \cdot z^{\gamma_k} \cdot t^{\delta_k} \end{aligned}$$

(3rd Addend Left Member—1st Equation)

$$
\begin{aligned}
a_{41} &= \rho \cdot w \frac{\partial u}{\partial z} \\
&= \rho \cdot \sum_{S_w=0}^{\infty} c_w(0) \cdot x^{\alpha_w} \cdot y^{\beta_w} \cdot z^{\gamma_w} \cdot t^{\delta_w} \cdot \sum_{S_u=0}^{\infty} c_u(0) \cdot x^{\alpha_u} \cdot y^{\beta_u} \cdot \gamma_u \cdot z^{\gamma_u - 1} \cdot t^{\delta_u} \\
&= \rho \cdot \sum_{S_k=0}^{\infty} \sum_{S_n=0}^{S_k} c_{w(n)}(0) \cdot c_{u(k-n)}(0) \cdot x^{\alpha_k} \cdot y^{\beta_k} \cdot (\gamma_k - \gamma_n) z^{\gamma_k - 1} \cdot t^{\delta_k}
\end{aligned}
$$

(4th Addend Left Member—1st Equation)

then right member

$$b_{11} = -\frac{\partial p}{\partial x} \quad \text{(1st Addend Right Member—1st Equation)}$$

$$b_{21} = \rho \cdot g_x \quad \text{(2nd Addend Right Member—1st Equation)}$$

$$b_{31} = \mu \frac{\partial^2 u}{\partial x^2} = \mu \sum_{S_u=0}^{\infty} c_u(0) \cdot \alpha_u \cdot (\alpha_u - 1) \cdot x^{\alpha_u - 2} \cdot y^{\beta_u} \cdot z^{\gamma_u} \cdot t^{\delta_u}$$

(3rdt Addend Right Member—1st Equation)

$$b_{14} = \mu \frac{\partial^2 u}{\partial y^2} = \mu \sum_{S_u=0}^{\infty} c_u(0) \cdot x^{\alpha_u} \cdot \beta_u \cdot (\beta_u - 1) \cdot y^{\beta_u - 2} \cdot z^{\gamma_u} \cdot t^{\delta_u}$$

(4th Addend Right Member—1st Equation)

$$b_{15} = \mu \frac{\partial^2 u}{\partial z^2} = \mu \sum_{S_u=0}^{\infty} c_u(0) \cdot x^{\alpha_u} \cdot y^{\beta_u} \cdot \gamma_u \cdot (\gamma_u - 1) \cdot z^{\gamma_u - 2} \cdot t^{\delta_u}$$

(5th Addend Right Member—1st Equation)

Then solve respect to p (1st Right Member) according to $\int \sum a \cdot \mathrm{d}x = \int \sum b \cdot \mathrm{d}x$ (that's possible for continuity hypothesis and for Taylor hypothesis, indeed see Reference [4] continuity imply uniform convergence that consent swapping) multiply the equation for dx and integrating without confuse dx with the δ parameter of *u*, *v*, *w*) adding a constant term in variables $h_1(y,z,t)+C$ and for sake of simplicity we consider the pressure p_0 in the point at the origin $(x_0, y_0, z_0) = (0,0,0)$. Further consider that if $\int f\mathrm{d}x < \infty$ and $\sum f < \infty$ then we can swap $\int \sum f\mathrm{d}x = \sum \int f\mathrm{d}x$. If we integrate respect one variable the other are taken as constant. We also denote indirectly a_n from $\int a\mathrm{d}x \to a$.

1st Equation after $\int$

$$
\begin{aligned}
&\rho \cdot \sum_{S_u=0}^{\infty} c_u(0) \cdot \frac{x^{\alpha_u + 1}}{\alpha_u + 1} \cdot y^{\beta_u} \cdot z^{\gamma_u} \cdot \delta t^{\delta_u - 1} \\
&+_{[\text{if } \alpha_k \neq 0]} \rho \cdot \sum_{S_k=0}^{\infty} \sum_{S_n=0}^{S_k} c_{u(n)}(0) \cdot c_{u(k-n)}(0) \cdot (\alpha_k - \alpha_n) \frac{x^{\alpha_k}}{\alpha_k} \cdot y^{\beta_k} \cdot z^{\gamma_k} \cdot t^{\delta_k} \\
&+ \rho \cdot \sum_{S_k=0}^{\infty} \sum_{S_n=0}^{S_k} c_{v(n)}(0) \cdot c_{u(k-n)}(0) \cdot \frac{x^{\alpha_k + 1}}{\alpha_k + 1} \cdot (\beta_k - \beta_n) y^{\beta_k - 1} \cdot z^{\gamma_k} \cdot t^{\delta_k} \\
&+ \rho \cdot \sum_{S_k=0}^{\infty} \sum_{S_n=0}^{S_k} c_{w(n)}(0) \cdot c_{u(k-n)}(0) \cdot \frac{x^{\alpha_k + 1}}{\alpha_k + 1} \cdot y^{\beta_k} \cdot (\gamma_k - \gamma_n) z^{\gamma_k - 1} \cdot t^{\delta_k}
\end{aligned}
$$

$$= p_0 - p + \rho \cdot g_x(x) + \mu \sum_{S_u=0}^{\infty} c_u(0) \cdot \alpha_u \cdot x^{\alpha_u - 1} \cdot y^{\beta_u} \cdot z^{\gamma_u} \cdot t^{\delta_u}$$

$$+ \mu \sum_{S_u=0}^{\infty} c_u(0) \cdot \frac{x^{\alpha_u+1}}{\alpha_u+1} \cdot \beta_u \cdot (\beta_u - 1) \cdot y^{\beta_u - 2} \cdot z^{\gamma_u} \cdot t^{\delta_u}$$

$$+ \mu \sum_{S_u=0}^{\infty} c_u(0) \cdot \frac{x^{\alpha_u+1}}{\alpha_u+1} \cdot y^{\beta_u} \cdot \gamma_u \cdot (\gamma_u - 1) \cdot z^{\gamma_u - 2} \cdot t^{\delta_u} + h_1(y,z,t) + C$$

(1st Equation after $\int$)

5. 2nd Equation of Navier Stokes

Now consider the second equation of Navier-Stokes

$$\rho \cdot \frac{\partial v}{\partial t} + \rho \cdot u \frac{\partial v}{\partial x} + \rho \cdot v \frac{\partial v}{\partial y} + \rho \cdot w \frac{\partial v}{\partial z} = -\frac{\partial p}{\partial y} + \rho \cdot g_y + \mu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right)$$

(2nd Equation)

In the same way values of V are substitute and integration is done with respect dy. Terms are integrated with respect to y gives

2nd Equation after $\int$

$$\rho \cdot \sum_{S_v=0}^{\infty} c_v(0) \cdot x^{\alpha_v} \cdot \frac{y^{\beta_v+1}}{\beta_v+1} \cdot z^{\gamma_v} \cdot \delta_v t^{\delta_v - 1}$$

$$+ \rho \cdot \sum_{S_k=0}^{\infty} \sum_{S_n=0}^{S_k} c_{u(n)}(0) \cdot c_{v(k-n)}(0) \cdot (\alpha_k - \alpha_n) x^{\alpha_k - 1} \cdot \frac{y^{\beta_k+1}}{\beta_k+1} \cdot z^{\gamma_k} \cdot t^{\delta_k}$$

$$+_{[\text{if } \beta_k \neq 0]} \rho \cdot \sum_{S_k=0}^{\infty} \sum_{S_n=0}^{S_k} c_{v(n)}(0) \cdot c_{v(k-n)}(0) \cdot x^{\alpha_k} \cdot (\beta_k - \beta_n) \frac{y^{\beta_k}}{\beta_k} \cdot z^{\gamma_k} \cdot t^{\delta_k}$$

$$+ \rho \cdot \sum_{S_k=0}^{\infty} \sum_{S_n=0}^{S_k} c_{w(n)}(0) \cdot c_{v(k-n)}(0) \cdot x^{\alpha_k} \cdot \frac{y^{\beta_k+1}}{\beta_k+1} \cdot (\gamma_k - \gamma_n) z^{\gamma_k - 1} \cdot t^{\delta_k}$$

$$= p_0 - p + \rho \cdot g_y \cdot y + \mu \sum_{S_v=0}^{\infty} c_v(0) \cdot \alpha_v \cdot (\alpha_v - 1) x^{\alpha_v - 2} \cdot \frac{y^{\beta_v+1}}{\beta_v+1} \cdot z^{\gamma_v} \cdot t^{\delta_v}$$

$$+ \mu \sum_{S_v=0}^{\infty} c_v(0) \cdot x^{\alpha_v} \cdot \beta_v \cdot y^{\beta_v - 1} \cdot z^{\gamma_v} \cdot t^{\delta_v}$$

$$+ \mu \sum_{S_v=0}^{\infty} c_v(0) \cdot x^{\alpha_v} \cdot \frac{y^{\beta_v+1}}{\beta_v+1} \cdot \gamma_v \cdot (\gamma_v - 1) z^{\gamma_v - 2} \cdot t^{\delta_v} + h_2(x,z,t) + C$$

(2nd Equation after $\int$)

6. 3rd Equation of Navier Stokes

Now consider the third equation of Navier-Stokes

$$\rho \cdot \frac{\partial w}{\partial t} + \rho \cdot u \frac{\partial w}{\partial x} + \rho \cdot v \frac{\partial w}{\partial y} + \rho \cdot w \frac{\partial w}{\partial z} = -\frac{\partial p}{\partial z} + \rho \cdot g_z + \mu \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right)$$

(3rd Equation)

analogously gives terms.

Left Member—3rd Equation after $\int$

$$\rho\cdot\sum_{S_w=0}^{\infty}c_w(0)\cdot x^{\alpha_w}\cdot y^{\beta_w}\cdot\frac{z^{\gamma_w+1}}{\gamma_w+1}\cdot\delta_w t^{\delta_w-1}$$
$$+\rho\cdot\sum_{S_k=0}^{\infty}\sum_{S_n=0}^{S_k}c_{u(n)}(0)\cdot c_{w(k-n)}(0)\cdot(\alpha_k-\alpha_n)x^{\alpha_k-1}\cdot y^{\beta_k}\cdot\frac{z^{\gamma_k+1}}{\gamma_k+1}\cdot t^{\delta_k}$$
$$+\rho\cdot\sum_{S_k=0}^{\infty}\sum_{S_n=0}^{S_k}c_{v(n)}(0)\cdot c_{w(k-n)}(0)\cdot x^{\alpha_k}\cdot(\beta_k-\beta_n)y^{\beta_k-1}\cdot\frac{z^{\gamma_k+1}}{\gamma_k+1}\cdot t^{\delta_k}$$
$$+_{[\text{if }\gamma_k\neq 0]}\rho\cdot\sum_{S_k=0}^{\infty}\sum_{S_n=0}^{S_k}c_{w(n)}(0)\cdot c_{w(k-n)}(0)\cdot x^{\alpha_k}\cdot y^{\beta_k}\cdot(\gamma_k-\gamma_n)\frac{z^{\gamma_k}}{\gamma_k}\cdot t^{\delta_k}$$
$$=p_0-p+\rho\cdot g_z z+\mu\sum_{S_w=0}^{\infty}c_w(0)\cdot\alpha_w\cdot(\alpha_w-1)x^{\alpha_w-2}\cdot y^{\beta_w}\cdot\frac{z^{\gamma_w+1}}{\gamma_w+1}\cdot t^{\delta_w}$$
$$+\mu\sum_{S_w=0}^{\infty}c_w(0)\cdot x^{\alpha_w}\cdot\beta_w\cdot(\beta_w-1)\cdot y^{\beta_w-2}\cdot\frac{z^{\gamma_w+1}}{\gamma_w+1}\cdot t^{\delta_w}$$
$$+\mu\sum_{S_w=0}^{\infty}c_w(0)\cdot x^{\alpha_w}\cdot y^{\beta_w}\cdot\gamma_w z^{\gamma_w-1}\cdot t^{\delta_w}+h_3(x,y,t)+C$$

(3rd Equation after $\int$)

7. Equations Comparison

Now we can make a comparison of three equation's terms to find $\alpha_{u,v,w},\beta_{u,v,w},\gamma_{u,v,w},\delta_{u,v,w}$. Consider the equality of p in the three equations: we match elements because they multiply for a specific physical element (see International System of Measure) like density ρ, viscosity μ or gravity g with dimension $[\rho]\neq[\mu]$ so we can match equation in those three groups separately. Elements grouped under μ should give the same result and terms grouped under ρ should give the same result respectively with the inclusion of 6th Right Member.

First consider the right member of the three group equations elements grouped by μ. Consider parameters of a given $t=t_0$ (where the other remain to determine x, y, z are fixed) is immediately see that equality hold if $\delta_u=\delta_v=\delta_w=\delta$. Under this case equations reduce to

1st Equation after $\int$

$$\rho\cdot\sum_{S_u=0}^{\infty}c_u(0)\cdot\frac{x^{\alpha_u+1}}{\alpha_u+1}\cdot y^{\beta_u}\cdot z^{\gamma_u}$$
$$+_{[\text{if }\alpha_k\neq 0]}\rho\cdot\sum_{S_k=0}^{\infty}\sum_{S_n=0}^{S_k}c_{u(n)}(0)\cdot c_{u(k-n)}(0)\cdot(\alpha_k-\alpha_n)\frac{x^{\alpha_k}}{\alpha_k}\cdot y^{\beta_k}\cdot z^{\gamma_k}\cdot t$$
$$+\rho\cdot\sum_{S_k=0}^{\infty}\sum_{S_n=0}^{S_k}c_{v(n)}(0)\cdot c_{u(k-n)}(0)\cdot\frac{x^{\alpha_k+1}}{\alpha_k+1}\cdot(\beta_k-\beta_n)y^{\beta_k-1}\cdot z^{\gamma_k}\cdot t$$
$$+\rho\cdot\sum_{S_k=0}^{\infty}\sum_{S_n=0}^{S_k}c_{w(n)}(0)\cdot c_{u(k-n)}(0)\cdot\frac{x^{\alpha_k+1}}{\alpha_k+1}\cdot y^{\beta_k}\cdot(\gamma_k-\gamma_n)z^{\gamma_k-1}\cdot t$$
$$=p_0-p+\rho\cdot g_x(x)+\mu\sum_{S_u=0}^{\infty}c_u(0)\cdot\alpha_u\cdot x^{\alpha_u-1}\cdot y^{\beta_u}\cdot z^{\gamma_u}\cdot t$$
$$+\mu\sum_{S_u=0}^{\infty}c_u(0)\cdot\frac{x^{\alpha_u+1}}{\alpha_u+1}\cdot\beta_u\cdot(\beta_u-1)\cdot y^{\beta_u-2}\cdot z^{\gamma_u}\cdot t$$

$$+\mu\sum_{S_u=0}^{\infty}c_u(0)\cdot\frac{x^{\alpha_u+1}}{\alpha_u+1}\cdot y^{\beta_u}\cdot\gamma_u\cdot(\gamma_u-1)\cdot z^{\gamma_u-2}\cdot t+h_1(y,z,t)+C$$

(1st Equation after $\int$)

$$\rho\cdot\sum_{S_v=0}^{\infty}c_v(0)\cdot x^{\alpha_v}\cdot\frac{y^{\beta_v+1}}{\beta_v+1}\cdot z^{\gamma_v}$$
$$+\rho\cdot\sum_{S_k=0}^{\infty}\sum_{S_n=0}^{S_k}c_{u(n)}(0)\cdot c_{v(k-n)}(0)\cdot(\alpha_k-\alpha_n)x^{\alpha_k-1}\cdot\frac{y^{\beta_k+1}}{\beta_k+1}\cdot z^{\gamma_k}\cdot t$$
$$+_{[\text{if }\beta_k\neq0]}\rho\cdot\sum_{S_k=0}^{\infty}\sum_{S_n=0}^{S_k}c_{v(n)}(0)\cdot c_{v(k-n)}(0)\cdot x^{\alpha_k}\cdot(\beta_k-\beta_n)\frac{y^{\beta_k}}{\beta_k}\cdot z^{\gamma_k}\cdot t$$
$$+\rho\cdot\sum_{S_k=0}^{\infty}\sum_{S_n=0}^{S_k}c_{w(n)}(0)\cdot c_{v(k-n)}(0)\cdot x^{\alpha_k}\cdot\frac{y^{\beta_k+1}}{\beta_k+1}\cdot(\gamma_k-\gamma_n)z^{\gamma_k-1}\cdot t$$
$$=p_0-p+\rho\cdot g_y\cdot y+\mu\sum_{S_v=0}^{\infty}c_v(0)\cdot\alpha_v\cdot(\alpha_v-1)x^{\alpha_v-2}\cdot\frac{y^{\beta_v+1}}{\beta_v+1}\cdot z^{\gamma_v}\cdot t$$
$$+\mu\sum_{S_v=0}^{\infty}c_v(0)\cdot x^{\alpha_v}\cdot\beta_v\cdot y^{\beta_v-1}\cdot z^{\gamma_v}\cdot t$$
$$+\mu\sum_{S_v=0}^{\infty}c_v(0)\cdot x^{\alpha_v}\cdot\frac{y^{\beta_v+1}}{\beta_v+1}\cdot\gamma_v\cdot(\gamma_v-1)z^{\gamma_v-2}\cdot t+h_2(x,z,t)+C$$

(2nd Equation after $\int$)

$$\rho\cdot\sum_{S_w=0}^{\infty}c_w(0)\cdot x^{\alpha_w}\cdot y^{\beta_w}\cdot\frac{z^{\gamma_w+1}}{\gamma_w+1}$$
$$+\rho\cdot\sum_{S_k=0}^{\infty}\sum_{S_n=0}^{S_k}c_{u(n)}(0)\cdot c_{w(k-n)}(0)\cdot(\alpha_k-\alpha_n)x^{\alpha_k-1}\cdot y^{\beta_k}\cdot\frac{z^{\gamma_k+1}}{\gamma_k+1}\cdot t$$
$$+\rho\cdot\sum_{S_k=0}^{\infty}\sum_{S_n=0}^{S_k}c_{v(n)}(0)\cdot c_{w(k-n)}(0)\cdot x^{\alpha_k}\cdot(\beta_k-\beta_n)y^{\beta_k-1}\cdot\frac{z^{\gamma_k+1}}{\gamma_k+1}\cdot t$$
$$+_{[\text{if }\gamma_k\neq0]}\rho\cdot\sum_{S_k=0}^{\infty}\sum_{S_n=0}^{S_k}c_{w(n)}(0)\cdot c_{w(k-n)}(0)\cdot x^{\alpha_k}\cdot y^{\beta_k}\cdot(\gamma_k-\gamma_n)\frac{z^{\gamma_k}}{\gamma_k}\cdot t$$
$$=p_0-p+\rho\cdot g_z z+\mu\sum_{S_w=0}^{\infty}c_w(0)\cdot\alpha_w\cdot(\alpha_w-1)x^{\alpha_w-2}\cdot y^{\beta_w}\cdot\frac{z^{\gamma_w+1}}{\gamma_w+1}\cdot t$$
$$+\mu\sum_{S_w=0}^{\infty}c_w(0)\cdot x^{\alpha_w}\cdot\beta_w\cdot(\beta_w-1)\cdot y^{\beta_w-2}\cdot\frac{z^{\gamma_w+1}}{\gamma_w+1}\cdot t$$
$$+\mu\sum_{S_w=0}^{\infty}c_w(0)\cdot x^{\alpha_w}\cdot y^{\beta_w}\cdot\gamma_w z^{\gamma_w-1}\cdot t+h_3(x,y,t)+C$$

(3rd Equation after $\int$)

Consider parameters of a given $z=z_0$, in this group (where the other remain to determine *x*, *y* are fixed), the three elements match in the first equation and the second equation if $\gamma_u=\gamma_v=\gamma$ (because of the power of *z*) whereas for a given $y=y_0$ in the first equation and third equation if $\beta_u=\beta_w=\beta$ (because of the power of *y*), whereas for a given $x=x_0$ in the second equation and the third equation if $\alpha_v=\alpha_w=\alpha$ (because of the power of *x*). Now confronting the elements with higher exponent of *z* of the first and second equation they match if $\alpha_u=\beta_v$ and if $\alpha_u+1=\alpha_v=1$ and $\beta_u=\beta_v+1=1$ (because of power of *z* is chosen): but in this way match the all two equations for this group. Then $c_u = c_v$. Analogous result can be obtained comparing the first equation with the third (starting from lower exponent y and because of power of y) $\beta_u = \beta_w$, $\gamma_w = \alpha_u$, $\alpha_w = \alpha_u + 1$, $\gamma_u = \gamma_w + 1$, $c_u = c_w$. The second equation with the third gives can be done as check of the other.

Satisfy Continuity Equation

Now consider the velocities with substitution of parameter found above.

$$u = \sum_{S_u}^{\infty} c_u(0) \cdot x^{\alpha-1} \cdot y^{\beta} \cdot z^{\gamma} \cdot t^{\delta_u}$$

$$v = \sum_{S_v}^{\infty} c_v(0) \cdot x^{\alpha} \cdot y^{\beta-1} \cdot z^{\gamma} \cdot t^{\delta_v}$$

$$w = \sum_{S_w}^{\infty} c_w(0) \cdot x^{\alpha} \cdot y^{\beta} \cdot z^{\gamma-1} \cdot t^{\delta_w}$$

From continuity equation result

$$\sum_{S_u}^{\infty} c_u(0) \cdot (\alpha-1) x^{\alpha-2} \cdot y^{\beta} \cdot z^{\gamma} \cdot t^{\delta_u} + \sum_{S_v}^{\infty} c_v(0) \cdot x^{\alpha} (\beta-1) \cdot y^{\beta-2} \cdot z^{\gamma} \cdot t^{\delta_v}$$
$$+ \sum_{S_w}^{\infty} c_w(0) \cdot x^{\alpha} \cdot y^{\beta} \cdot (\gamma-1) z^{\gamma-2} \cdot t^{\delta_w} = 0$$

This equation result satisfied for all (x_0, y_0, z_0, t_0) if $\alpha = \beta = \gamma = 1$.

Parameters		
$\alpha_u = 0$	$\alpha_v = 1$	$\alpha_w = 1$
$\beta_u = 1$	$\beta_v = 0$	$\beta_w = 1$
$\gamma_u = 1$	$\gamma_v = 1$	$\gamma_w = 0$
$c_u = c$	$c_v = c$	$c_w = c$

Further note that parameter $c_{(\alpha_u, \beta_u, \gamma_u, \delta)} = c_{(0,1,1,\delta)}$ gives u expression

$$u = c_{(0,1,1,\delta)} \cdot y \cdot z \cdot t^{\delta}$$

and substituting the value of u in (A.1) with those parameters

$$c_{(\alpha)}(0) = c_{(0,1,1,\delta)} \cdot \delta \cdot t^{\delta-1}$$

This term in point $a = (0,0,0,0)$ is constant different from zero if and only if $\delta = 1$

Parameters		
$\delta_u = 1$	$\delta_v = 1$	$\delta_w = 1$

Group under ρ

With this choice of parameter all first elements of the three groups under ρ became the same. For other elements they have an element in each group that goes to zero: that is the element $(\alpha_k - \alpha_n) = \alpha_u$ and $(\beta_k - \beta_n) = \beta_v$, $(\gamma_k - \gamma_n) = \gamma_w$ respectively. Then the other two elements in each of the three group take two possible values from three values a_1, a_2, a_3 but not the same couple for two different groups: $(a_1, a_2), (a_1, a_3), (a_2, a_3)$ with

$$a_1 = \rho \cdot c_0^2 \cdot \frac{x^2 y^2 t^2}{2}$$

$$a_2 = \rho \cdot c_0^2 \cdot \frac{x^2 z^2 t^2}{2}$$

$$a_3 = \rho \cdot c_0^2 \cdot \frac{y^2 z^2 t^2}{2}$$

The values are then matched because of 6th Right Member

$h_1(y,z,t), h_2(x,z,t), h_3(x,y,t)$

Group under μ

All $\alpha, \beta, \gamma \le 1$ and $\alpha_u = \beta_v = \gamma_w = 0$ therefore those terms are all zero.

Group under g

The second right member if we substitute $\mathbf{g} = (0, 0, -g)$ gives the element only for the component of z versor: $-\rho \cdot g \cdot z$ which can be matched in the 6th Right Member of the 1st and 2nd equation $h_1(y,z,t)$ and $h_2(x,z,t)$.

Consideration

Exist infinite groups of parameters and function that solve Navier-Stokes in continuum. There is at least one because of Parameters choice, whereas they are infinite for terms $h_1(y,z,t)$ and $h_2(x,z,t)$ and $h_3(x,y,t)$.

8. Result

Substituting the parameter in the V equations we obtain:

$$\begin{aligned} V &= u \cdot \mathbf{i} + v \cdot \mathbf{j} + w \cdot \mathbf{k} \\ u &= c_o \cdot y \cdot z \cdot t \\ v &= c_o \cdot x \cdot z \cdot t \\ w &= c_o \cdot x \cdot y \cdot t \end{aligned} \qquad \text{(9. Velocities)}$$

where we stated c_o the coefficient respect the origin O of space and time with $[c_0] = [\mathrm{m}^{-1} \cdot \mathrm{s}^{-2}]$: c_0 can be named wave constant. When elements are on the surface $z = 0$ only $w \ne 0$ and motion is towards upward if $c > 0$ and downwards if $c < 0$. If we think a point as origin $z = 0$ and $0 \le x, y \vee x, y \le 0$ we see a movement upward and downward respect the four quarters (x, y) that should be wave motion (Cartesian plane on sea surface). Velocity can be normalized to give stream

$$\begin{aligned} u_s &= \frac{y \cdot z}{\sqrt{(xy)^2 + (xz)^2 + (yz)^2}} \\ v_s &= \frac{x \cdot z}{\sqrt{(xy)^2 + (xz)^2 + (yz)^2}} \\ w_s &= \frac{x \cdot y}{\sqrt{(xy)^2 + (xz)^2 + (yz)^2}} \end{aligned} \qquad \text{(10. Stream)}$$

x, y, z became the shift between two different streams and convergence is possible because quantities are limited. Time is periodic, water that periodically drops into sea. The continuity equation became

$$\nabla \cdot \mathbf{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \qquad \text{(11.Continuity)}$$

Elements in one equation from comparison section became

$$\rho \cdot c_o \cdot x \cdot y \cdot z + \rho \cdot c_o^2 \cdot \frac{y^2}{2} \cdot z^2 \cdot t^2 + \rho \cdot c_o^2 \cdot \frac{x^2}{2} \cdot z^2 \cdot t^2 + \rho \cdot c_o^2 \cdot \frac{x^2}{2} \cdot y^2 \cdot t^2$$
$$= p_0 - p - \rho \cdot g \cdot z$$

So the pressure is

$$p = p_0 - \rho \cdot c_o \cdot x \cdot y \cdot z - \rho \cdot c_o^2 \cdot \frac{y^2}{2} \cdot z^2 \cdot t^2 - \rho \cdot c_o^2 \cdot \frac{x^2}{2} \cdot z^2 \cdot t^2$$
$$- \rho \cdot c_o^2 \cdot \frac{x^2}{2} \cdot y^2 \cdot t^2 - \rho \cdot g \cdot z$$

If we set to $c_0 \cdot x \cdot y = \frac{\partial w}{\partial t} = w_t$ acceleration along z axis

$$p = p_0 + \rho \cdot \left[\frac{|\mathbf{V}|^2}{2} - z \cdot \left(w_t + g \right) \right] \qquad \text{(12. Pressure)}$$

with p_0 pressure at origin an p pressure at point (x, y, z) in time t, module Velocity $|V| = c \cdot t \sqrt{(xy)^2 + (xz)^2 + (yz)^2}$ and dimension in $[p] = [\mathrm{kg} \cdot \mathrm{m}^{-1} \cdot \mathrm{s}^{-2}]$, $[\rho] = [\mathrm{kg} \cdot \mathrm{m}^{-3}]$ where t is periodic. Pressure increases proportionally to the square of velocity and decrease proportionally to gravity and z-axis acceleration.

Observation

Further consider that in hydrostatic case when $\mathbf{V} = 0$ then $w_t = 0$ and $p = p_0 - \rho \cdot g \cdot z$ that is well known Stevino formula $p = -\rho g z + p_0$ where the negative sign multiplies negative z.

9. Conclusion

Pressure with Velocity Module and Stream is completing where every physical quantity is finite with respect to convergence.

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Appendix

Some Previous Law

Consider the following system in form:

$$\nabla \cdot \mathbf{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$\rho \cdot \frac{D\mathbf{V}}{Dt} = -\nabla p + \rho \cdot \mathbf{g} + \mu \nabla^2 \mathbf{V}$$

where C is the continuity equation, whereas NS are the Navier-Stokes equation for incompressible fluid. Unknown are V,p whereas $\mathbf{g}$ is the gravity and μ is the viscosity constant.

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$$

with $u = \frac{\mathrm{d}x}{\mathrm{d}t}; v = \frac{\mathrm{d}y}{\mathrm{d}t}; w = \frac{\mathrm{d}z}{\mathrm{d}t}$ and $\mathbf{g} = g_x\mathbf{i} + g_y\mathbf{j} + g_z\mathbf{k}$ usually $\mathbf{g} = -g\mathbf{k}$ and $\mathbf{V} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$ where i, j, k are the unit vector in the axis direction.

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

is the gradient and

$$\nabla^2 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)$$

is the Laplacian operator.

For example respect to u Navier Stokes first equation became

$$\rho \cdot \frac{\partial u}{\partial t} + \rho u\frac{\partial u}{\partial x} + \rho v\frac{\partial u}{\partial y} + \rho w\frac{\partial u}{\partial z} = -\frac{\partial p}{\partial x} + \rho \cdot g_x + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

(1^{st} Navier Stokes)