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Four New Examples of the Non-Elementary Expo-Elliptic Functions That Are Giving Solutions to Some Second-Order Nonlinear Autonomous ODEs

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Abstract

In this paper, we define four new examples of the non-elementary expo-elliptic functions. This is an exponential function whose exponent is the product of a real number and the upper limit of integration in a non-elementary integral that can be arbitrary. We are using Abel's methods, described by Armitage and Eberlein. We will study some of the second-order nonlinear ODEs, especially those that exhibit limit cycles, and systems of nonlinear ODEs that these functions are giving solutions to.

Keywords

Non-Elementary Functions, Second-Order Nonlinear Autonomous ODE

1. Introduction

On page 1 in the book [1] we find the sentences: Very few ordinary differential equations have explicit solutions expressible in finite terms. This is not because ingenuity fails, but because the repertory of standard functions (polynomials, exp, sin and so on) in terms of which solutions may be expressed is too limited to accommodate the variety of differential equations encountered in practice.

This is the main reason for this work. It should be possible to do something about this problem. If we don't have enough tools in our mathematical toolbox, we must make the tools first. For this problem we will attempt to define some new functions. In this paper I want to share some of these results with you. The numbers I have given the ODEs and the integral functions (IF) in the text, are the numbers they have in my collection.

In the Introduction to [2] we can read: Nonlinear dynamical systems exhibiting limit cycles are found in a large variety of fields including biology, chemistry, mechanics and electronics.

In the phase diagrams in this paper we will see many different limit cycles with 3 - 17 equilibrium points. They appear in the study of the second-order nonlinear ODEs and systems of nonlinear ODEs that have the expo-elliptic functions defined in this paper as solutions.

Wolfram Math World describes three nonlinear second-order ODEs that have the Jacobi elliptic functions sn, cn and dn as solutions. Define a solution $x(t) = cn(t)$ and differentiate twice, and you will obtain the ODE:

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = \left(2k^2 - 1\right)x - 2k^2 x^3, \quad 0 \le k < 1 \tag{1}$$

And if we use the Jacobi amplitude function $am(t,k)$ as a solution $x(t)$ and differentiate twice, we will obtain the ODE:

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = -k^2 \sin(x)\cos(x) \tag{2}$$

This causes us to think that other second-order nonlinear ODEs have functions made by the same methods as Jacobi elliptic functions, as their solutions. It should be possible to make more non-elementary functions by changing the non-elementary integral. In this paper, we will work in the same way: First define some non-elementary functions, and then differentiate them twice in order to see what kind of ODEs these functions are giving solutions to.

We will use the methods described by Armitage and Eberlein [3] in their book Elliptic Functions. Especially Section 1.6 and 1.7. They apply what they call the Abel's methods. "Eberlein sought to relate the ideas of Abel to the later work of Jacobi."

During the last 30 years there have been done a lot of progress in finding solutions to nonlinear ODEs and PDEs. The progress is mostly made by using different methods like the Prelle-Singer method [4], Abel's equations [5] [6], the new Jacobi elliptic functions [7] [8], the old Jacobi elliptic functions [9] [10], a new method [11], revised methods [12], Jacobi elliptic function expansion method [13], expo-elliptic functions [14].

With exception of the new Jacobi elliptic functions and the expo-elliptic functions, it seems to me that nobody has tried to make new non-elementary functions that can give solutions to second-order nonlinear ODEs. In this paper we will attempt to take a step further.

Through this paper we will study four new examples of the expo-elliptic functions, give each of them symbols, find their derivatives and investigate which second order nonlinear ODEs these functions are giving solutions to by differentiating them twice. In order to discover some of the qualities of the expo-elliptic functions we will make some phase diagrams. The behavior of the solution curves in the phase diagrams reflects the qualities of the solution functions. The functions defined in this paper are new to the literature, at least to my knowledge.

2. The Expo-Elliptic Functions

This is a large group of functions and very useful as both solutions to second-order nonlinear ODEs and systems of nonlinear ODEs. In this paper we will study four new examples.

In order to work with functions, we must give them some symbols. I have used two letters where the first one is the same for a set of two functions, and the second letters are *s* and *d*, for example *rs* and *rd*, or *ks* and *kd*.

2.1. Definition

Define an exponential function $\mu = \mu(u) = \mathrm{e}^{a\varphi}$, $a, \varphi \in R$, $-\infty < a, \varphi < \infty$, $\varphi = \varphi(u)$ is the amplitude or upper limit of integration in a non-elementary integral. This connection to the elliptic functions is the reason for this special name. They may also be named μ-functions, but that name tells nothing about these functions. The integral may also be elementary. Then the solution $x(t) = \mathrm{e}^{a\varphi}$ is an elementary solution. When $\frac{\mathrm{d}\varphi}{\mathrm{d}t} = 1$, is $\mathrm{e}^{a\varphi} = \mathrm{e}^{at}$. The expo-elliptic function is more general than the elementary exponential function. The expo-elliptic function is not periodic, and it is continuous and differentiable on the whole *R*, for the limitations of the parameters.

2.2. Four Subgroups of the Expo-Elliptic Functions

These subgroups are defined by how the integral functions u are:

$$\text{A1:} \quad u = u(\varphi) = \int_0^{\varphi} \frac{\mathrm{d}\theta}{\sqrt{f(\mathrm{e}^{a\theta})}} \tag{3}$$

The function f can be whatever.

$$\text{A2:} \quad u = u(\varphi) = \int_0^{\varphi} \frac{\mathrm{e}^{a\theta}}{\sqrt{f(\mathrm{e}^{a\theta})}} \mathrm{d}\theta \tag{4}$$

$$\text{B1:} \quad u = u(\varphi) = \int_0^{\varphi} \frac{\mathrm{d}\theta}{f(\mathrm{e}^{a\theta})} \tag{5}$$

$$\text{B2:} \quad u = u(\varphi) = \int_0^{\varphi} \frac{\mathrm{e}^{a\theta}}{f(\mathrm{e}^{a\theta})} \mathrm{d}\theta \tag{6}$$

We find the most interesting functions in the subgroup B2. The examples in this paper are chosen from this group.

2.3. The Derivative of the Expo-Elliptic Function

$$\frac{\mathrm{d}}{\mathrm{d}u}\mu(u) = a\mathrm{e}^{a\varphi}\frac{\mathrm{d}\varphi}{\mathrm{d}u} = a\mu(u)\frac{\mathrm{d}\varphi}{\mathrm{d}u} \tag{7}$$

2.4. Four Examples

Let us take a look at four examples of expo-elliptic functions, and some second-order nonlinear ODEs and systems of nonlinear ODEs that have these functions as solutions. I think that the behavior of the solution curves in a phase diagram reflects the qualities of the solution functions. And by studying the behavior of the solution curves we can discover some of the qualities of the expo-elliptic functions.

2.4.1. The Functions *rs* and *rd*

Just like the Jacobi elliptic functions we start with a non-elementary integral.

Define an integral function u (IF 121):

$$u = u(\varphi) = \int_0^{\varphi} \frac{\mathrm{e}^{a\theta}}{n + f\mathrm{e}^{a\theta} + h\cos\left(g\mathrm{e}^{2a\theta} + p\mathrm{e}^{a\theta} + b\right) + k\sin\left(c\mathrm{e}^{2a\theta} + v\mathrm{e}^{a\theta} + d\right)} \mathrm{d}\theta \quad (8)$$

a,b,c,d,f,g,h,k,n,p,v are parameters defined on R.

The denominator can become 0. In order to avoid that we can make some restrictions to the parameters: $n > 3$, $-1 \le f,h,k \le 1$, $a < 0$

$$\frac{\mathrm{d}u}{\mathrm{d}\varphi} = \frac{\mathrm{e}^{a\varphi}}{n + f\mathrm{e}^{a\varphi} + h\cos\left(g\mathrm{e}^{2a\varphi} + p\mathrm{e}^{a\varphi} + b\right) + k\sin\left(c\mathrm{e}^{2a\varphi} + v\mathrm{e}^{a\varphi} + d\right)} \quad (9)$$

Inverting $\frac{\mathrm{d}u}{\mathrm{d}\varphi}$:

$$\frac{\mathrm{d}\varphi}{\mathrm{d}u} = \mathrm{e}^{-a\varphi}\left(n + f\mathrm{e}^{a\varphi} + h\cos\left(g\mathrm{e}^{2a\varphi} + p\mathrm{e}^{a\varphi} + b\right) + k\sin\left(c\mathrm{e}^{2a\varphi} + v\mathrm{e}^{a\varphi} + d\right)\right) \quad (10)$$

Define a set of 2 functions *rs* and *rd*, so that

$$rs(u) = \mathrm{e}^{a\varphi} \quad (11)$$

$$\begin{aligned} rd(u) &= \mathrm{e}^{-a\varphi}\left(n + f\mathrm{e}^{a\varphi} + h\cos\left(g\mathrm{e}^{2a\varphi} + p\mathrm{e}^{a\varphi} + b\right) + k\sin\left(c\mathrm{e}^{2a\varphi} + v\mathrm{e}^{a\varphi} + d\right)\right) \\ &= \frac{1}{rs(u)}\left(n + f\,rs(u) + h\cos\left(g\,rs^2(u) + p\,rs(u) + b\right)\right. \\ &\quad \left. + k\sin\left(c\,rs^2(u) + v\,rs(u) + d\right)\right) \end{aligned} \quad (12)$$

The connection between the functions *rs* and *rd*:

$$\begin{aligned} & rd(u)rs(u) - f\,rs(u) - h\cos\left(g\,rs^2(u) + p\,rs(u) + b\right) \\ & -k\sin\left(c\,rs^2(u) + v\,rs(u) + d\right) = n \end{aligned} \quad (13)$$

We see that the function $rd(u)$ exists for all values of the parameters a,b,c,d,f,g,h,n,p,v even though the integral IF 121 is not defined for values that make the denominator = 0. The functions $rs(u)$ and $rd(u)$ are continuous and differentiable on the whole *R*.

The derivatives to these functions are:

$$\frac{\mathrm{d}}{\mathrm{d}u} rs(u) = a\,rs(u)\,rd(u) \quad (14)$$

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}u} rd(u) = &-\frac{a}{rs^2(u)}\left(n + f\,rs(u) + h\cos\left(g\,rs^2(u) + p\,rs(u) + b\right)\right. \\ & \left. + k\sin\left(c\,rs^2(u) + v\,rs(u) + d\right)\right)^2 + \frac{a}{rs(u)}\left(n + f\,rs(u)\right. \end{aligned}$$

$$+h\cos\left(g\,rs^2(u)+p\,rs(u)+b\right)+k\sin\left(c\,rs^2(u)+v\,rs(u)+d\right)\Big)$$
$$\times\Big[f-h\sin\left(g\,rs^2(u)+p\,rs(u)+b\right)\left(2g\,rs(u)+p\right) \tag{15}$$
$$+k\cos\left(c\,rs^2(u)+v\,rs(u)+d\right)\left(2c\,rs(u)+v\right)\Big]$$

Define a solution

$$x(t)=rs(t) \tag{16}$$

$$\frac{\mathrm{d}x}{\mathrm{d}t}=a\,rs(t)rd(t)=a\left(n+fx+h\cos\left(gx^2+px+b\right)+k\sin\left(cx^2+vx+d\right)\right) \tag{17}$$

$$\begin{aligned}\frac{\mathrm{d}^2x}{\mathrm{d}t^2}&=a^2\left(n+fx+h\cos\left(gx^2+px+b\right)+k\sin\left(cx^2+vx+d\right)\right)\\&\quad\times\Big[f-h\sin\left(gx^2+px+b\right)\left(2gx+p\right)\\&\quad+k\cos\left(cx^2+vx+d\right)\left(2cx+v\right)\Big]\end{aligned} \tag{2880}$$

$$\begin{aligned}\frac{\mathrm{d}^2x}{\mathrm{d}t^2}&=avk\frac{\mathrm{d}x}{\mathrm{d}t}\cos\left(cx^2+vx+d\right)+a^2\Big(n+fx+h\cos\left(gx^2+px+b\right)\\&\quad+k\sin\left(cx^2+vx+d\right)\Big)\times\Big[f-h\sin\left(gx^2+px+b\right)\left(2gx+p\right)\\&\quad+2ckx\cos\left(cx^2+vx+d\right)\Big]\end{aligned} \tag{2881}$$

The solution curves of the differential Equation (2881) have some very interesting behavior. For some of the values of the parameters, we become equations where the solution curves form limit cycles with 3, 5, 7, 9, 11, 13 or even 17 equilibrium points. They are spiral sources, spiral sinks or saddle points. See the Figures below.

In **Figure 1** are the parameter-values:

$$c=\frac{1}{4},v=2,d=-1,n=6,f=-\frac{1}{10},h=-\frac{1}{2},g=-\frac{1}{2},p=\frac{1}{2},b=3,k=1,a=1 \tag{18}$$

A limit cycle (LC) with 3 equilibrium points: 2 spiral source and one saddle point.

In **Figure 2** and **Figure 3** are the parameter-values:

$$\begin{aligned}&c=\frac{1}{4},v=5,d=-1,n=5,f=-\frac{1}{50},h=-\frac{1}{2},g=-\frac{1}{4},\\&p=\frac{1}{2},b=5,k=\frac{2}{5},a=1\end{aligned} \tag{19}$$

Figure 3 shows a large LC containing 2 smaller stable LC and one unstable LC, with 17 equilibrium points: 5 spiral sources, 5 spiral sinks and 7 saddle points, as I can see.

Figure 2 shows only the large LC with one inside initial point and one outside initial point.

The equilibrium points do not need to be on the x-axis. In **Figure 4** the equilibrium points are located on a slope straight line.

System (3005):

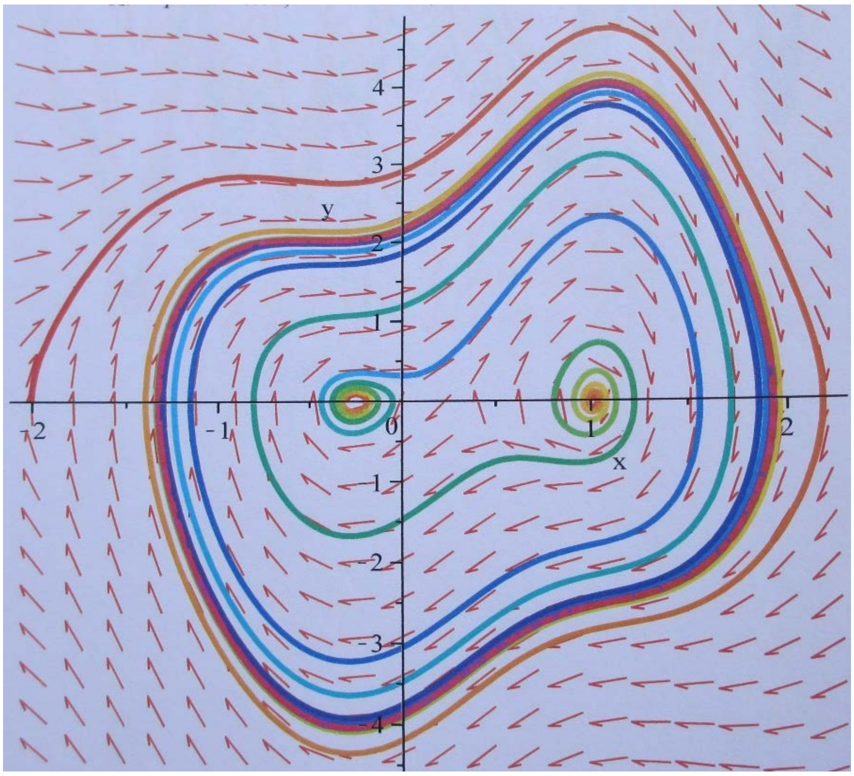

Figure 1. LC with 3 equilibrium points.

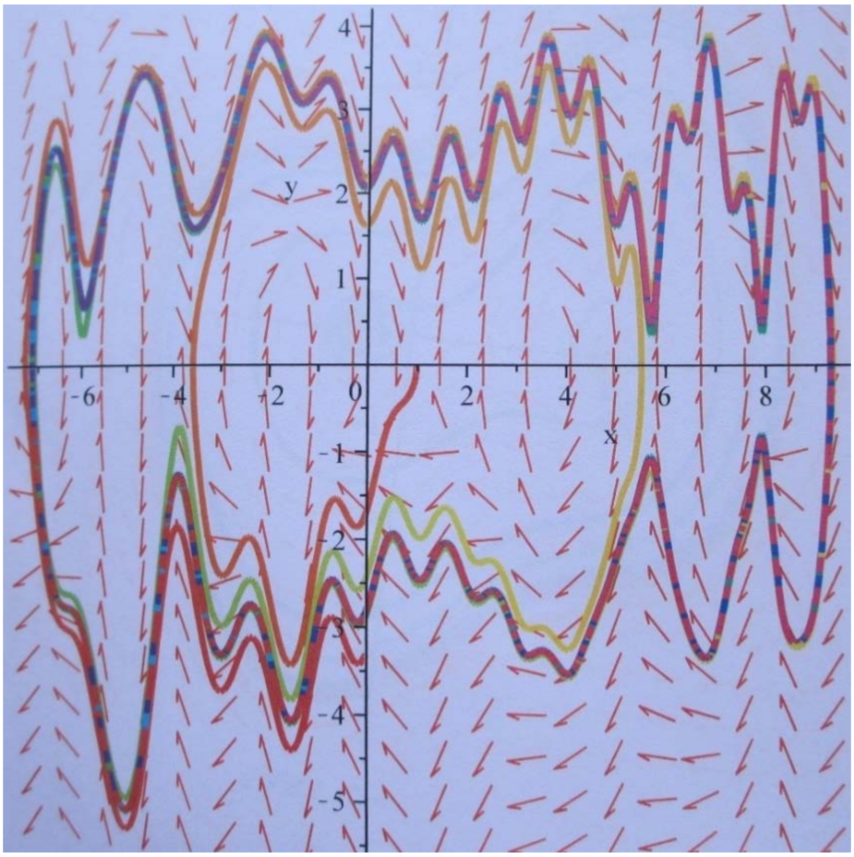

Figure 2. A large irregular LC.

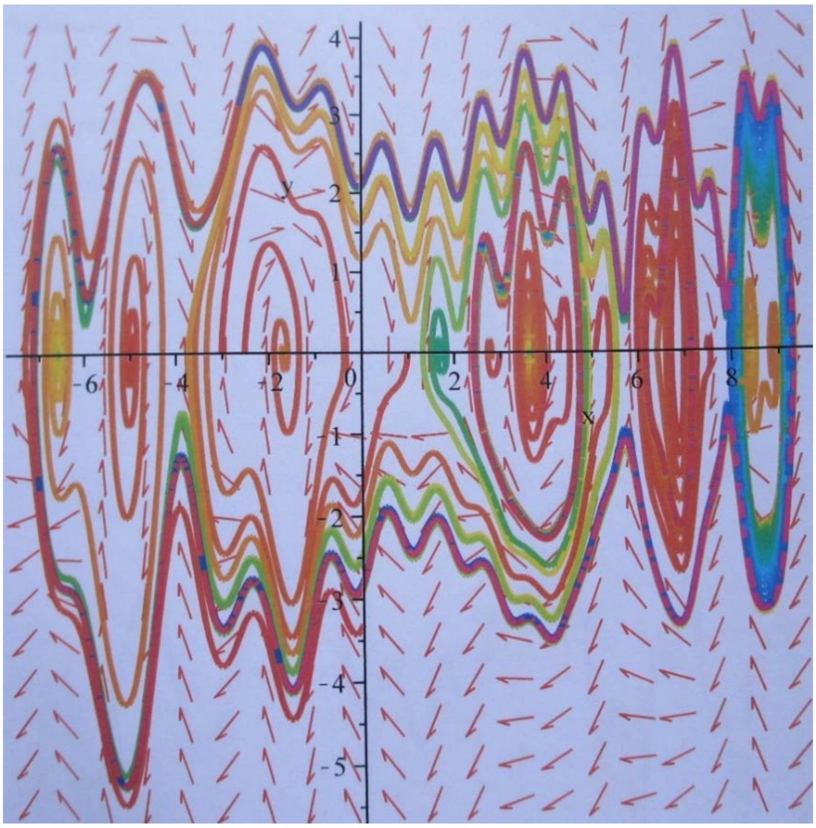

Figure 3. A LC with 17 equilibrium points.

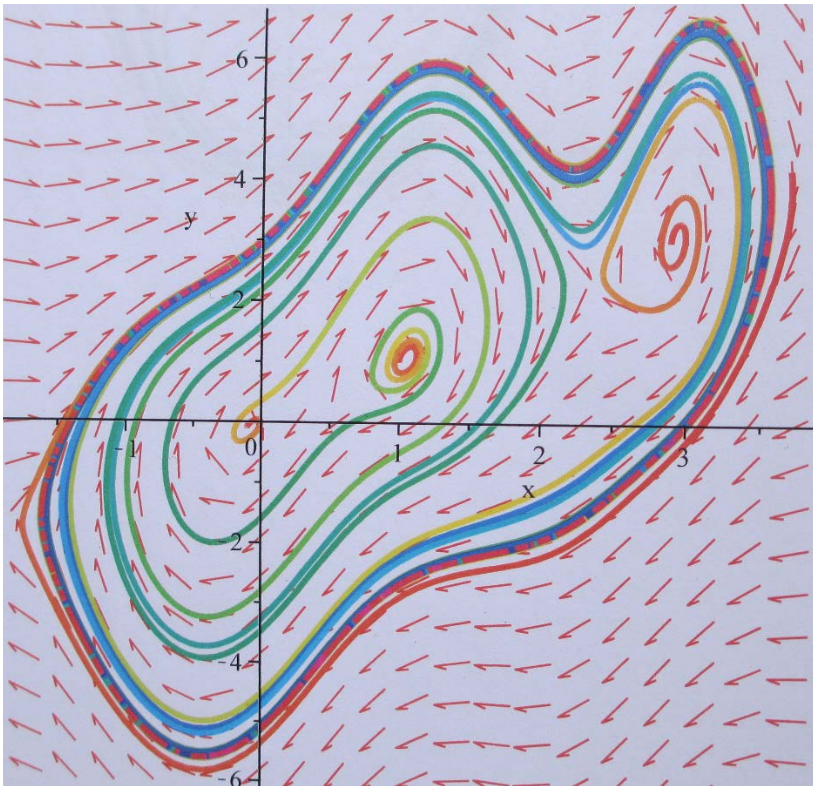

Figure 4. A LC with 5 equilibrium points on the line $y = x$.

$$\frac{\mathrm{d}x}{\mathrm{d}t} = sx + ly$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -\frac{s^2}{l}x - sy + \frac{a}{l}vk(sx+ly)\cos\left(cx^2+vx+d\right)$$
$$+\frac{a^2}{l}\left(n+fx+h\cos\left(gx^2+px+b\right)+k\sin\left(cx^2+vx+d\right)\right)$$
$$\times\left[f-h\sin\left(gx^2+px+b\right)(2gx+p)+2kcx\cos\left(cx^2+vx+d\right)\right]$$

The system (3005) has the solutions:

$$x(t) = rs(t) \tag{20}$$

$$y(t) = -\frac{s}{l}rs(t)+\frac{a}{l}\Big(n+f\,rs(t)+h\cos\left(g\,rs^2(t)+p\,rs(t)+b\right) + k\sin\left(c\,rs^2(t)+v\,rs(t)+d\right)\Big) \tag{21}$$

In **Figure 4** are the parameter-values:

$$s=-1, l=1, a=1, f=-\frac{1}{10}, p=1, n=6, v=2,$$
$$d=-1, b=3, k=1, h=-\frac{1}{2}, c=\frac{1}{4}, g=-\frac{1}{2} \tag{22}$$

Now we will make a system where the equilibrium points are located on a curve line, as we can see in **Figure 5**.

The system (3014):

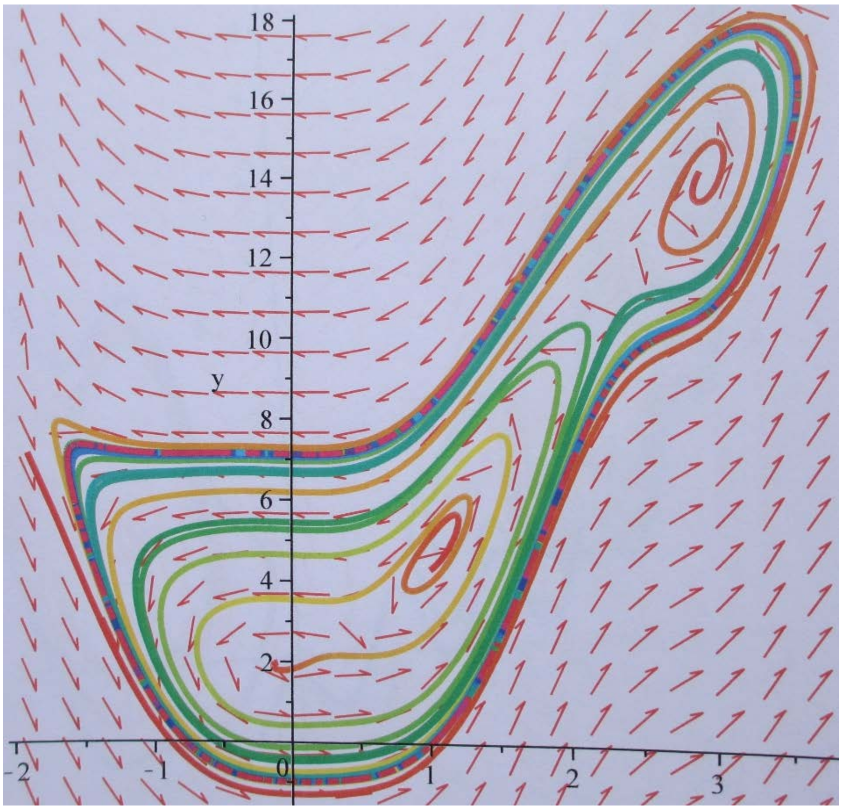

Figure 5. A LC with 5 equilibrium points on a curve line.

$$\frac{\mathrm{d}x}{\mathrm{d}t} = sx + ux^2 + jx^3 + ly + q$$

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}t} = &-\frac{1}{l}\left(sx + ly + ux^2 + jx^3 + q\right)\left(s + 2ux + 3jx^2\right)\\&+\frac{a}{l}vk\left(sx + ly + ux^2 + jx^3 + q\right)\cos\left(cx^2 + vx + d\right)\\&+\frac{a^2}{l}\left(n + fx + h\cos\left(gx^2 + px + b\right) + k\sin\left(cx^2 + vx + d\right)\right)\\&\times\left[f - h\sin\left(gx^2 + px + b\right)\left(2gx + p\right) + 2kcx\cos\left(cx^2 + vx + d\right)\right]\end{aligned}$$

System (3014) has the solutions:

$$x(t) = rs(t) \tag{23}$$

$$\begin{aligned}y(t) = &-\frac{1}{l}\left(s\,rs(t) + u\,rs^2(t) + j\,rs^3(t) + q\right)\\&+\frac{a}{l}\left(n + f\,rs(t) + h\cos\left(g\,rs^2(t) + p\,rs(t) + b\right)\right.\\&\left.+k\sin\left(c\,rs^2(t) + v\,rs(t) + d\right)\right)\end{aligned} \tag{24}$$

The parameter-values are:

$$\begin{aligned}&s = 1, l = -1, u = 2, j = -\frac{1}{3}, q = 2, a = 1, f = -\frac{1}{10}, p = 1,\\&n = 6, v = 2, d = -1, b = 3, k = 1, h = -\frac{1}{2}, c = \frac{1}{4}, g = -\frac{1}{2}\end{aligned} \tag{25}$$

System (3039) as a last example:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = sx^2 + ly + q$$

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}t} = &-\frac{2s}{l}x\left(sx^2 + ly + q\right) + \frac{a}{l}vk\left(sx^2 + ly + q\right)\cos\left(cx^2 + vx + d\right)\\&+\frac{a^2}{l}\left(n + fx + h\cos\left(gx^2 + px + b\right) + k\sin\left(cx^2 + vx + d\right)\right)\\&\times\left[f - h\sin\left(gx^2 + px + b\right)\left(2gx + p\right) + 2kcx\cos\left(cx^2 + vx + d\right)\right]\end{aligned}$$

System (3039) has the solutions:

$$x(t) = rs(t) \tag{26}$$

$$\begin{aligned}y(t) = &-\frac{1}{l}\left(s\,rs^2(t) + q\right) + \frac{a}{l}\left(n + f\,rs(t) + h\cos\left(g\,rs^2(t) + p\,rs(t) + b\right)\right.\\&\left.+k\sin\left(c\,rs^2(t) + v\,rs(t) + d\right)\right)\end{aligned} \tag{27}$$

The parameter-values are:

$$\begin{aligned}&a = -1, l = 1, s = -\frac{1}{4}, q = -1, v = 5, f = -\frac{1}{50}, k = \frac{2}{5},\\&h = -\frac{1}{2}, p = \frac{1}{2}, d = -1, b = 5, g = -\frac{1}{4}, c = \frac{1}{4}, n = 5\end{aligned} \tag{28}$$

All the second-order nonlinear ODEs and the systems of nonlinear ODEs above have the expo-elliptic function $rs(t)$ as solutions. And a lot more is possible to make (**Figure 6**).

2.4.2. The Functions *ks* and *kd*

And again we start with a non-elementary integral.

Define a function u (IF 171):

$$u = u(\varphi) = \int_0^{\varphi} \frac{e^{a\theta}}{n + h\cos\left(se^{2a\theta} - sb^2\right) + k\left(e^{2a\theta} - b^2\right)\sin\left(pe^{2a\theta} - pb^2\right)} d\theta \quad (29)$$

a, b, h, k, p, n, s are parameters defined on *R*.

The denominator can become 0. To avoid that we can make some restrictions to the parameters: $a < 0$, $n > 2$, $-1 \le h, k \le 1$, $-\infty < b, p, s < \infty$

$$\frac{du}{d\varphi} = \frac{e^{a\varphi}}{n + h\cos\left(se^{2a\varphi} - sb^2\right) + k\left(e^{2a\varphi} - b^2\right)\sin\left(pe^{2a\varphi} - pb^2\right)} \quad (30)$$

Inverting $\dfrac{du}{d\varphi}$:

$$\frac{d\varphi}{du} = e^{-a\varphi}\left(n + h\cos\left(se^{2a\varphi} - sb^2\right) + k\left(e^{2a\varphi} - b^2\right)\sin\left(pe^{2a\varphi} - pb^2\right)\right) \quad (31)$$

Define a set of 2 functions: *ks* and *kd*, so that

$$ks(u) = e^{a\varphi}, \quad (32)$$

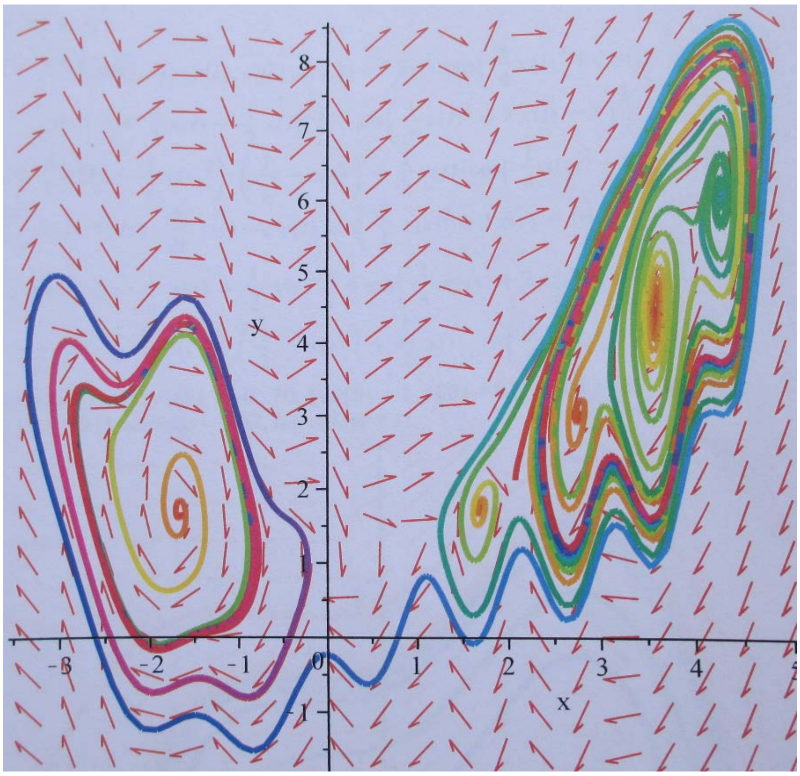

Figure 6. Two LC on the line $y = \frac{1}{4}x^2 + 1$.

$$kd(u) = e^{-a\varphi}\left(n + h\cos\left(se^{2a\varphi} - sb^2\right) + k\left(e^{2a\varphi} - b^2\right)\sin\left(p\,e^{2a\varphi} - pb^2\right)\right)$$
$$= \frac{1}{ks(u)}\left(n + h\cos\left(s\,ks^2(u) - sb^2\right) + k\left(ks^2(u) - b^2\right)\sin\left(p\,ks^2(u) - pb^2\right)\right) \tag{33}$$

The connection between the functions *ks* and *kd* is:

$$kd(u)ks(u) - h\cos\left(s\,ks^2(u) - sb^2\right)$$
$$-k\left(ks^2(u) - b^2\right)\sin\left(p\,ks^2(u) - pb^2\right) = n \tag{34}$$

We see that the functions $ks(u)$ and $kd(u)$ exist for all values of the parameters a, b, h, k, n, p, s even though the integral IF 171 is not defined for values that make the denominator = 0. The functions $ks(u)$ and $kd(u)$ are continuous and differentiable on the whole *R*.

The derivatives to these functions are:

$$\frac{\mathrm{d}}{\mathrm{d}u}ks(u) = ae^{a\varphi}\frac{\mathrm{d}\varphi}{\mathrm{d}u} = a\,ks(u)kd(u) \tag{35}$$

$$\frac{\mathrm{d}}{\mathrm{d}u}kd(u) = -\frac{a}{ks^2(u)}\Big(n + h\cos\left(s\,ks^2(u) - sb^2\right) + k\left(ks^2(u) - b^2\right)$$
$$\times\sin\left(p\,ks^2(u) - pb^2\right)\Big)^2 + 2a\Big(n + h\cos\left(s\,ks^2(u) - sb^2\right)$$
$$+k\left(ks^2(u) - b^2\right)\sin\left(p\,ks^2(u) - pb^2\right)\Big)\Big[-hs\sin\left(s\,ks^2 - sb^2\right)$$
$$+k\sin\left(p\,ks^2(u) - pb^2\right) + pk\left(ks^2(u) - b^2\right)\cos\left(p\,ks^2(u) - pb^2\right)\Big] \tag{36}$$

Define a solution $x(t) = ks(t)$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = a\left(n + h\cos\left(sx^2 - sb^2\right) + k\left(x^2 - b^2\right)\sin\left(px^2 - pb^2\right)\right) \tag{37}$$

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -2ahsx\frac{\mathrm{d}x}{\mathrm{d}t}\sin\left(sx^2 - sb^2\right) + 2a^2kx\Big(n + h\cos\left(sx^2 - sb^2\right)$$
$$+k\left(x^2 - b^2\right)\sin\left(px^2 - pb^2\right)\Big)\Big[\sin\left(px^2 - pb^2\right)$$
$$+p\left(x^2 - b^2\right)\cos\left(px^2 - pb^2\right)\Big]$$

$(0,0)$ and $(\pm b,0)$ are equilibrium points. Define 2 functions *f* and *g*, so that $f(x,y) = y$ and

$$g(x,y) = -2ahsxy\sin\left(sx^2 - sb^2\right) + 2a^2kx\Big(n + h\cos\left(sx^2 - sb^2\right)$$
$$+k\left(x^2 - b^2\right)\sin\left(px^2 - pb^2\right)\Big)\Big[\sin\left(px^2 - pb^2\right) \tag{38}$$
$$+p\left(x^2 - b^2\right)\cos\left(px^2 - pb^2\right)\Big]$$

Jacobian matrix:

$$J = \begin{bmatrix} \frac{\partial f}{\partial x}(x_0, y_0) & \frac{\partial f}{\partial y}(x_0, y_0) \\ \frac{\partial g}{\partial x}(x_0, y_0) & \frac{\partial g}{\partial y}(x_0, y_0) \end{bmatrix} \tag{39}$$
$$\det\left(J(\pm b, 0) - \lambda I\right) = \det\begin{bmatrix} 0-\lambda & 1 \\ 8a^2b^2kp(n+h) & 0-\lambda \end{bmatrix}$$

$$(-\lambda)^2 - 8a^2b^2kp(n+h) = 0$$

$$\lambda = \pm 2ab\sqrt{2kp(n+h)} \tag{40}$$

Complex eigenvalues with zero real part when $kp(n+h) < 0$.

$(\pm b, 0)$ is center when $kp(n+h) < 0$ and saddle when $kp(n+h) > 0$.

In Figure 7 is (−3, 0) a stable center, and has an area around it with stable closed curves. We can also see a limit cycle to the left. The parameter-values are:

$$a = -1, k = -1, h = 1, n = 5, s = 1, p = \frac{1}{9}, b = 3 \tag{41}$$

$$\det\left(J(0,0) - \lambda I\right)$$
$$= \det\begin{bmatrix} 0-\lambda & 1 \\ -2a^2k\left(\sin\left(pb^2\right) + pb^2\cos\left(pb^2\right)\right)\left(n + h\cos\left(sb^2\right) + kb^2\sin\left(pb^2\right)\right) & 0-\lambda \end{bmatrix} \tag{42}$$

$$\lambda = \pm a\sqrt{-2k\left(\sin\left(pb^2\right) + pb^2\cos\left(pb^2\right)\right)\left(n + h\cos\left(sb^2\right) + kb^2\sin\left(pb^2\right)\right)} \tag{43}$$

Complex eigenvalues with zero real part when $-2k\left(\sin\left(pb^2\right) + pb^2\cos\left(pb^2\right)\right)\left(n + \cos\left(sb^2\right) + kb^2\sin\left(pb^2\right)\right) < 0$. Then is $(0,0)$ center.

It is possible to make phase diagrams where three of the equilibrium points are stable centers, while the others are saddles, spiral sinks and spiral sources

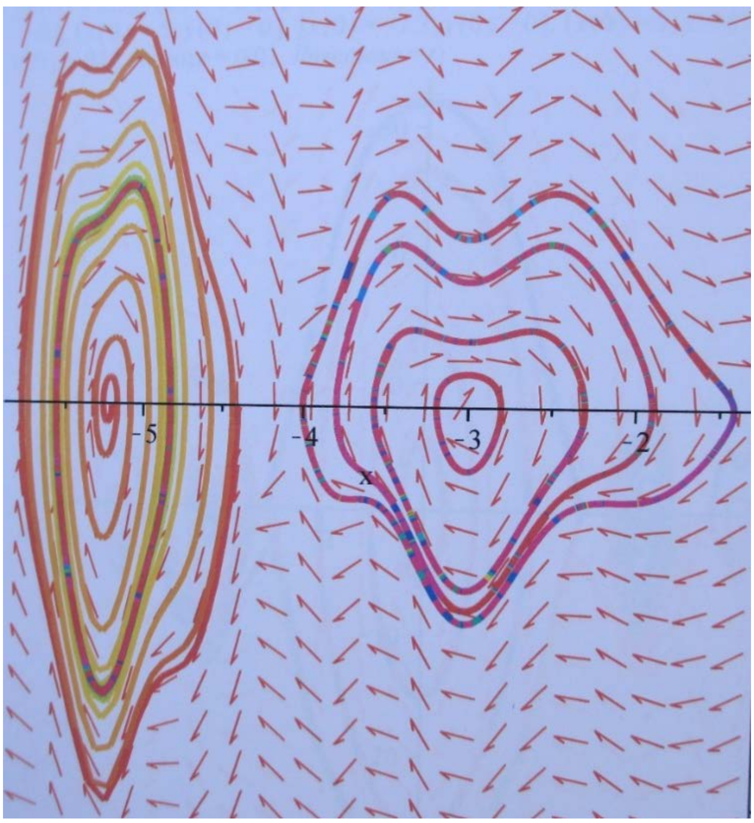

Figure 7. One stable center and one stable LC.

and some limit cycles, as we can see in Figure 8.

In Figure 8 is the parameter $b = 6$, the other values are the same as in Figure 7.

We can see a big area with stable closed curves with no attraction behavior surrounding $(0,0)$, and 4 limit cycles.

In Figures 7-9 we see these equilibrium points: center, saddle point, spiral sink and spiral source. And we also see some limit cycles (LC).

The behavior of the solution curves in the phase diagrams reflects the qualities of the functions *ks* and *kd*.

2.4.3. The functions *fs* and *fd*.

Define an integral function u (IF 153):

$$u = u(\varphi) = \int_0^{\varphi} \frac{e^{a\theta}}{n + h\cos\left(se^{a\theta}\right) + f\sin\left(ve^{a\theta}\right) + ke^{a\theta}\sin\left(pe^{a\theta}\right)} d\theta \quad (44)$$

a, h, f, k, n, s, v, p are parameters defined on *R*.

The denominator can become 0. In order to avoid that we can make some restrictions to the parameters: $a < 0$, $n > 3$, $-1 \le h, f, k \le 1$, $-\infty < s, v, p < \infty$

$$\frac{du}{d\varphi} = \frac{e^{a\varphi}}{n + h\cos\left(se^{a\varphi}\right) + f\sin\left(ve^{a\varphi}\right) + ke^{a\varphi}\sin\left(pe^{a\varphi}\right)} \quad (45)$$

Inverting:

$$\frac{d\varphi}{du} = e^{-a\varphi}\left(n + h\cos\left(se^{a\varphi}\right) + f\sin\left(ve^{a\varphi}\right) + ke^{a\varphi}\sin\left(pe^{a\varphi}\right)\right) \quad (46)$$

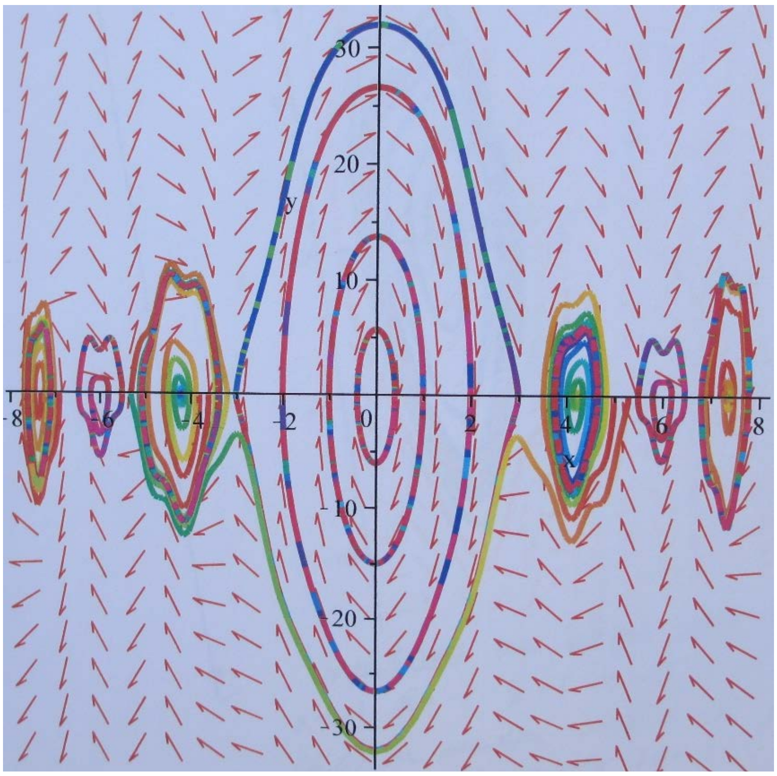

Figure 8. Three stable centers and four stable LC.

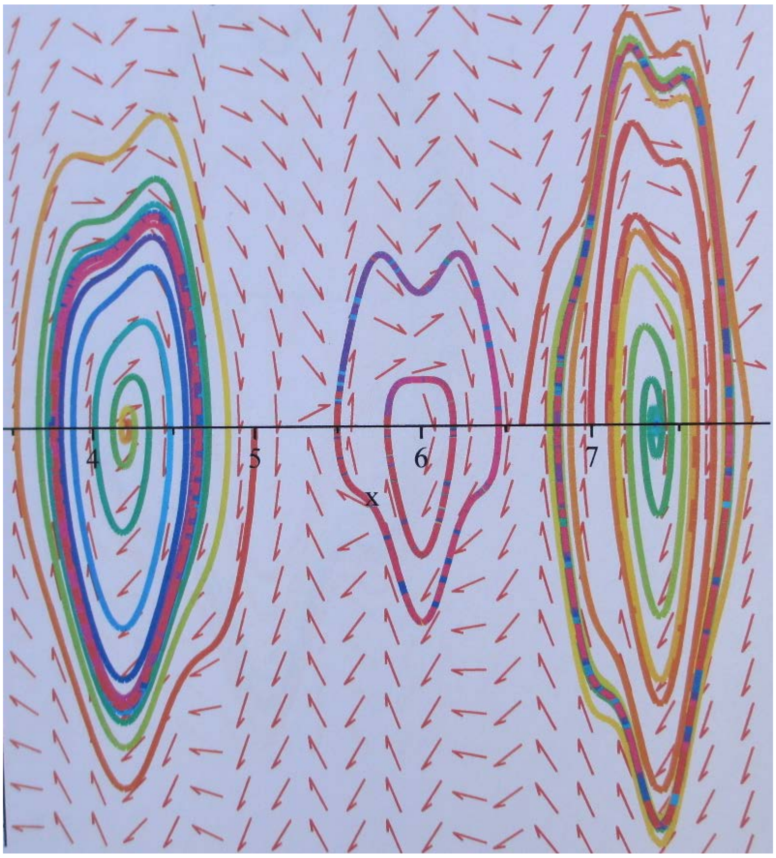

Figure 9. A close-up of **Figure 8**.

Define a set of 2 functions *fs* and *fd*, so that

$$fs(u) = \mathrm{e}^{a\varphi} \tag{47}$$

$$\begin{aligned} fd(u) &= \mathrm{e}^{-a\varphi}\left(n + h\cos\left(s\mathrm{e}^{a\varphi}\right) + f\sin\left(v\mathrm{e}^{a\varphi}\right) + k\mathrm{e}^{a\varphi}\sin\left(p\mathrm{e}^{a\varphi}\right)\right) \\ &= \frac{1}{fs(u)}\left(n + h\cos\left(s\ fs(u)\right) + f\sin\left(v\ fs(u)\right) + k\ fs(u)\sin\left(p\ fs(u)\right)\right) \end{aligned} \tag{48}$$

The connection between the functions *fs* and *fd* is:

$$fd(u)\ fs(u) - h\cos\left(s\ fs(u)\right) - f\sin\left(v\ fs(u)\right) - k\ fs(u)\sin\left(p\ fs(u)\right) = n \tag{49}$$

Notice that the function $fd(u)$ exist for all values of the parameters a, h, f, k, s, v, n, p, even thought the integral IF153 don't exist for values that make the denominator = 0. The functions $fs(u)$ and $fd(u)$ are continues and differentiable on the whole *R*.

The derivatives to these functions are:

$$\frac{\mathrm{d}}{\mathrm{d}u} fs(u) = a\ fs(u)\ fd(u) \tag{50}$$

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}u} fd(u) = &-\frac{a}{fs^2(u)}\left(n + h\cos\left(s\ fs(u)\right) + f\sin\left(v\ fs(u)\right)\right. \\ &\left. + k\ fs(u)\sin\left(p\ fs(u)\right)\right)^2 + \frac{a}{fs(u)}\left(n + h\cos\left(s\ fs(u)\right)\right) \end{aligned}$$

$$+f\sin\left(v\,fs(u)\right)+k\,fs(u)\sin\left(p\,fs(u)\right)\Big)\Big[-hs\sin\left(s\,fs(u)\right)$$
$$+fv\cos\left(v\,fs(u)\right)+k\sin\left(p\,fs(u)\right)+kp\,fs(u)\cos\left(p\,fs(u)\right)\Big] \quad (51)$$

Define a solution $x(t)=fs(t)$

$$\frac{\mathrm{d}x}{\mathrm{d}t}=a\left(n+h\cos(sx)+f\sin(vx)+kx\sin(px)\right) \quad (52)$$

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2}=-ahs\frac{\mathrm{d}x}{\mathrm{d}t}\sin(sx)+afv\frac{\mathrm{d}x}{\mathrm{d}t}\cos(vx)+ak\frac{\mathrm{d}x}{\mathrm{d}t}\sin(px)+akpx\frac{\mathrm{d}x}{\mathrm{d}t}\cos(px) \quad (53)$$

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2}=afv\frac{\mathrm{d}x}{\mathrm{d}t}\cos(vx)-a^2\left(n+h\cos(sx)+f\sin(vx)+kx\sin(px)\right)\times\left[hs\sin(sx)-k\sin(px)-kpx\cos(px)\right] \quad (3291)$$

Equation (3291) is one of at least 5 second-order nonlinear ODEs that is possible to make from (53) that have the function $fs(t)$ as solution.

In **Figure 10** the parameter-values are:

$$a=-1, v=4, s=1, f=1, k=-1, p=1, n=4, h=-1 \quad (54)$$

The 5 equilibrium points in **Figure 10** are 3 spiral sink and 2 saddle points. There are also 2 unstable limit cycles (LC).

In **Figure 11** are the parameter-values:

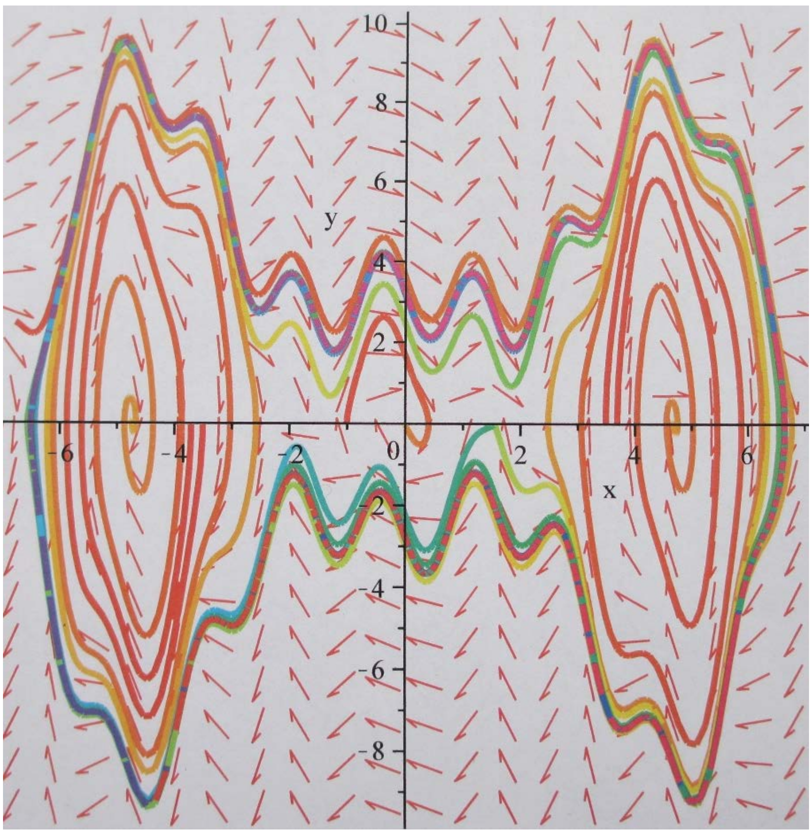

Figure 10. One large LC with five equilibrium points.

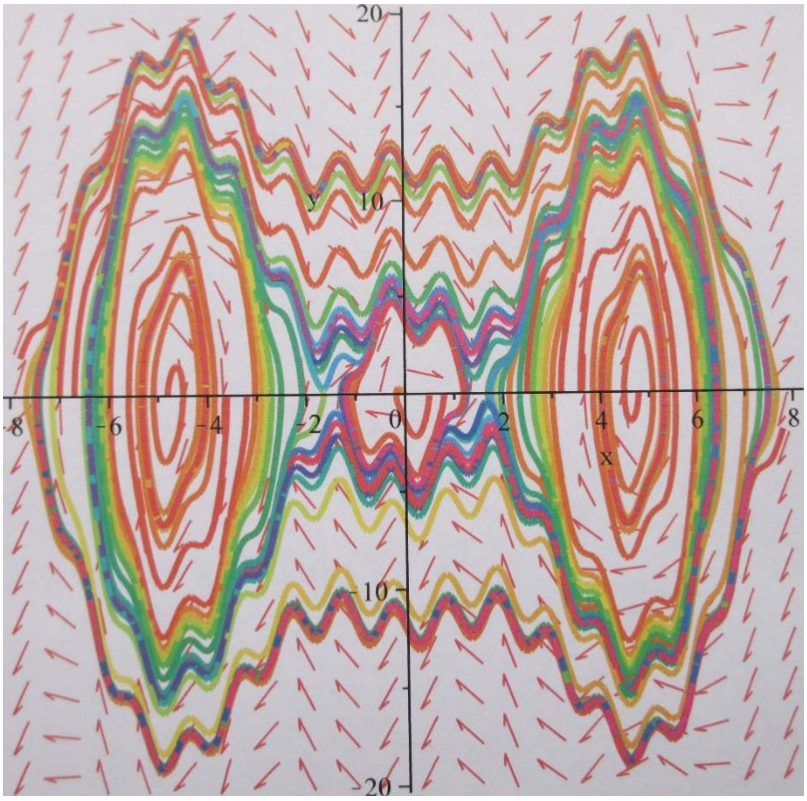

Figure 11. Two large LC surrounding three small LC.

$$a = -1, v = 6, s = 1, f = 1, k = -1, p = 1, n = 15, h = -1 \tag{55}$$

In **Figure 11** we see the same 5 equilibrium points as in **Figure 10**, but they have changed to 2 spiral source, 1 spiral sink and 2 saddle points. 2 large LC are surrounding 3 small LC. One large unstable LC between the 2 large stable LC. There are also 3 unstable LC.

2.4.4. The Functions *ts* and *td*

This is a set of functions that are giving solutions to some dynamical systems with a funny behavior that reminds a bit of the Lorenz' equations.

Define an integral function u (IF 143):

$$u = u(\varphi) = \int_0^{\varphi} \frac{e^{a\theta}}{n + f e^{a\theta} + k e^{a\theta} \sin\left(p e^{a\theta}\right)} d\theta \tag{56}$$

a, f, k, n, p are parameters defined on R.

In order to avoid the denominator to become zero we can make some restrictions to the parameters: $n > 2$, $a < 0$, $-1 \le f, k, \le 1$, $-\infty < p < \infty$

$$\frac{du}{d\varphi} = \frac{e^{a\varphi}}{n + f e^{a\varphi} + k e^{a\varphi} \sin\left(p e^{a\varphi}\right)} \tag{57}$$

Inverting:

$$\frac{d\varphi}{du} = e^{-a\varphi}\left(n + f e^{a\varphi} + k e^{a\varphi} \sin\left(p e^{a\varphi}\right)\right) \tag{58}$$

Define a set of 2 functions *ts* and *td*, so that

$$ts(u) = \mathrm{e}^{a\varphi}$$

$$\begin{aligned} td(u) &= \mathrm{e}^{-a\varphi}\left(n + f\,\mathrm{e}^{a\varphi} + k\mathrm{e}^{a\varphi}\sin\left(p\mathrm{e}^{a\varphi}\right)\right) \\ &= \frac{1}{ts(u)}\left(n + f\,ts(u) + k\,ts(u)\sin\left(p\,ts(u)\right)\right) \end{aligned} \quad (59)$$

The connection between the functions *ts* and *td:*

$$td(u)ts(u) - f\,ts(u) - k\,ts(u)\sin\left(p\,ts(u)\right) = n \quad (60)$$

We see that the function $td(u)$ exists for all values of the parameters a, f, k, n, p, even though the integral IF 143 doesn't exist for values that make the denominator = 0. The functions $ts(u)$ and $td(u)$ are continuous and differentiable on the whole *R*.

The derivatives to the functions $ts(u)$ and $td(u)$:

$$\frac{\mathrm{d}}{\mathrm{d}u}ts(u) = a\,ts(u)td(u) \quad (61)$$

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}u}td(u) &= -\frac{a}{ts^2(u)}\left(n + f\,ts(u) + k\,ts(u)\sin\left(p\,ts(u)\right)\right)^2 \\ &+ \frac{a}{ts(u)}\left(n + f\,ts(u) + k\,ts(u)\sin\left(p\,ts(u)\right)\right) \\ &\times\left[f + k\sin\left(p\,ts(u)\right) + kp\,ts(u)\cos\left(p\,ts(u)\right)\right] \end{aligned} \quad (62)$$

Define a solution $x(t) = ts(t)\frac{\mathrm{d}x}{\mathrm{d}t} = a\left(n + fx + kx\sin(px)\right)$ (63)

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = a\frac{\mathrm{d}x}{\mathrm{d}t}\left(f + k\sin(px)\right) + a^2kpx\cos(px)\left(n + fx + kx\sin(px)\right) \quad (3366)$$

We can make a 3D system of this equation by defining a bit of it as the solution $z(t) = ax\cos(px)$.

Then we become the system (3367):

$$\frac{\mathrm{d}x}{\mathrm{d}t} = y$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = ay\left(f + k\sin(px)\right) + akpz\left(n + fx + kx\sin(px)\right)$$

$$\frac{\mathrm{d}z}{\mathrm{d}t} = -apxy\sin(px) + a^2\cos(px)\left(n + fx + kx\sin(px)\right)$$

This system has the solutions:

$$\begin{aligned} x(t) &= ts(t) \\ y(t) &= a\left(n + f\,ts(t) + k\,ts(t)\sin\left(p\,ts(t)\right)\right) \\ z(t) &= a\,ts(t)\cos\left(p\,ts(t)\right) \end{aligned} \quad (64)$$

In **Figures 12-16** are the parameter-values:

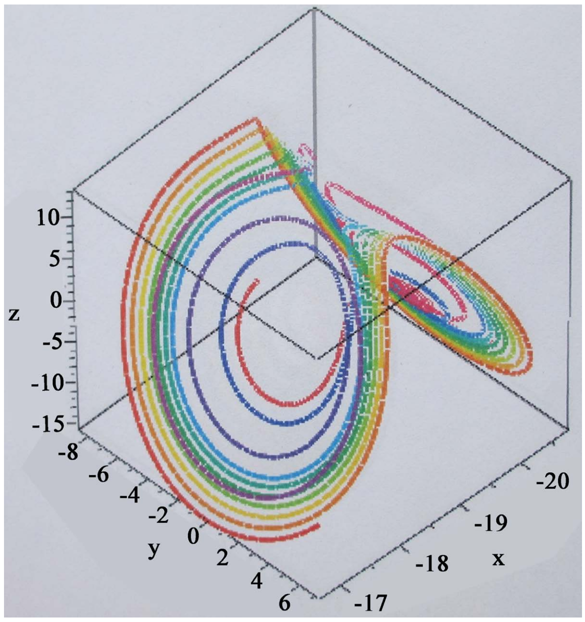

Figure 12. Initial point (−17, 6, −10), $t = 0...20$.

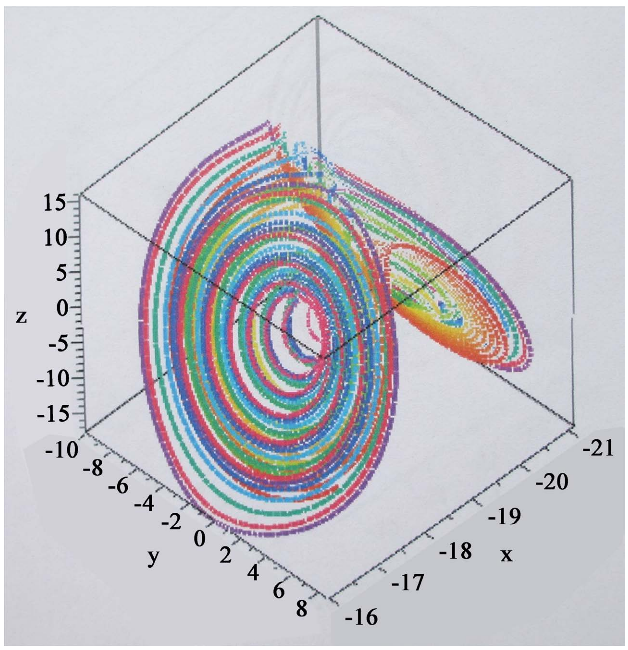

Figure 13. Initial point (−17, 6, −10), $t = 0...80$.

$$a = -1, n = 3, k = -\frac{1}{2}, p = 1, f = \frac{1}{7} \tag{65}$$

This system has a variation in behavior that depends on the initial values. For some values, the curves approach a spiral sink, for other values the solution curve will go to infinity, and for other initial values will the solution curve have a chaotic behavior. In this case the solution curve is behaving like a restless man finding peace nowhere.

The solution curves are very sensitive to the initial conditions, to the initial

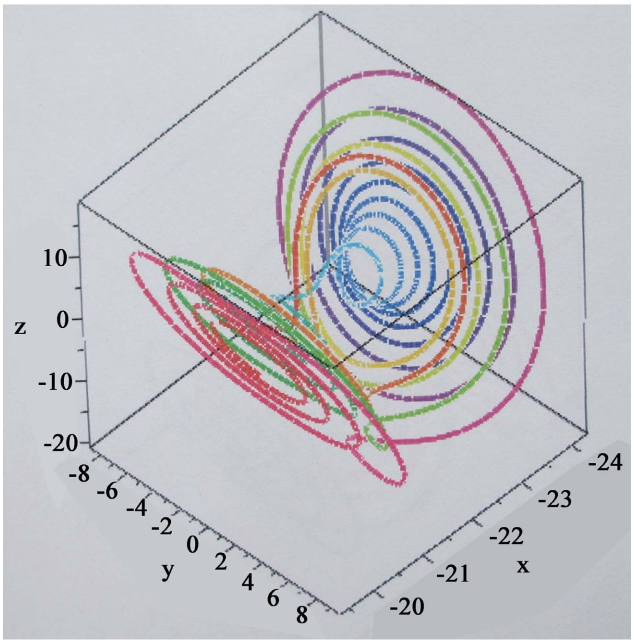

Figure 14. Initial point (−21, −6, 0), $t = 0...18$.

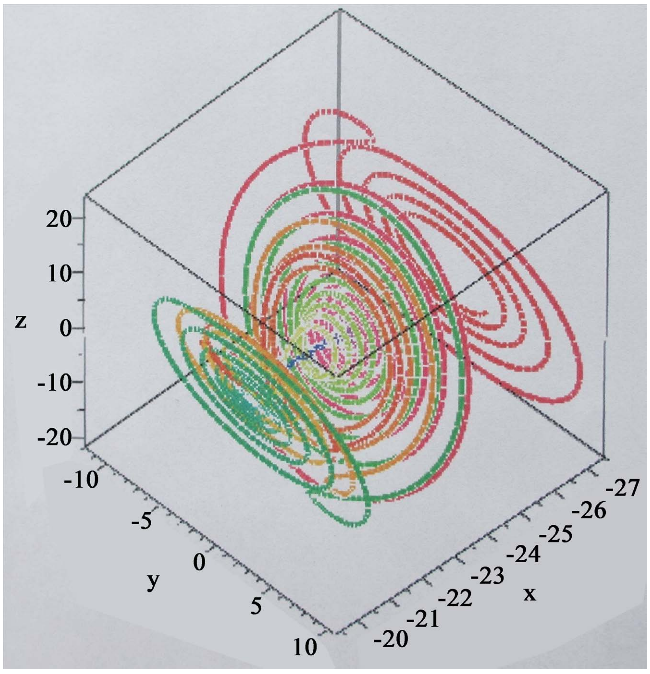

Figure 15. Initial point (−21, −6, 0), $t = 0...48$.

values x_0, y_0 and z_0. Sometimes a change in initial value of 0.002 is enough to make a big change in the long-term behavior, and at other times a change in 0.1 will bring a change in the long-term behavior. The paths exhibit sensitive dependence to initial conditions [1].

Figure 16 shows both solution curves with the initial points (−17, 6, −10) and (−21, −6, 0). Both solution curves sometimes circulate along the same spiral in the middle of the picture. This is so far my math-program can work.

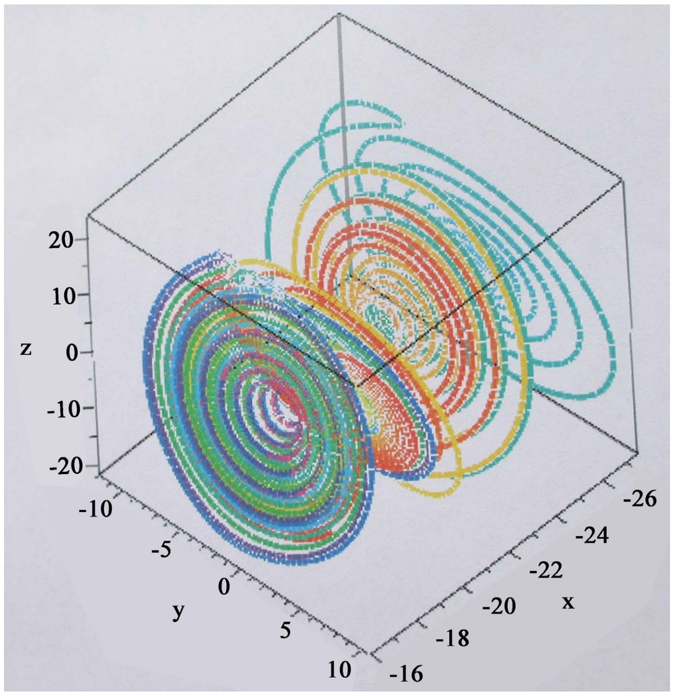

Figure 16. Two initial points (−17, 6, −10), (−21, −6, 0), $t = 0...100$.

3. Conclusions

My purpose with all these pictures is to show you some of the variations in behavior and qualities of the expo-elliptic functions. I don't know any other functions that have these qualities. If the behavior of the solution curves in the phase diagrams reflects the qualities of the solution functions, then we can see some of these qualities in the pictures in this paper.

What is new in this paper are the four sets of non-elementary functions *rs* and *rd*, *ks* and *kd*, *fs* and *fd*, *ts* and *td*. They are useful as both solutions to second-order nonlinear ODEs and systems of nonlinear ODEs. Some of them are exhibiting limit cycles with a few or many equilibrium points, or have limit cycles inside each other with different sizes and shapes. It is amazing to see what properties some of the expo-elliptic functions have.

It is possible to make a lot of non-elementary functions using the Abel's methods described by Armitage and Eberlein, by how they define the Jacobi elliptic functions. In the same way as Jacobi's functions *sn*, *cn*, *dn* and *am* give solutions to a few ODEs, the expo-elliptic functions described in this paper and a lot more, give solutions to many other different kinds of ODEs. I don't see any limit for this subject. The only limit is our imagination.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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