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The Quality Properties of Integral Type Problems for Wave Equations and Applications

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Abstract

In this paper, the integral problem for linear and nonlinear wave equations is studied. The equation involves abstract operator A in Hilbert space H. Here, assuming enough smoothness on the initial data and the operators, the existence, uniqueness, regularity properties of solutions are established. By choosing the space H and A, the regularity properties of solutions of a wide class of wave equations in the field of physics are obtained.

Keywords

Abstract Differential Equations, Boussinesq Equations, Wave Equations, Regularity Property of Solutions, Fourier Multipliers

1. Introduction, Definitions and Background

The aim here, is to study the existence, uniqueness, regularity properties of solutions of the integral problem (IP) for abstract wave equation (WE)

$$u_{tt} - a\Delta u + Au = f(u), (x,t) \in \mathbb{R}_T^n = \mathbb{R}^n \times (0,T), \quad (1.1)$$

$$u(x,0) = \varphi(x) + \int_0^T \eta(\sigma) u(x,\sigma) \mathrm{d}\sigma, \quad (1.2)$$

$$u_t(x,0) = \psi(x) + \int_0^T \beta(\sigma) u_t(x,\sigma) \mathrm{d}\sigma,$$

where A is a linear and $f(u)$ is a nonlinear operator in a Hilbert space H, $\eta(\sigma)$, $\beta(\sigma)$ are measurable functions on $(0,T)$, a is a complex number, $T \in (0,\infty]$. Here, Δ denotes the Laplace operator with respect to $x \in \mathbb{R}^n$, $\varphi(x)$ and $\psi(x)$ are the given H-valued initial functions.

Wave type equations occur in a wide variety of physical systems, such as in the propagation of longitudinal deformation waves in an elastic rod, hydro-dynamical process in plasma, in materials science which describes spinodal decomposition and in the absence of mechanical stresses (see [1] [2] [3] [4]). The nonlocal theory of elasticity was introduced (see [5] [6] [7] [8] [9] and the references cited therein). The global existence of the Cauchy problem for Boussinesq type equations has been studied by many authors (see [10] [11] [12]). Note that, the existence and uniqueness of solutions and regularity properties of a wide class of wave equations were considered e.g. in [13]-[22]. The abstract evolution equations were studied e.g. in [23]-[32]. Unlike in these studies, in this paper the abstract wave equation (1.1) is considered. The L^p well-posedness of the Cauchy problem (1.1)-(1.2) depends crucially on the presence of the linear operator A and nonlinear operator $f(u)$. Then the question that naturally arises is which of the possible forms of the operator functions and kernel functions are relevant for the global well-posedness of the Cauchy problem (1.1)-(1.2). We find the class of operator A such that provides the existence, uniqueness, regularity properties and blow up of solutions (1.1)-(1.2) in terms of fractional powers of operator A. By choosing the space H, operator A in (1.1)-(1.2), we obtain a wide class of wave equations which occur in application. Let we put $H = L^2(0,1)$ and consider the operator $A = A_1$ defined by

$$D(A_1) = W^{2,2}(0,1,L_k), A_1 u = b_1 u^{(2)} + b_0 u, \tag{1.3}$$

$$L_k u = \left[\alpha_k u^{(m_k)}(0) + \beta_k u^{(m_k)}(1)\right] = 0, k = 1,2,$$

where $b_1(.), b_0(.)$ are VMO functions (see definitions below), $m_k \in \{0,1\}$, α_k, β_k are complex numbers.

Consider the following mixed problem for WE with discontinuous coefficients

$$\frac{\partial^2 u}{\partial t^2} - a\Delta_x u + b_1 \frac{\partial^2 u}{\partial y^2} + b_0 u = f(u), t \in (0,T), x \in \mathbb{R}^n, \tag{1.4}$$

$$u(x,y,0) = \varphi(x,y) + \int_0^T \eta(\sigma) u(x,y,\sigma) \mathrm{d}\sigma,$$

$$u_t(x,y,0) = \psi(x,y) + \int_0^T \beta(\sigma) u_t(x,y,\sigma) \mathrm{d}\sigma,$$

$$\alpha_k u^{(m_k)}(x,0,t) + \beta_k u^{(m_k)}(x,1,t) = 0, k = 1,2,$$

where a is a complex number. From our results we obtain the existence, uniqueness, regularity properties and blow up of solutions of (1.4) in $L^{\mathbf{p}}(\mathbb{R}^n \times (0,1))$ with terms of fractional powers of the operator A_1, where $\mathbf{p} = (2, p, p)$ and $L^{\mathbf{p}}(\mathbb{R}^n \times (0,1))$ denotes the space of all $\mathbf{p}$-summable complex-valued measurable functions f defined on Ω with the mixed norm

$$\|f\|_{L^{\mathbf{p}}(\Omega)} = \left(\int_{\mathbb{R}^n} \int_0^T \left(\int_0^1 |f(x,y,t)|^{p_1} \mathrm{d}y \right)^{\frac{2}{p_1}} \mathrm{d}x\mathrm{d}t \right)^{\frac{1}{2}} < \infty.$$

Let E be a Banach space. $L^p(\Omega;E)$ denotes the space of strongly measurable E-valued functions that are defined on the measurable subset $\Omega \subset \mathbb{R}^n$ with the norm

$$\|f\|_p = \|f\|_{L^p(\Omega;E)} = \left(\int_\Omega \|f(x)\|_E^p \, \mathrm{d}x\right)^{\frac{1}{p}}, 1 \le p < \infty,$$

$$\|f\|_{L^\infty(\Omega;E)} = \operatorname*{ess\,sup}_{x\in\Omega} \|f(x)\|_E .$$

Let E_1 and E_2 be two Banach spaces. $(E_1, E_2)_{\theta,p}$ for $\theta \in (0,1)$, $p \in [1,\infty]$ denotes the real interpolation spaces defined by K-method ([33], Section 1.3.2). Let E_1 and E_2 be two Banach spaces. $B(E_1, E_2)$ will denote the space of all bounded linear operators from E_1 to E_2. For $E_1 = E_2 = E$ it will be denoted by $B(E)$.

Here,

$$S_\phi = \{\lambda \in \mathbb{C}, \lambda \ne 0, |\arg\lambda| \le \phi, 0 \le \phi < \pi\}.$$

A closed linear operator A is said to be sectorial in a Banach space E with bound $M > 0$ if $D(A)$ and $R(A)$ are dense on E, $N(A) = \{0\}$ and

$$\left\|(A + \lambda I)^{-1}\right\|_{B(E)} \le M |\lambda|^{-1}$$

for any $\lambda \in S_\phi$, $0 \le \phi < \pi$, where I is the identity operator in E, $D(A)$ and $R(A)$ denote domain and range of the operator A, respectively. It is known that (see e.g. [33], Section 1.15.1) there exist the fractional powers A^θ of a sectorial operator A. Let $E(A^\theta)$ denote the space $D(A^\theta)$ with the graphical norm

$$\|u\|_{E(A^\theta)} = \left(\|u\|^p + \left\|A^\theta u\right\|^p\right)^{\frac{1}{p}}, 1 \le p < \infty, 0 < \theta < \infty.$$

A sectorial operator $A(\xi)$ is said to be uniformly sectorial in E for $\xi \in \mathbb{R}^n$, if $D(A(\xi))$ is independent of ξ and the following uniform estimate

$$\left\|(A + \lambda I)^{-1}\right\|_{B(E)} \le M |\lambda|^{-1}$$

holds for any $\lambda \in S_\phi$.

A function $\Psi \in L^\infty(\mathbb{R}^n)$ is called a Fourier multiplier from $L^p(\mathbb{R}^n;E)$ to $L^q(\mathbb{R}^n;E)$ if the map $P: u \to \mathbb{F}^{-1}\Psi(\xi)\mathbb{F}u$ is well defined for $u \in S(\mathbb{R}^n;E)$ and extends to a bounded linear operator.

Definition 1.1. Let U be an open set in a Banach space X, let Y be a Banach space. A function $f: U \to Y$ is called (Frechet) differentiable at $x \in U$ if there is a bounded linear operator $Df(x): X \to Y$, called the derivative of f at a, such that

$$\lim_{h\to 0} \frac{\|f(x+h) - f(x) - Df(x)h\|_Y}{\|h\|_X} = 0$$

If f is differentiable at each $x \in U$, then f is called differentiable. This function may also have a derivative, the second order derivative of f, which, by the definition of derivative, will be a map

$$D^2 f : U \to L\left(X, L\left(X, Y\right)\right).$$

Let E be a Banach space. $S = S\left(\mathbb{R}^n; E\right)$ denotes E-valued Schwartz class, *i.e.* the space of all E-valued rapidly decreasing smooth functions on $\mathbb{R}^n$ equipped with its usual topology generated by seminorms. $S\left(\mathbb{R}^n; \mathbb{C}\right)$ denoted by S. Let $S'\left(\mathbb{R}^n; E\right)$ denote the space of all continuous linear functions from S into E, equipped with the bounded convergence topology. Recall $S\left(\mathbb{R}^n; E\right)$ is norm dense in $L^p\left(\mathbb{R}^n; E\right)$ when $1 \le p < \infty$. Let m be a positive integer. $W^{m,p}\left(\Omega; E\right)$ denotes an E-valued Sobolev space of all functions $u \in L^p\left(\Omega; E\right)$ that have the generalized derivatives $\frac{\partial^m u}{\partial x_k^m} \in L^p\left(\Omega; E\right)$ with the norm

$$\|u\|_{W^{m,p}(\Omega;E)} = \|u\|_{L^p(\Omega;E)} + \sum_{k=1}^{n} \left\| \frac{\partial^m u}{\partial x_k^m} \right\|_{L^p(\Omega;E)} < \infty.$$

Let $W^{s,p}\left(\mathbb{R}^n; E\right)$ denotes the fractional Sobolev space of order $s \in \mathbb{R}$, that is defined as:

$$W^{s,p}\left(E\right) = W^{s,p}\left(\mathbb{R}^n; E\right)$$
$$= \left\{ u \in S'\left(\mathbb{R}^n; E\right), \|u\|_{W^{s,p}(E)} = \left\| \mathbb{F}^{-1}\left(I + |\xi|^2\right)^{\frac{s}{2}} \hat{u} \right\|_{L^p\left(\mathbb{R}^n; E\right)} < \infty \right\}.$$

It is clear that $W^{0,p}\left(\mathbb{R}^n; E\right) = L^p\left(\mathbb{R}^n; E\right)$. Let E_0 and E be two Banach spaces and E_0 is continuously and densely embedded into E. Here, $W^{s,p}\left(\mathbb{R}^n; E_0, E\right)$ denote the Sobolev-Lions type space *i.e.*,

$$W^{s,p}\left(\mathbb{R}^n; E_0, E\right) = \left\{ u \in W^{s,p}\left(\mathbb{R}^n; E\right) \cap L^p\left(\mathbb{R}^n; E_0\right), \right.$$
$$\left. \|u\|_{W^{s,p}\left(\mathbb{R}^n; E_0, E\right)} = \|u\|_{L^p\left(\mathbb{R}^n; E_0\right)} + \|u\|_{W^{s,p}\left(\mathbb{R}^n; E\right)} < \infty \right\}.$$

In a similar way, we define the following Sobolev-Lions type space:

$$W^{2,s,p}\left(\mathbb{R}_T^n; E_0, E\right) = \left\{ u \in L^p\left(\mathbb{R}_T^n; E_0\right), \partial_t^2 u \in L^p\left(\mathbb{R}_T^n; E\right), \right.$$
$$\mathbb{F}_x^{-1}\left(I + |\xi|^2\right)^{\frac{s}{2}} \hat{u} \in L^p\left(\mathbb{R}_T^n; E\right), \|u\|_{W^{2,s,p}\left(\mathbb{R}_T^n; E_0, E\right)}$$
$$\left. = \|u\|_{L^p\left(\mathbb{R}_T^n; E_0\right)} + \left\|\partial_t^2 u\right\|_{L^p\left(\mathbb{R}_T^n; E\right)} + \left\| \mathbb{F}_x^{-1}\left(I + |\xi|^2\right)^{\frac{s}{2}} \hat{u} \right\|_{L^p\left(\mathbb{R}_T^n; E\right)} < \infty \right\}.$$

Let $L_q^*\left(E\right)$ denote the space of all E-valued function space such that

$$\|u\|_{L_q^*(E)} = \left(\int_0^\infty \|u(t)\|_E^q \frac{\mathrm{d}t}{t} \right)^{\frac{1}{q}} < \infty, 1 \le q < \infty, \|u\|_{L_\infty^*(E)} = \sup_{0<t<\infty} \|u(t)\|_E.$$

Let $s>0$. Fourier-analytic representation of E-valued Besov space on $\mathbb{R}^n$ is defined as:

$$B_{p,q}^{s}\left(\mathbb{R}^n;E\right)=\Big\{u\in S'\left(\mathbb{R}^n;E\right),$$

$$\|u\|_{B_{p,q}^{s}\left(\mathbb{R}^n;E\right)}=\left\|\mathbb{F}^{-1}\sum_{k=1}^{n}t^{\varkappa-s}\left(1+|\xi|^2\right)^{\frac{\varkappa}{2}}\mathrm{e}^{-t|\xi|^2}\mathbb{F}u\right\|_{L_q^*\left(L^p\left(\mathbb{R}^n;E\right)\right)},$$

$$p\in(1,\infty),q\in[1,\infty],\varkappa>s\Big\}.$$

It should be noted that, the norm of Besov space does not depend on $\varkappa$ (see e.g. [33], Section 2.3 for $E=\mathbb{C}$).

Let A be a sectorial operator in H. Here,

$$X_p=L^p\left(\mathbb{R}^n;H\right),X_p\left(A^\gamma\right)=L^p\left(\mathbb{R}^n;H\left(A^\gamma\right)\right),1\le p,q\le\infty,$$

$$Y^{s,p}=Y^{s,p}(H)=W^{s,p}\left(\mathbb{R}^n;H\right),Y_q^{s,p}(H)=Y^{s,p}(H)\cap X_q,$$

$$\|u\|_{Y_q^{s,p}}=\|u\|_{W^{s,p}\left(\mathbb{R}^n;H\right)}+\|u\|_{X_q}<\infty,$$

$$W^{s,p}\left(A^\gamma\right)=W^{s,p}\left(\mathbb{R}^n;H\left(A^\gamma\right)\right),0<\gamma\le1,$$

$$Y^{s,p}=Y^{s,p}(A,H)=W^{s,p}\left(\mathbb{R}^n;H(A),H\right),$$

$$Y^{2,s,p}=Y^{2,s,p}(A,H)=W^{2,s,p}\left(\mathbb{R}_T^n;H(A),H\right),$$

$$Y_q^{s,p}(A;H)=Y^{s,p}(H)\cap X_q(A),$$

$$\|u\|_{Y_q^{s,p}(A,H)}=\|u\|_{Y^{s,p}(H)}+\|u\|_{X_q(A)}<\infty,$$

$$\mathbb{E}_{0p}=\left(Y^{s,p}(A,H),X_p\right)_{\frac{1}{2p},p},\mathbb{E}_{1p}=\left(Y^{s,p}(A,H),X_p\right)_{\frac{1+p}{2p},p},$$

where $\left(Y^{s,p},X_p\right)_{\theta,p}$ denotes the real interpolation space between $Y^{s,p}$ and X_p for $\theta\in(0,1)$, $p\in[1,\infty]$ (see e.g. [33], Section 1.3).

Remark 1.1. By Fubini's theorem we get

$$L^p\left(\mathbb{R}_T^n;H\right)=L^p\left(0,T;X_p\right)\text{ for }X_p=L^p\left(\mathbb{R}^n;H\right).$$

Then by definition of spaces $Y^{2,s,p}$, $Y^{s,p}=H^{s,p}\left(\mathbb{R}^n;H(A),H\right)$ and X_p we have

$$Y^{2,s,p}=\Big\{u:u\in W^{2,p}\left(0,T;Y^{s,p},X_p\right),\|u\|_{W^{2,p}\left(0,T;Y^{s,p},X_p\right)}$$

$$=\|u\|_{L^p\left(0,T;Y^{s,p}\right)}+\left\|u^{(2)}\right\|_{L^p\left(0,T;X_p\right)}\Big\}.$$

By J. lions-J. Peetre result (see e.g. [33], Section 1.8.2) for $u\in W^{2,p}\left(0,T;Y^{s,p},X_p\right)$ the trace operator $u\to\frac{\mathrm{d}^iu}{\mathrm{d}t^i}(t_0)=\frac{\partial^iu}{\partial t^i}(.,t_0)$ is bounded from $Y^{2,s,p}$ into

$$C\left(0,T;\left(Y^{s,p},X_p\right)_{\theta_j,p}\right),\theta_j=\frac{1+jp}{2p},j=0,1.$$

Moreover, if $u(x,.)\in\left(Y^{s,p},X_p\right)_{\theta_j,p}$, then under some assumptions that will be stated in Section 3, $f(u)\in H$ for all x, $t\in\mathbb{R}_T^n$ and the map $u\to f(u)$ is bounded from $\left(Y^{s,p},X_p,\right)_{\frac{1}{2p},p}$ into E. Hence, the nonlinear Equation (1.1) is satisfied in the Banach space H. Here, $H(A)$ denotes a domain of A equipped with graphical norm.

Sometimes we use one and the same symbol C without distinction in order to denote positive constants which may differ from each other even in a single context. When we want to specify the dependence of such a constant on a parameter, say α, we write C_α. Moreover, for $u,\upsilon>0$ the relations $u\lesssim\upsilon$, $u\approx\upsilon$ means that there exist positive constants C,C_1,C_2 independent on u and υ such that, respectively

$$u\leq C\upsilon, C_1\upsilon\leq u\leq C_2\upsilon.$$

The paper is organized as follows: In Section 1, some definitions and background are given. In Section 2, we obtain the existence of unique solution and a priory estimates for solution of the linearized problem (1.1)-(1.2). In Section 3, we show the existence and uniqueness of local strong solution of the problem (1.1)-(1.2). In Section 4, the existence and uniqueness of global strong solution of the problem (1.1)-(1.2) is derived. Section 5 is devoted to blow up property of the solution of (1.1)-(1.2). In Section 6, we show some applications of the problem (1.1)-(1.2).

Sometimes we use one and the same symbol C without distinction in order to denote positive constants which may differ from each other even in a single context. When we want to specify the dependence of such a constant on a parameter, say h, we write C_h.

2. Estimates for Linearized Equation

In this section, we make the necessary estimates for solutions of the integral problem for linear WE

$$u_{tt}-a\Delta u+Au=g(x,t), x\in\mathbb{R}^n, t\in(0,T), T\in(0,\infty], \tag{2.1}$$

$$u(x,0)=\varphi(x)+\int_0^T\eta(\sigma)u(x,\sigma)\mathrm{d}\sigma, \tag{2.2}$$

$$u_t(x,0)=\psi(x)+\int_0^T\beta(\sigma)u_t(x,\sigma)\mathrm{d}\sigma,$$

where A is a linear operator in a Banach space E, a is a complex number and $\eta(s)$, $\beta(s)$ are measurable functions on $(0,T)$.

Remark 2.1. By properties of real interpolation of Banach spaces and interpolation of the intersection of the spaces (see e.g. [33], Section 1.3) we obtain

$$\mathbb{E}_{0p}=\left(Y^{s,p}(A,H)\cap X_p, X_p\right)_{\frac{1}{2p},p}$$
$$=\left(Y^{s,p}(H),X_p\right)_{\frac{1}{2p},p}\cap\left(X_p(A),X_p\right)_{\frac{1}{2p},p}$$

$$= W^{s\left(1-\frac{1}{2p}\right),p}\left(\mathbb{R}^n;H\right) \cap L^p\left(\mathbb{R}^n;\left(H(A),H\right)_{\frac{1}{2p},p}\right)$$

$$= W^{s\left(1-\frac{1}{2p}\right),p}\left(\mathbb{R}^n;\left(H(A),H\right)_{\frac{1}{2p},p},H\right).$$

In a similar way, we have

$$\mathbb{E}_{1p} = \left(Y^{s,p}(A,H) \cap X_p, X_p\right)_{\frac{1+p}{2p},p} = W^{\frac{s(p-1)}{2p},p}\left(\mathbb{R}^n;\left(H(A),H\right)_{\frac{1+p}{2p},p},H\right).$$

Remark 2.2. Let A be a sectorial operator in a Banach space E. In view of interpolation of sectorial operators (see e.g. [33], Section 1.8.2) we have the following relation

$$E\left(A^{1-\theta-\varepsilon}\right) \subset \left(E(A),E\right)_{\theta,p} \subset E\left(A^{1-\theta+\varepsilon}\right)$$

for $0<\theta<1$ and $0<\varepsilon<1-\theta$.

Note that from J. lions-J. Peetre result (see e.g. [33], Section 1.8.2) we obtain the following result.

Lemma A$_1$. The trace operator $u \to \frac{\partial^i u}{\partial t^i}(x,t)$ is bounded from $Y^{2,s,p}(A,H)$ into

$$C\left(\mathbb{R}^n;\left(Y^{s,p}(A,H),X_p\right)_{\theta_j,p}\right), \theta_j = \frac{1+jp}{2p}, j=0,1.$$

We assume that A is a sectorial operator in a Hilbert space H. Let A be a generator of a strongly continuous cosine operator function in a Banach space E defined by formula

$$C(t) = C_A(t) = \frac{1}{2}\left(\mathrm{e}^{itA^{\frac{1}{2}}} + \mathrm{e}^{-itA^{\frac{1}{2}}}\right)$$

(see e.g. [25], Section 11 or [23], Section 3). Then, from the definition of sine operator-function $S(t)$ we have

$$S(t) = S_A(t) = \int_0^t C(\sigma)\mathrm{d}\sigma, \text{ i.e. } S(t) = \frac{1}{2i}A^{-\frac{1}{2}}\left(\mathrm{e}^{itA^{\frac{1}{2}}} - \mathrm{e}^{-itA^{\frac{1}{2}}}\right).$$

Remark 2.3. Let A be a densely defined operator in H. By virtue of ([23], Theorem 3.15.3) if A be the generator of a cosine function $C(t)$, *i.e.*

$$R\left(\lambda^2,A\right) = \frac{1}{\lambda}\int_0^\infty \mathrm{e}^{-\lambda t}C(t)\mathrm{d}t \text{ for } \lambda > \omega.$$

Let

$$A_\pm(\xi) = \mathrm{e}^{itA(\xi)} \pm \mathrm{e}^{-itA(\xi)}, C(t) = C(\xi,t) = \frac{A_+(\xi)}{2}, \tag{2.3}$$

$$S(t) = S(\xi,t) = S(\xi,t,A) = \frac{1}{2i}A^{-1}(\xi)A_-(\xi).$$

Condition 2.1. Assume: 1)

$$\left|1+\int_0^T \eta(\sigma)\beta(\sigma)\mathrm{d}\sigma\right| > \int_0^T \left(\left|\eta(\sigma)\right|+\left|\beta(\sigma)\right|\right)\mathrm{d}\sigma; \tag{2.0}$$

2) A is a ϕ-sectorial operator in the Hilbert space H and A is a generator of a cosine function; 3) $a \in S_{\phi_1}$ for $0 \le \phi_1 < \pi$, $\phi_1 < \pi - \phi$; 4) $\varphi \in \mathbb{E}_{0p}$ and $\psi \in \mathbb{E}_{1p}$.

Definition 1.1. Let $T > 0$, $\varphi \in \mathbb{E}_{0p}$ and $\psi \in \mathbb{E}_{1p}$. The function $u \in C^2\left(Y_1^{s,p}(A)\right)$ satisfies of the problem (1.1)-(1.2) is called the continuous solution or the strong solution of (1.1)-(1.2). If $T < \infty$, then $u(x,t)$ is called the local strong solution of (1.1)-(1.2). If $T = \infty$, then $u(x,t)$ is called the global strong solution of (1.1)-(1.2).

First we need the following lemmas:

Lemma 2.1. Let the Condition 2.1 holds. Then, problem (2.1)-(2.2) has a solution.

Proof. By using of the Fourier transform, we get from (2.1)-(2.2):

$$\hat{u}_{tt}(\xi,t) + A_\xi^2 \hat{u}(\xi,t) = \hat{g}(\xi,t), \tag{2.4}$$

$$\hat{u}(\xi,0) = \hat{\varphi}(\xi) + \int_0^T \eta(\sigma)\hat{u}(\xi,\sigma)\mathrm{d}\sigma, \tag{2.5}$$

$$\hat{u}_t(\xi,0) = \hat{\psi}(\xi) + \int_0^T \beta(\sigma)\hat{u}_t(\xi,\sigma)\mathrm{d}\sigma,$$

where $\hat{u}(\xi,t)$ is a Fourier transform of $u(x,t)$ in x and $\hat{\varphi}(\xi)$, $\hat{\psi}(\xi)$ are Fourier transform of φ and ψ, respectively and

$$A_\xi = \left[a|\xi|^2 + A\right]^{\frac{1}{2}}.$$

Consider first, the Cauchy problem

$$\hat{u}_{tt}(\xi,t) + A_\xi^2 \hat{u}(\xi,t) = \hat{g}(\xi,t), \tag{2.6}$$

$$\hat{u}(\xi,0) = u_0(\xi), \hat{u}_t(\xi,0) = u_1(\xi), \xi \in \mathbb{R}^n, t \in [0,T],$$

where $u_0(\xi), u_1(\xi) \in D(A)$ for $\xi \in \mathbb{R}^n$. By virtue of ([25], Section 11.2, 11.4) we obtain that A_ξ is a generator of a strongly continuous cosine operator function and the Cauchy problem (2.6) has a unique solution for all $\xi \in \mathbb{R}^n$. Moreover, the solution of (2.6) can be expressed as

$$\hat{u}(\xi,t) = C(t)u_0(\xi) + S(t)u_1(\xi) + \int_0^T S(t-\tau,\xi,A)\hat{g}(\xi,\tau)\mathrm{d}\tau, t \in (0,T), \tag{2.7}$$

where $C(t)$ is a cosine and $S(t)$ is a sine operator-functions generated by A_ξ, *i.e.*

$$C(t) = C(t,\xi,A) = \frac{1}{2}\left(\mathrm{e}^{tA_\xi} + \mathrm{e}^{-tA_\xi}\right),$$

$$S(t) = S(t,\xi,A) = \frac{1}{2}A_\xi^{-1}\left(\mathrm{e}^{tA_\xi} - \mathrm{e}^{-tA_\xi}\right).$$

Using the formula (2.7) and the first integral condition (2.5) we get

$$u_0(\xi)=\hat{\varphi}(\xi)+\int_0^T \eta(\sigma)\left[u_0(\xi)+\frac{1}{2i}A^{-1}(\xi)u_1(\xi)\right]\mathrm{d}\sigma$$
$$+\int_0^T \eta(\sigma)\left[C(\sigma)u_0(\xi)+S(\sigma)u_1(\xi)\right]\mathrm{d}\sigma$$
$$+\int_0^T\int_0^T \eta(\sigma)S(\sigma-\tau,\xi,A)\hat{g}(\sigma,\xi)\mathrm{d}\tau\mathrm{d}\sigma,\tau\in(0,T),$$

i.e. we obtain the first equation with respect to $u_0(\xi)$, $u_1(\xi)$:

$$b_{10}(\xi)u_0(\xi)+b_{11}(\xi)u_1(\xi)=g_{10}(\xi), \tag{2.8}$$

where

$$b_{10}(\xi)=\left[1-\int_0^T \eta(\sigma)\left[1+C(\sigma)\right]\mathrm{d}\sigma\right],$$
$$b_{11}(\xi)=-\frac{1}{2i}A_\xi^{-1}\int_0^T \eta(\sigma)C(\sigma)\mathrm{d}\sigma-\int_0^T \eta(\sigma)S(\sigma)\mathrm{d}\sigma,$$
$$g_{10}(\xi)=\hat{\varphi}(\xi)+\int_0^T\int_0^T \eta(\sigma)S(\sigma-\tau,\xi,A)\hat{g}(\sigma,\xi)\mathrm{d}\tau\mathrm{d}\sigma$$

Differentiating both sides of formula (2.7) and using the seconf integral condition (2.5), we have

$$u_1(\xi)=\hat{\psi}(\xi)+\int_0^T \beta(\sigma)\left[\frac{1}{2i}u_0(\xi)+u_1(\xi)\right]\mathrm{d}\sigma$$
$$+\int_0^T\int_0^T \beta(\sigma)C(\sigma-\tau,\xi,A)\hat{g}(\xi,\sigma)\mathrm{d}\tau\mathrm{d}\sigma,$$

i.e. we get the second equation with respect to $u_0(\xi)$, $u_1(\xi)$:

$$b_{20}(\xi)u_0(\xi)+b_{21}(\xi)u_1(\xi)=g_{20}(\xi), \tag{2.9}$$

where

$$b_{20}(\xi)=-\frac{1}{2i}\int_0^T \beta(\sigma)\mathrm{d}\sigma, b_{21}(\xi)=1-\int_0^T \beta(\sigma)\mathrm{d}\sigma,$$
$$g_{20}(\xi)=\hat{\psi}(\xi)+\int_0^T\int_0^T \beta(\sigma)C(\sigma-\tau,\xi,A)\hat{g}(\xi,\sigma)\mathrm{d}\tau\mathrm{d}\sigma.$$

Now, we consider the system of Equations (2.8)-(2.9) in $u_0(\xi)$ and $u_1(\xi)$. By assumption (2.0) and due to uniformly boundedness of A_ξ^{-1}, the main determinant of this system

$$D(\xi)=\begin{vmatrix} b_{10}(\xi) & b_{11}(\xi) \\ b_{20}(\xi) & b_{21}(\xi)\end{vmatrix}=\left[1-\int_0^T \eta(\sigma)\left[1+C(\sigma)\right]\mathrm{d}\sigma\right]\left[1-\int_0^T \beta(\sigma)\mathrm{d}\sigma\right]$$
$$-\left[-\frac{1}{2i}\int_0^T \beta(\sigma)\mathrm{d}\sigma\right]\left[-\frac{1}{2i}A_\xi^{-1}\int_0^T \eta(\sigma)C(\sigma)\mathrm{d}\sigma-\int_0^T \eta(\sigma)S(\sigma)\mathrm{d}\sigma\right]$$
$$=1-\int_0^T \beta(\sigma)\mathrm{d}\sigma-\int_0^T \eta(\sigma)\left[1+C(\sigma)\right]\mathrm{d}\sigma+\left(\int_0^T \beta(\sigma)\mathrm{d}\sigma\right)\int_0^T \eta(\sigma)\left[1+C(\sigma)\right]\mathrm{d}\sigma$$
$$+-\frac{1}{4}A_\xi^{-1}\left(\int_0^T \beta(\sigma)\mathrm{d}\sigma\right)\left(\int_0^T \eta(\sigma)C(\sigma)\mathrm{d}\sigma\right)-\frac{1}{2i}\left(\int_0^T \beta(\sigma)\mathrm{d}\sigma\right)\left[\int_0^T \eta(\sigma)S(\sigma)\mathrm{d}\sigma\right]$$

$$=1-\int_0^T \beta(\sigma)\mathrm{d}\sigma+\left[\int_0^T \eta(\sigma)\left[1+C(\sigma)\right]\mathrm{d}\sigma\right]\left[\int_0^T \beta(\sigma)\mathrm{d}\sigma-1\right]$$
$$-\left(\int_0^T \beta(\sigma)\mathrm{d}\sigma\right)\left[\frac{1}{4}A_\xi^{-1}\int_0^T \eta(\sigma)C(\sigma)\mathrm{d}\sigma+\frac{1}{2i}\int_0^T \eta(\sigma)S(\sigma)\mathrm{d}\sigma\right]$$
$$\neq 0$$

for all $\xi\in\mathbb{R}^n$. By solving the system (2.8)-(2.9) we get

$$u_0(\xi)=D_1(\xi)D^{-1}(\xi), u_1(\xi)=D_2(\xi)D^{-1}(\xi), \tag{2.10}$$
$$D_1(\xi)=b_{21}(\xi)g_{10}(\xi)-b_{11}(\xi)g_{20}(\xi),$$
$$D_2(\xi)=b_{10}(\xi)g_{20}(\xi)-b_{20}(\xi)g_{10}(\xi).$$

By substituting the values $u_0(\xi)$ and $u_1(\xi)$ in (2.7), we obtain

$$\hat{u}(\xi,t)=C(\xi,t)D_1(\xi)D^{-1}(\xi)+S(\xi,t)D_2(\xi)D^{-1}(\xi)$$
$$+\int_0^t S(\xi,t-\tau)\hat{g}(\xi,\tau)\mathrm{d}\tau, \tag{2.11}$$

i.e. problem (2.1)-(2.2) has a unique solution

$$u(x,t)=C_1(t)\varphi+S_1(t)\psi+Qg, \tag{2.12}$$

where $C_1(t)$, $S_1(t)$, Q are linear operator functions defined by

$$C_1(t)\varphi=\mathbb{F}^{-1}\left[C(\xi,t)D_1(\xi)\right], S_1(t)\psi=\mathbb{F}^{-1}\left[S(\xi,t)D_2(\xi)\right],$$
$$Qg=\mathbb{F}^{-1}\tilde{Q}(\xi,t), \tilde{Q}(\xi,t)=\int_0^t \mathbb{F}^{-1}\left[S(\xi,t-\tau)\hat{g}(\xi,\tau)\right]\mathrm{d}\tau.$$

Theorem 2.1. Assume the Condition 2.1 holds and

$$s>\frac{2pn}{2p-1}\left(\frac{2}{q}+\frac{1}{p}\right) \tag{2.13}$$

for $p\in[1,\infty]$ and for a $q\in[1,2]$. Let $0\leq\alpha<1-\frac{1}{2p}$. Then for $\varphi\in\mathbb{E}_{0p}\cap X_1(A^\alpha)$, $\psi\in\mathbb{E}_{1p}\cap X_1\left(A^{\alpha-\frac{1}{2}}\right)$, $g(.,t)\in Y_1^{s,p}$ for $t\in[0,T]$ and $g(x,.)\in L^1(0,T;Y_1^{s,p})$ for $x\in\mathbb{R}^n$ problem (2.1)-(2.2) has a unique solution $u(x,t)\in C^2([0,T];X_\infty)$. Moreover, the following estimate holds

$$\left\|A^\alpha u\right\|_{X_\infty}+\left\|A^\alpha u_t\right\|_{X_\infty}\leq C_0\left[\left\|\varphi\right\|_{\mathbb{E}_{0p}}+\left\|A^\alpha\varphi\right\|_{X_1}+\left\|\psi\right\|_{\mathbb{E}_{1p}}+\left\|A^{\alpha-\frac{1}{2}}\psi\right\|_{X_1}\right.$$
$$\left.+\int_0^t\left(\left\|g(.,\tau)\right\|_{Y_1^{s,p}}+\left\|g(.,\tau)\right\|_{X_1}\right)\mathrm{d}\tau\right], \tag{2.14}$$

uniformly in $t\in[0,T]$, where the constant $C_0>0$ depends only on A, the space H and initial data.

Proof. By Lemma 2.1, the problem (2.1)-(2.2) has a solution $u(x,t)\in C^2([0,T];Y^{s,p}(A;H))$ for $\varphi\in\mathbb{E}_{0p}$, $\psi\in\mathbb{E}_{1p}$ and $g(.,t)\in Y_1^{s,p}$. Let $N\in\mathbb{N}$ and

$$\Pi_N = \{\xi : \xi \in \mathbb{R}^n, |\xi| \le N\}, \Pi'_N = \{\xi : \xi \in \mathbb{R}^n, |\xi| \ge N\}.$$

From (2.12) we deduced that

$$\begin{aligned} \|A^\alpha u\|_{X_\infty} \lesssim & \|\mathbb{F}^{-1} C(\xi,t) A^\alpha D_1(\xi) D^{-1}(\xi)\|_{L^\infty(\Pi_N)} \\ & + \|\mathbb{F}^{-1} S(\xi,t) A^\alpha D_2(\xi) D^{-1}(\xi)\|_{L^\infty(\Pi_N)} \\ & + \|\mathbb{F}^{-1} C(\xi,t) A^\alpha D_1(\xi) D^{-1}(\xi)\|_{L^\infty(\Pi'_N)} \\ & + \|\mathbb{F}^{-1} S(\xi,t) A^\alpha D_2(\xi) D^{-1}(\xi)\|_{L^\infty(\Pi'_N)} \\ & + \frac{1}{2}\|\mathbb{F}^{-1} A^\alpha \tilde{Q}(\xi,t) \hat{g}(\xi,\tau)\|_{L^\infty(\Pi_N)} \\ & + \frac{1}{2}\|\mathbb{F}^{-1} A^\alpha \tilde{Q}(\xi,t) \hat{g}(\xi,\tau)\|_{L^\infty(\Pi'_N)}. \end{aligned} \tag{2.15}$$

By virtue of Remakes 2.1, 2.2 and the properties of sectorial operators we get the following uniform estimate

$$\|\mathbb{F}^{-1} A^\alpha \tilde{Q}(\xi,t) \hat{g}(\xi,\tau)\|_{L^\infty(\Pi_N)} \le C\|g\|_{X_1}.$$

Hence, due to uniform boundedness of operator functions $C(\xi,t)$, $S(\xi,t)$, by (2.3), in view of (2.8)-(2.10) and by Minkowski's inequality for integrals we get the uniform estimate

$$\begin{aligned} & \|\mathbb{F}^{-1} C(\xi,t) A^\alpha D_1(\xi) D^{-1}(\xi)\|_{L^\infty(\Pi_N)} \\ & + \|\mathbb{F}^{-1} S(\xi,t) A^\alpha D_2(\xi) D^{-1}(\xi)\|_{L^\infty(\Pi_N)} \\ & \lesssim \left[\|A^\alpha \varphi\|_{X_1} + \|A^\alpha \psi\|_{X_1} + \|g\|_{X_1}\right]. \end{aligned} \tag{2.16}$$

Let

$$l = s\left(1 - \frac{1}{2p}\right) - \delta \text{ for a } \delta > 0.$$

Moreover, in a similar way, we deduced that

$$\begin{aligned} & \|\mathbb{F}^{-1} C(\xi,t) A^\alpha D_1(\xi) D^{-1}(\xi)\|_{L^\infty(\Pi'_N)} + \|\mathbb{F}^{-1} S(\xi,t) A^\alpha D_2(\xi) D^{-1}(\xi)\|_{L^\infty} \\ & \lesssim \|\mathbb{F}^{-1} C(\xi,t) A^\alpha D_1(\xi) D^{-1}(\xi)\|_{L^\infty} + \|\mathbb{F}^{-1} S(\xi,t) A^\alpha D_2(\xi) D^{-1}(\xi)\|_{L^\infty} \\ & \quad + \|\mathbb{F}^{-1} S(\xi,t) A^\alpha \tilde{Q}(\xi,t) \hat{g}(\xi,\tau)\|_{L^\infty} \\ & \lesssim \left\|\mathbb{F}^{-1}\left(1+|\xi|^2\right)^{-\frac{l}{2}} C(\xi,t)\left(1+|\xi|^2\right)^{\frac{l}{2}} A^\alpha \hat{\varphi}(\xi)\right\|_{L^\infty} \\ & \quad + \left\|\mathbb{F}^{-1}\left(1+|\xi|^2\right)^{-\frac{l}{2}} S(\xi,t)\left(1+|\xi|^2\right)^{\frac{l}{2}} A^\alpha \hat{\psi}(\xi)\right\|_{L^\infty} \\ & \quad + \left\|\mathbb{F}^{-1}\left(1+|\xi|^2\right)^{-\frac{l}{2}} S(\xi,t)\left(1+|\xi|^2\right)^{\frac{l}{2}} A^\alpha \tilde{Q}(\xi,t) \hat{g}(\xi,\tau)\right\|_{L^\infty}, \end{aligned} \tag{2.17}$$

here, the space $L^\infty(\Omega; H)$ is denoted by L^∞. Let

$$\Phi_0(\xi)=\left[A^{1-\frac{1}{2p}-\varepsilon_0}+\left(1+|\xi|^2\right)^{s\left(1-\frac{1}{2p}\right)-\varepsilon_0}\right]^{-1}, 0<\varepsilon_0<1-\frac{1}{2p}, \tag{2.18}$$

$$\Phi_1(\xi)=\left[A^{\frac{1}{2}-\frac{1}{2p}-\varepsilon}+\left(1+|\xi|^2\right)^{s\left(\frac{1}{2}-\frac{1}{2p}\right)-\varepsilon_1}\right]^{-1}, 0<\varepsilon_1<\frac{1}{2}-\frac{1}{2p},$$

$$\Phi_{01}(\xi)=2\xi_k s\left(1-\frac{1}{2p}-\varepsilon_0\right)\left[\left(1+|\xi|^2\right)^{s\left(1-\frac{1}{2p}\right)-\varepsilon_0-1}\right]$$

$$\times\left[A^{1-\frac{1}{2p}-\varepsilon_0}+\left(1+|\xi|^2\right)^{s\left(1-\frac{1}{2p}\right)-\varepsilon_0}\right]^{-2},$$

$$\Phi_{11}(\xi)=2\xi_k s\left(s\left(\frac{1}{2}-\frac{1}{2p}\right)-\varepsilon_1\right)\left[\left(1+|\xi|^2\right)^{s\left(\frac{1}{2}-\frac{1}{2p}\right)-\varepsilon_1-1}\right]$$

$$\times\left[A^{\frac{1}{2}-\frac{1}{2p}-\varepsilon}+\left(1+|\xi|^2\right)^{s\left(\frac{1}{2}-\frac{1}{2p}\right)-\varepsilon_1}\right]^{-2}.$$

By using the resolvent properties of sectorial operators, we have

$$\left\|A^{\alpha}\Phi_i(\xi)\right\|_{B(E)}\lesssim|\xi|^{-\varepsilon}, 0<\varepsilon<\frac{1}{2}-\frac{1}{2p}, i=1,2, \tag{2.19}$$

$$\left\|A^{\alpha}C(\xi,t)\Phi_0(\xi)\right\|_{B(E)}\le C\left\|A^{\alpha}A^{-\left(1-\frac{1}{2p}-\varepsilon_0\right)}(\xi)\right\|_{B(E)}\le C_0,$$

$$\left\|A^{\alpha}S(\xi,t)\Phi_1(\xi)\right\|_{B(E)}\le\left\|A^{\frac{1}{2}}\eta^{-1}(\xi)\right\|_{B(E)}\left\|A^{\alpha}A^{-\frac{1}{2}}\Phi_1(\xi)\right\|_{B(E)}$$

$$\le C\left\|A^{\alpha}A^{-\left(1-\frac{1}{2p}-\varepsilon_0\right)}(\xi)\right\|_{B(E)}\le C_1.$$

Then by calculating $\frac{\partial}{\partial\xi_k}\Phi_0(\xi)$, $\frac{\partial}{\partial\xi_k}\Phi_1(\xi)$, we obtain

$$A^{\alpha}\frac{\partial}{\partial\xi_k}\Phi_0(\xi)\in B(H), A^{\alpha}\frac{\partial}{\partial\xi_k}\Phi_1(\xi)\in B(H).$$

Let we show that $G_i(.,t)\in B_{q,1}^{n\left(\frac{1}{q}+\frac{1}{p}\right)}(\mathbb{R}^n;H)$ for some $q\in(1,2)$ and for all $t\in[0,T]$, where

$$G_i(\xi,t)=\left(1+|\xi|^2\right)^{-\frac{l}{2}}AC(\xi,t)\Phi_i(\xi), i=0,1.$$

By embedding properties of Sobolev and Besov spaces it is sufficient to derive that $G_i\in W_q^{n\left(\frac{1}{q}+\frac{1}{p}\right)+\varepsilon}(\mathbb{R}^n)$ for some $\varepsilon>0$. Indeed by contraction, by Condition 2.2 and by (2.18) we get $G_i\in L^q(\mathbb{R}^n)$. Let $\sigma>n\left(\frac{1}{r}+\frac{1}{p}\right)$. For deriving the embedding relations $G_i\in W_q^{\sigma+\varepsilon}(\mathbb{R}^n)$, it sufficient to show

$$\left(1+|\xi|^2\right)^{\frac{\sigma}{2}} G_i(.,t) \in L^{\sigma}\left(\mathbb{R}^n\right) \text{ for al } t \in [0,T].$$

Indeed, in view of (2.18), $\left(1+|\xi|^2\right)^{\frac{\sigma}{2}} \Phi_i(\xi)$ are uniformly bounded for $\xi \in \mathbb{R}^n$. By virtue of (2.3), (2.19), by Condition 2.2 for $l < s\left(1-\frac{1}{2p}\right)$ and $(l-\sigma)q > n$ we have

$$\begin{aligned}
&\int_{\mathbb{R}^n} \left(1+|\xi|^2\right)^{\frac{\sigma}{2}q} \left|G_i(\xi,t)\right|^q \mathrm{d}\xi \\
&= \int_{\mathbb{R}^n} \left(1+|\xi|^2\right)^{-\frac{l-\sigma}{2}q} \left\|C(\xi,t)\right\|^q \left\|A^{\alpha}\Phi_i(\xi)\right\|_{B(E)}^q \mathrm{d}\xi \\
&\lesssim \int_{\mathbb{R}^n} \left(1+|\xi|^2\right)^{\frac{\sigma-l+\varepsilon}{2}q} |\xi|^{-\varepsilon q} \mathrm{d}\xi \\
&\lesssim \int_{\mathbb{R}^n} \left(1+|\xi|^2\right)^{-\left(\frac{l-\sigma}{2}\right)q} \mathrm{d}\xi < \infty.
\end{aligned}$$

Hence, by Fourier multiplier theorems (see e.g. [32], Theorem 4.3) we get that the functions $G_i(\xi,t)$ are Fourier multipliers from $L^p\left(\mathbb{R}^n;H\right)$ to $L^{\infty}\left(\mathbb{R}^n;H\right)$. In a similar way we obtain that

$$\left(1+|\xi|^2\right)^{-\frac{s}{2}} S(\xi,t)\left(1+|\xi|^2\right)^{\frac{s}{2}} A^{\alpha}\hat{\psi}(\xi),$$

$$\left(1+|\xi|^2\right)^{-\frac{s}{2}} S(\xi,t)\left(1+|\xi|^2\right)^{\frac{s}{2}} A^{\alpha}\tilde{Q}(\xi,t)\hat{g}(\xi,\tau)$$

are $L^p\left(\mathbb{R}^n;H\right) \to L^{\infty}\left(\mathbb{R}^n;H\right)$ Fourier multipliers. Then by Minkowski's inequality for integrals, from (2.3), (2.16)-(2.18) and by Remake 2.3 we have

$$\begin{aligned}
&\left\|F^{-1}C(\xi,t)A^{\alpha}\hat{\varphi}(\xi)\right\|_{L^{\infty}} + \left\|\mathbb{F}^{-1}S(\xi,t)A^{\alpha}\hat{\psi}(\xi)\right\|_{L^{\infty}} \\
&\lesssim \left\|F^{-1}C(\xi,t)\eta^{-2}\hat{\varphi}\right\|_{L^{\infty}} + \left\|\mathbb{F}^{-1}S(\xi,t)\eta^{-1}\hat{\psi}\right\|_{L^{\infty}} \\
&\lesssim \left[\|\varphi\|_{\mathbb{E}_{0p}} + \|\psi\|_{\mathbb{E}_{1p}} + \|g\|_{W^{s,p}}\right].
\end{aligned} \tag{2.20}$$

Moreover, by virtue of Remakes 2.1 - 2.3 and by reasoning as the above, we have the following estimate

$$\left\|F^{-1}A^{\alpha}\tilde{Q}(\xi,t)\right\|_{X_{\infty}} \le C_0^t\left(\left\|g(.,\tau)\right\|_{W^{s,p}} + \left\|g(.,\tau)\right\|_{X_1}\right)\mathrm{d}\tau \tag{2.21}$$

uniformly in $t \in [0,T]$. Thus, from (2.12), (2.20) and (2.21) we obtain

$$\begin{aligned}
\left\|A^{\alpha}u\right\|_{X_{\infty}} \le C\bigg[&\|\varphi\|_{\mathbb{E}_{0p}} + \left\|A^{\alpha}\varphi\right\|_{X_1} + \|\psi\|_{\mathbb{E}_{1p}} + \left\|A^{\alpha}\psi\right\|_{X_1} \\
&+ \int_0^t \left(\left\|g(.,\tau)\right\|_{Y^{s,p}} + \left\|g(.,\tau)\right\|_{X_1}\right)\mathrm{d}\tau\bigg].
\end{aligned} \tag{2.22}$$

By differentiating (2.12) in a similar way, we get

$$\left\|A^{\alpha}u_t\right\|_{X_\infty} \le C\left[\|\varphi\|_{\mathbb{E}_{0p}} + \left\|A^{\alpha}\varphi\right\|_{X_1} + \left\|A^{\alpha}\psi\right\|_{\mathbb{E}_{1p}} + \left\|A^{\alpha}\psi\right\|_{X_1} + \int_0^t \left(\|g(.,\tau)\|_{Y^{s,p}} + \|g(.,\tau)\|_{X_1}\right)\mathrm{d}\tau\right]. \tag{2.23}$$

Then from (2.22) and (2.23) in view of Remarks 2.1, 2.2 we obtain the estimate (2.14).

Let now show that problem (2.1) has a unique solution $u \in C^{(1)}\left([0,T];Y^{s,p}\right)$. Let's admit it is the opposite. So let's assume that the problem (2.1) has two solutions $u_1, u_2 \in C^{(1)}\left([0,T];Y^{s,p}\right)$. Then by linearity of (2.1), we get that $\upsilon = u_1 - u_2$ is also a solution of the corresponding homogenous equation

$$u_{tt} - a\Delta u + Au = 0, \upsilon(x,0) = 0, \upsilon_t(x,0) = 0, x \in \mathbb{R}^n, t \in (0,T).$$

Moreover, by (2.7) we have the following estimate

$$\left\|A^{\alpha}\upsilon\right\|_{X_\infty} \le 0.$$

Since $N(A) = \{0\}$, the above estimate implies that $\upsilon = 0$, *i.e.* $u_1 = u_2$.

Theorem 2.2. Assume the Condition 2.1 and (2.13) is satisfied. Let $0 \le \alpha < 1 - \frac{1}{2p}$. Then for $\varphi \in \mathbb{E}_{0p}$, $\psi \in \mathbb{E}_{1p}$, $g(.,t) \in Y^{s,p}$ for $t \in [0,T]$ and $g(x,.) \in L^1\left(0,T;Y^{s,p}\right)$ for $x \in \mathbb{R}^n$ problem (2.1)-(2.2) has a unique solution $u \in C^2\left([0,T];Y^{s,p}\right)$ and the following estimate holds

$$\left(\left\|A^{\alpha}u\right\|_{Y^{s,p}} + \left\|A^{\alpha}u_t\right\|_{Y^{s,p}}\right) \le C_0\left[\|\varphi\|_{\mathbb{E}_{0p}} + \|\psi\|_{\mathbb{E}_{1p}} + \int_0^t \|g(.,\tau)\|_{Y^{s,p}}\,\mathrm{d}\tau\right] \tag{2.24}$$

for all $t \in [0,T]$.

Proof. From (2.11) and (2.17) we get the following uniform estimate

$$\begin{aligned}
&\left(\left\|\mathbb{F}^{-1}\left(1+|\xi|^2\right)^{\frac{s}{2}} A^{\alpha}\hat{u}\right\|_{X_p} + \left\|\mathbb{F}^{-1}\left(1+|\xi|^2\right)^{\frac{s}{2}} A^{\alpha}\hat{u}_t\right\|_{X_p}\right) \\
&\le C\left\{\left\|\mathbb{F}^{-1}\left(1+|\xi|^2\right)^{\frac{s}{2}} C(\xi,t)A^{\alpha}\hat{\varphi}\right\|_{X_p} + \left\|\mathbb{F}^{-1}\left(1+|\xi|^2\right)^{\frac{s}{2}} A^{\alpha}S(\xi,t)\hat{\psi}\right\|_{X_p}\right. \\
&\left. + \int_0^t \left\|\left(1+|\xi|^2\right)^{\frac{s}{2}} A^{\alpha}\tilde{Q}(\xi,t)\hat{g}(\xi,\tau)\right\|_{X_p} \mathrm{d}\tau\right\}.
\end{aligned} \tag{2.25}$$

By using the Fourier multiplier theorem ([32], Theorem 4.3) and by reasoning as in Theorem 2.1 we get that $\left(1+|\xi|^2\right)^{-\frac{s}{2}} C(\xi,t)$, $\left(1+|\xi|^2\right)^{-\frac{s}{2}} S(\xi,t)$ and $\left(1+|\xi|^2\right)^{-\frac{s}{2}} A^{\alpha}S(\xi,t)$ are Fourier multipliers in $L^p\left(\mathbb{R}^n;H\right)$ uniformly with respect to $t \in [0,T]$. So, the estimate (2.25) by using the Minkowski's inequality for integrals implies (2.24).

The uniqueness of (2.1)-(2.2) is obtained by reasoning as in Theorem 2.1.

3. Local Well Posedness of IVP for Nonlinear WE

In this section, we will show the local existence and uniqueness of solution of the nonlinear problem (1.1)-(1.2).

For this aim we need the following lemmas. By reasoning as in [7] [18] [35], we show the following lemmas concerning the behaviour of the nonlinear term in *E*-valued space $Y^{s,p}$. Here, let *E* be a Banach algebra.

Lemma 3.1. Let $s \geq 0$, $f \in C^{[s]+1}(\mathbb{R}; E)$ with $f(0) = 0$. Then for any $u \in Y^{s,p} \cap L^\infty$, we have $f(u) \in Y^{s,p} \cap X_\infty$. Moreover, there is some constant $A(M)$ depending on *M* such that for all $u \in Y^{s,p} \cap L^\infty$ with $\|u\|_{X_\infty} \leq M$,

$$\|f(u)\|_{Y^{s,p}} \leq C(M)\|u\|_{Y^{s,p}}. \tag{3.1}$$

Proof. For $s = 0$ in view of $f(0) = 0$, we get

$$f(u) = \int_0^1 f^{(1)}(\sigma u)\,\mathrm{d}\sigma.$$

It follows that

$$\|f(u)\|_{X_p} \leq C(M)\|u\|_{X_p}.$$

If *s* is a positive integer, we have

$$\|f(u)\|_{Y^{s,p}} \leq C\left[\|f(u)\|_{X_p} + \sum_{k=1}^{n}\left\|\frac{\partial^s}{\partial x_k} f(u)\right\|_{X_p}\right]. \tag{3.2}$$

By calculation of derivative and applying Holder inequality, we get

$$\begin{aligned}\left\|\frac{\partial^s}{\partial x_i} f(u)\right\|_{X_p} &\leq \sum_{l=1}^{s}\sum_{\alpha}\left\|f^{(l)}(u)\frac{\partial^{\beta_1}u}{\partial x_i}\frac{\partial^{\beta_2}u}{\partial x_i}\cdots\frac{\partial^{\beta_l}u}{\partial x_i}\right\|_{X_p} \\ &\leq \sum_{l=1}^{s}\sum_{\alpha}\left\|f^{(l)}(u)\right\|_{X_\infty}\prod_{k=1}^{l}\left\|\frac{\partial^{\beta_k}u}{\partial x_i}\right\|_{X_{p_k}}, i = 1, 2, \cdots, n,\end{aligned} \tag{3.3}$$

where

$$\beta = (\beta_1, \beta_2, \cdots, \beta_l), \beta_k \geq 1, \beta_1 + \beta_2 + \cdots + \beta_l = l, p_k = \frac{pl}{\beta_k}.$$

Applying Gagliardo-Nirenberg's inequality in *E*-valued X_p spaces, we have

$$\left\|\frac{\partial^{\beta_k}u}{\partial x_i}\right\|_{X_{p_k}} \leq C\|u\|_{X_\infty}^{1-\frac{\beta_k}{l}}\left\|\frac{\partial^s u}{\partial x_i^s}\right\|_{X_p}^{\frac{\beta_k}{l}}. \tag{3.4}$$

Hence, from (3.3) and (3.4) we get

$$\left\|\frac{\partial^s}{\partial x_i} f(u)\right\|_{X_p} \leq C(M)\left\|\frac{\partial^s u}{\partial x_i^s}\right\|_{X_p}. \tag{3.5}$$

Then combining (3.2), (3.3) and (3.5) we obtain (3.1).

Let *s* is not integer number and $m = [s]$. From the above proof, we have

$$\|f(u)\|_{Y^{m,p}} \leq C(M)\|u\|_{Y^{m,p}}, \|f(u)\|_{Y^{m+1,p}} \leq C(M)\|u\|_{Y^{m+1,p}}.$$

Then using interpolation between $W^{m+1,p}$ and $W^{m,p}$ yields (3.1) for all $s \geq 0$.

By using Lemma 3.1 and properties of convolution operators we obtain.

Corollary 3.1. Let $s \geq 0$, $f \in C^{[s]+1}(\mathbb{R};H)$ with $f(0)=0$. Moreover, assume $\Phi \in L^{\infty}(\mathbb{R}^n;B(H))$. Then for any $u \in Y^{s,p} \cap L^{\infty}$ we have, $f(u) \in Y^{s,p} \cap X_{\infty}$. Moreover, there is some constant $A(M)$ depending on M such that for all $u \in Y^{s,p} \cap L^{\infty}$ with $\|u\|_{X_{\infty}} \leq M$,

$$\|\Phi * f(u)\|_{Y^{s,p}} \leq C(M)\|u\|_{Y^{s,p}}.$$

Lemma 3.2. Let $s \geq 0$, $f \in C^{[s]+1}(\mathbb{R};H)$. Then for any M there is some constant $K(M)$ depending on M such that for all u, $\upsilon \in Y^{s,p} \cap X_{\infty}$ with $\|u\|_{X_{\infty}} \leq M$, $\|\upsilon\|_{X_{\infty}} \leq M$, $\|u\|_{Y^{s,p}} \leq M$, $\|\upsilon\|_{Y^{s,p}} \leq M$,

$$\|f(u)-f(\upsilon)\|_{Y^{s,p}} \leq K(M)\|u-\upsilon\|_{Y^{s,p}}, \|f(u)-f(\upsilon)\|_{X_{\infty}} \leq K(M)\|u-\upsilon\|_{X_{\infty}}.$$

Corollary 3.2. Let $s > \frac{n}{2}$, $f \in C^{[s]+1}(\mathbb{R};H)$. Then for any positive M there is a constant $K(M)$ depending on M such that for all $u, \upsilon \in Y^{s,p}$ with $\|u\|_{Y^{s,p}} \leq M$, $\|\upsilon\|_{Y^{s,p}} \leq M$,

$$\|f(u)-f(\upsilon)\|_{Y^{s,p}} \leq K(M)\|u-\upsilon\|_{Y^{s,p}}.$$

Lemma 3.3. If $s > 0$, then $Y_{\infty}^{s,p}$ is an algebra. Moreover, for $f, g \in Y_{\infty}^{s,p}$,

$$\|fg\|_{Y^{s,p}} \leq C\left[\|f\|_{X_{\infty}} + \|g\|_{Y^{s,p}} + \|f\|_{Y^{s,p}} + \|g\|_{X_{\infty}}\right].$$

By using, the Corollary 3.1 and Lemma 3.3 we obta*in*.

Lemma 3.4. Let $s \geq 0$, $f \in C^{[s]+1}(\mathbb{R};H)$ and $f(u) = O\left(|u|^{\gamma+1}\right)$ for $u \to 0$, $\gamma \geq 1$ be a positive integer. If $u \in Y_{\infty}^{s,p}$ and $\|u\|_{X_{\infty}} \leq M$, then

$$\|f(u)\|_{Y^{s,p}} \leq C(M)\left[\|u\|_{Y^{s,p}} \|u\|_{X_{\infty}}^{\gamma}\right],$$

$$\|f(u)\|_{X_1} \leq C(M)\|u\|_{X_p}^{p} \|u\|_{X_{\infty}}^{\gamma-1}.$$

Corollary 3.3. Let $s \geq 0$, $f \in C^{[s]+1}(\mathbb{R};H)$ and $f(u) = O\left(|u|^{\gamma+1}\right)$ for $u \to 0$, $\gamma \geq 1$ be a positive integer. Moreover, assume $\Phi \in L^{\infty}(\mathbb{R}^n;B(E))$. If $u \in Y_{\infty}^{s,p}$ and $\|u\|_{X_{\infty}} \leq M$, then

$$\|\Phi * f(u)\|_{Y^{s,p}} \leq C(M)\left[\|u\|_{Y^{s,p}} \|u\|_{X_{\infty}}^{\gamma}\right],$$

$$\|\Phi * f(u)\|_{X_1} \leq C(M)\|u\|_{X_p}^{p} \|u\|_{X_{\infty}}^{\gamma-1}.$$

Lemma 3.5. Let $s \geq 0$, $f \in C^{[s]+1}(\mathbb{R};H)$ and $f(u) = O\left(|u|^{\gamma+1}\right)$ for $u \to 0$. Moreover, let $\gamma \geq 0$ be a positive integer. If $u, \upsilon \in Y_{\infty}^{s,p}$, $\|u\|_{Y^{s,p}} \leq M$, $\|\upsilon\|_{Y^{s,p}} \leq M$ and $\|u\|_{X_{\infty}} \leq M$, $\|\upsilon\|_{X_{\infty}} \leq M$, then

$$\|f(u)-f(\upsilon)\|_{Y^{s,p}} \leq C(M)\left[\left(\|u\|_{X_{\infty}} - \|\upsilon\|_{X_{\infty}}\right)\left(\|u\|_{Y^{s,p}} + \|\upsilon\|_{Y^{s,p}}\right)\left(\|u\|_{X_{\infty}} + \|\upsilon\|_{X_{\infty}}\right)^{\gamma-1},\right.$$

$$\|f(u)-f(\upsilon)\|_{X_1} \leq C(M)\left(\|u\|_{X_{\infty}} + \|\upsilon\|_{X_{\infty}}\right)^{\gamma-1}\left(\|u\|_{X_p} + \|\upsilon\|_{X_p}\right)\|u-\upsilon\|_{X_p}.$$

Let $\mathbb{E}_0$ denotes the real interpolation space between $Y^{s,p}(A,H)$ and X_p with $\theta=\frac{1}{2p}$, *i.e.*

$$\mathbb{E}_{0p}=\left(Y^{s,p}(A,H),X_p\right)_{\frac{1}{2p},p}.$$

Remark 3.1. By using J. Lions-J. Peetre result (see e.g. [33], Section 1.8) we obtain that the map $u\to u(t_0)$, $t_0\in[0,T]$ is continuous and surjective from $Y^{2,s,p}(A,H)$ onto $\mathbb{E}_{0p}$ and there is a constant C_1 such that

$$\|u(t_0)\|_{\mathbb{E}_{0p}}\leq C_1\|u\|_{Y^{2,s,p}(A,E)},1\leq p\leq\infty. \tag{3.6}$$

Let

$$C^2\left(Y_1^{s,p}(A)\right)=C^{(2)}\left([0,T];Y_1^{s,p}(A,H)\right),C^{2,s}(A,H)=C^{(2)}\left([0,T];Y^{s,p}(A,H)\right).$$

Condition 3.1. Assume:

1) the Condition 2.1 holds for $s>\frac{2pn}{2p-1}\left(\frac{2}{q}+\frac{1}{p}\right)$, $p\in[1,\infty]$, for a $q\in[1,2]$ and $0\leq\alpha<1-\frac{1}{2p}$;

2) the function $u\to f(u)$: continuous from $u\in\mathbb{E}_{0p}$ into *H*, $f\in C^k(\mathbb{R};H)$ with *k* an integer, $k\geq s>\frac{n}{p}$ and $f(u)=O\left(|u|^{\gamma+1}\right)$ for $u\to 0$, $\gamma\geq 1$ be a positive integer.

Let

$$Y_1^{s,p}\left(A^\alpha;H\right)=Y^{s,p}\left(A^\alpha;H\right)\cap X_1\left(A^\alpha\right),Y^{s,p}\left(A^\alpha;H\right)=\Big\{u\in Y^{s,p}\left(A^\alpha;H\right),$$

$$\|u\|_{Y^{s,p}(A^\alpha;E)}=\left\|A^\alpha u\right\|_{X_p}+\left\|\mathbb{F}^{-1}\left(1+|\xi|^2\right)^{\frac{s}{2}}\hat{u}\right\|_{X_p}<\infty\Big\}.$$

Main aim of this section is to prove the following results:

Theorem 3.1. Let the Condition 3.1 holds. Then there exists a constant $\delta>0$ such that for any $\varphi\in Y_0\left(A^\alpha\right)$ and $\psi\in Y_1\left(A^\alpha\right)$ satisfying

$$\|\varphi\|_{\mathbb{E}_{0p}}+\left\|A^\alpha\varphi\right\|_{X_1}+\|\psi\|_{\mathbb{E}_{1p}}+\left\|A^\alpha\psi\right\|_{X_1}\leq\delta, \tag{3.7}$$

problem (1.1)-(1.2) has a unique local strange solution $u\in C^2\left(Y_1^{s,p}(A)\right)$. Moreover,

$$\sup_{t\in[0,T]}\left(\|u(.,t)\|_{\hat{Y}_1^{s,p}(A^\alpha,H)}+\|u_t(.,t)\|_{\hat{Y}_1^{s,p}(A^\alpha;H)}\right)\leq C\delta, \tag{3.8}$$

where the constant *C* depends only on *A*, *E*, *g*, *f* and initial values.

Proof. By (2.5), (2.6) the problem of finding a solution *u* of (1.1)-(1.2) is equivalent to finding a fixed point of the mapping

$$G(u)=C_1(t)\varphi(x)+S_1(t)\psi(x)+Q(u), \tag{3.9}$$

where $C_1(t)$, $S_1(t)$ are defined by (2.6) and $Q(u)$ is a map defined by

$$Q(u) = -\int_0^t \mathbb{F}^{-1}\left[U(\xi, t-\tau)\hat{f}(u)(\xi,\tau)\right]\mathrm{d}\tau.$$

We define the metric space

$$C(T,A) = C_\delta^2\left(Y_1^{s,p}(A)\right) = \left\{u \in C^{2,s}(A,E), \|u\|_{C^{2,s,p}(T,A)} \le 5C_0\delta\right\}$$

equipped with the norm defined by

$$\|u\|_{C(T,A)} = \sup_{t\in[0,T]}\left[\left\|A^\alpha u(.,t)\right\|_{X_\infty} + \|u(.,t)\|_{Y^{s,p}} + \left\|A^\alpha u_t(.,t)\right\|_{X_\infty} + \|u_t(.,t)\|_{Y^{s,p}}\right],$$

where $\delta > 0$ satisfies (3.7) and C_0 is a constant in Theorem 2.1 and 2.2. It is easy to prove that $C(T,A)$ is a complete metric space. From imbedding in Sobolev-Lions space $Y^{s,p}(A,E)$ (see e.g. [27], Theorem 1) and trace result (3.6) we got that $\|u\|_{X_\infty} \le 1$ if we take that δ is enough small. For $\varphi \in Y_0(A^\alpha)$ and $\psi \in Y_1(A^\alpha)$, let

$$\|\varphi\|_{\mathbb{E}_{0p}} + \left\|A^\alpha\varphi\right\|_{X_1} + \|\psi\|_{\mathbb{E}_{1p}} + \left\|A^\alpha\psi\right\|_{X_1} = \delta.$$

So, we will find T and M so that G is a contraction in $C^{2,s,p}(T,A)$. By Theorems 2.1, 2.2 and Corollary 3.3 $f(u) \in Y_1^{s,p}$. So, problem (1.1)-(1.2) has a solution that satisfies the following

$$G(u)(x,t) = C_1(t)\varphi + S_1(t)\psi + Q(u), \tag{3.10}$$

where $C_1(t)$, $S_1(t)$ are defined by (2.5) and (2.6). By assumptions, it is easy to see that the map G is well defined for $f \in C^{[s]+1}(\mathbb{E}_{0p}; H)$. First, let us prove that the map G has a unique fixed point in $C(T,A)$. For this aim, it is sufficient to show that the operator G maps $C(T,A)$ into $C(T,A)$ and G is strictly contractive if δ is suitable small. In fact, by (2.7) in Theorem 2.1, Corollary 3.3 and in view of (3.7), we have

$$\begin{aligned}
&\left\|A^\alpha G(u)\right\|_{X_\infty} + \left\|A^\alpha G_t(u)\right\|_{X_\infty} \\
&\le 2C_0\left[\|\varphi\|_{Y_0^\alpha(A^\alpha)} + \|\psi\|_{Y_1^\alpha(A^\alpha)} + \int_0^t\left(\left\|\hat{f}((u))\right\|_{Y^{s,p}} + \left\|\hat{f}((u))\right\|_{X_1}\right)\mathrm{d}\tau\right] \\
&\le 2C_0\delta + C\int_0^t\left(\|u(\tau)\|_{Y^{s,p}}\|u(\tau)\|_{X_\infty}^\gamma + \|u(\tau)\|_{X_p}^p\|u(\tau)\|_{X_\infty}^{\gamma-1}\right)\mathrm{d}\tau \\
&\le 2C_0\delta + C\|u\|_{C^{2,s,p}(T,A)}^{\gamma+1}.
\end{aligned} \tag{3.11}$$

On the other hand, by (2.17), Corollary 3.3 and (3.7), we get

$$\begin{aligned}
&\left(\left\|A^\alpha G(u)\right\|_{Y^{s,p}} + \left\|A^\alpha G_t(u)\right\|_{Y^{s,p}}\right) \\
&\le 2C_0\left(\|\varphi\|_{\mathbb{E}_{0p}} + \|\psi\|_{\mathbb{E}_{1p}} + \int_0^t\left\|\hat{f}((u))\right\|_{Y^{s,p}}\mathrm{d}\tau\right) \\
&\le 2C_0\delta + \int_0^t\left[\|u(\tau)\|_{Y^{s,p}}\|u(\tau)\|_{X_\infty}^\gamma\right]\mathrm{d}\tau \\
&\le 2C_0\delta + C\|u\|_{C^{2,s,p}(T,A)}^{\gamma+1}.
\end{aligned} \tag{3.12}$$

Hence, combining (3.11) with (3.12) we obtain

$$\left\|A^{\alpha}G(u)\right\|_{Y_{\infty}^{s,p}}+\left\|A^{\alpha}G_{t}(u)\right\|_{Y_{\infty}^{s,p}}\leq 4C_{0}\delta+C\|u\|_{C^{2,s,p}(T,A)}^{\gamma+1}. \quad (3.13)$$

So, taking that δ is enough small such that $C(5C_{8}\delta)^{\gamma}<\frac{1}{5}$, by Theorems 2.1, 2.2 and (3.13), G maps $C(T,A)$ into $C(T,A)$.

Now, we are going to prove that the map G is strictly contractive. Let $u_{1},u_{2}\in C(T,A)$ given. From (3.10) we get

$$G(u_{1})-G(u_{2})=\int_{0}^{T}\left[S(x,t-\tau)\left(\hat{f}(u_{1})(\tau)-\hat{f}(u_{2})(\tau)\right)\right]\mathrm{d}\tau, t\in(0,T).$$

By (2.7) in Theorem 2.1 and Corollary 3.3, we have

$$\begin{aligned}&\left\|A^{\alpha}\left[G(u_{1})-G(u_{2})\right]\right\|_{X_{\infty}}+\left\|A^{\alpha}\left[G(u_{1})-G(u_{2})\right]_{t}\right\|_{X_{\infty}}\\&\leq\int_{0}^{t}\left(\left\|\left[\hat{f}(u_{1})-\hat{f}(u_{2})\right]\right\|_{Y^{s,p}}+\left\|\left[\hat{f}(u_{1})-\hat{f}(u_{2})\right]\right\|_{X_{1}}\right)\mathrm{d}\tau\\&\leq\int_{0}^{t}\left\{\|u_{1}-u_{2}\|_{X_{\infty}}\left(\|u_{1}\|_{Y^{s,p}}+\|u_{2}\|_{Y^{s,p}}\right)\left(\|u_{1}\|_{X_{\infty}}+\|u_{2}\|_{X_{\infty}}\right)^{\gamma-1}\right.\\&\quad+\|u_{1}-u_{2}\|_{Y^{s,p}}\left(\|u_{1}\|_{X_{\infty}}+\|u_{2}\|_{X_{\infty}}\right)^{\gamma}\\&\quad\left.+\left(\|u_{1}\|_{X_{\infty}}+\|u_{2}\|_{X_{\infty}}\right)^{\gamma-1}\|u_{1}+u_{2}\|_{X_{p}}\|u_{1}-u_{2}\|_{X_{p}}\right\}\\&\leq C\left(\|u_{1}\|_{C(T,A)}+\|u_{2}\|_{C(T,A)}\right)^{\gamma}\|u_{1}-u_{2}\|_{C(T,A)}.\end{aligned} \quad (3.14)$$

On the other hand, by (2.17) in Theorem 2.2, Corollary 3.3 and (3.7), we get

$$\begin{aligned}&\left(\left\|A^{\alpha}\left[G(u_{1})-G(u_{2})\right]\right\|_{Y^{s,p}}+\left\|A^{\alpha}\left[G(u_{1})-G(u_{2})\right]_{t}\right\|_{Y^{s,p}}\right)\\&\leq C\int_{0}^{t}\left\|\hat{f}(u_{1})(\tau)-\hat{f}(u_{2})(\tau)\right\|_{Y^{s,p}}\mathrm{d}\tau\\&\leq C\int_{0}^{t}\left\{\|u_{1}-u_{2}\|_{X_{\infty}}\left(\|u_{1}\|_{Y^{s,p}}+\|u_{2}\|_{Y^{s,p}}\right)\left(\|u_{1}\|_{X_{\infty}}+\|u_{2}\|_{X_{\infty}}\right)^{\gamma-1}\right.\\&\quad\left.+\|u_{1}-u_{2}\|_{Y^{s,p}}\left(\|u_{1}\|_{X_{\infty}}+\|u_{2}\|_{X_{\infty}}\right)^{\gamma}\right\}\mathrm{d}\tau\\&\leq C\left(\|u_{1}\|_{C(T,A)}+\|u_{2}\|_{C(T,A)}\right)^{\gamma}\|u_{1}-u_{2}\|_{C(T,A)}.\end{aligned} \quad (3.15)$$

Combining (3.14) with (3.15) yields

$$\left\|G(u_{1})-G(u_{2})\right\|_{C(T,A)}\leq C\left(\|u_{1}\|_{C(T,A)}+\|u_{2}\|_{C(T,A)}\right)^{\gamma}\|u_{1}-u_{2}\|_{C(T,A)}. \quad (3.16)$$

Taking δ is enough small, from (3.16) we obtain that G is strictly contractive in $C(T,A)$. Using the contraction mapping principle, we get that $G(u)$ has a unique fixed point $u(x,t)\in C(T,A)$ and $u(x,t)$ is the solution of (1.1)-(1.2).

Let us show that this solution is a unique in $C^{2,s}(A,H)$. Let $u_{1},u_{2}\in C^{2,s}(A,H)$ are two solutions of (1.1)-(1.2). Then for $u=u_{1}-u_{2}$, we have

$$u_{tt}-a\Delta u+Au=\left[f(u_{1})-f(u_{2})\right]. \quad (3.17)$$

Hence, by Minkowski's inequality for integrals and by Theorem 2.2 from (3.17) we obtain

$$\left\|u_1-u_2\right\|_{Y^{s,p}} \le C_2(T)\int_0^t \left\|u_1-u_2\right\|_{Y^{s,p}} \mathrm{d}\tau. \tag{3.18}$$

From (3.18) and Gronwall's inequality, we have $\left\|u_1-u_2\right\|_{Y^{s,p}}=0$, *i.e.* problem (1.1)-(1.2) has a unique solution in $C^{2,s}(A,H)$.

Consider the problem (1.1)-(1.2), when $\varphi\in\mathbb{E}_{0p}$ and $\psi\in\mathbb{E}_{1p}$. Let

$$C^{(i)}\left(Y^{s,2}\right)=C^{(i)}\left([0,\infty);Y^{s,2}(A,H)\right), i=0,1,2.$$

4. Application

Consider the problem (1.4). Let

$$X_{p,2}=L^p\left(\mathbb{R}^n;L^2(0,1)\right), Y^{s,p,2}=H^{s,p}\left(\mathbb{R}^n;L^2(0,1)\right),$$

$$Y_q^{s,p,2}=H^{s,p}\left(\mathbb{R}^n;L^2(0,1)\right)\cap L^q\left(\mathbb{R}^n;L^2(0,1)\right),$$

$$Y^{s,p,2}=H^{s,p}\left(\mathbb{R}^n;H^{2,2}(0,1),L^2(0,1)\right), 1\le p,q\le\infty,$$

$$E_{0p,2}=\left(Y^{s,p}\left(A,L^2(0,1)\right)\cap X_{p.2},X_{p,2}\right)_{\frac{1}{2p},p},$$

$$E_{1p,2}=\left(Y^{s,p}\left(A,L^2(0,1)\right)\cap X_{p,p_1},X_{p,2}\right)_{\frac{1+p}{2p},p}.$$

Let $\omega_1=\omega_1(y)$, $\omega_2=\omega_2(y)$ be roots of equation $b_1(y)\omega^2+1=0$. Let

$$\nu(y)=\begin{vmatrix}(-\omega_1)^{m_1}\alpha_1 & \beta_1\omega_1^{m_1}\\ (-\omega_2)^{m_2}\alpha_2 & \beta_2\omega_2^{m_2}\end{vmatrix}, \eta_1(\xi)=\left[a|\xi|^2+A_1\right]^{\frac{1}{2}}.$$

Here,

$$E_{ip}\left(L^2(0,1)\right)=W^{s(1-\theta_i),p}\left(\mathbb{R}^n;L^2(0,1)\right)\cap L^p\left(\mathbb{R}^n;H^{2(1-\theta_i),2}(0,1)\right),$$

$$\theta_i=\frac{1+ip}{2p}, i=0,1, p_1\in(1,\infty).$$

From Theorem 3.1 we obtain the following result.

Theorem 4.1. Suppose the following conditions are satisfied:

1) $a\in S_{\phi_1}$ for $0\le\phi_1<\pi$, $0\le\alpha<1-\frac{1}{2p}$, $p\in[1,\infty]$ and $\nu(y)\ne 0$ for all $y\in[0,1]$;

2) $b_1\in VMO\cap L^\infty(0,1)$, $Re\omega_k\ne 0$ and $\frac{\lambda}{\omega_k}\in S(\phi_1)$ for a.e. $x\in(0,1)$, $\phi_1\in[0,\pi)$; $b_0\in VMO\cap L^\infty(0,1)$, $b_1(0)=b_1(1)$, $b_0(0)=b_0(1)$.

3) $\varphi\in Y_1^{s,p,2}$, $\psi\in Y_1^{s-1,p,2}$ and $f(.,t)\in Y_1^{s,p,2}$ for $s>\frac{2pn}{2p-1}\left(\frac{2}{r}+\frac{1}{p}\right)$ for $p\in[1,\infty]$, $r\in[1,2]$ and $t\in[0,T]$.

4) The function $u\to F(u)$ is continuous in $u\in E_{02}$ for $x,t\in\mathbb{R}^n\times[0,T]$;

moreover $F(u) \in C^{(1)}\left(E_{02}; L^2(0,1)\right)$.

Then problem (1.9)-(1.10) has a unique local strange solution

$$u \in C^{(2)}\left(\left[0, T_0\right); Y_\infty^{s,p,2}\right),$$

where T_0 is a maximal time interval that is appropriately small relative to M. Moreover, if

$$\|\varphi\|_{\mathbb{E}_{0p,p_1}} + \left\|A^\alpha \varphi\right\|_{X_1} + \|\psi\|_{\mathbb{E}_{1p,p_1}} + \left\|A^\alpha \psi\right\|_{X_1} \le \delta,$$

then $T_0 = \infty$.

Proof. By virtue of [30], $L^2(0,1)$ is a Fourier type space. By virtue of [30], the operator A_1 defined by (1.3) is sectorial in $L^2(0,1)$. Moreover, by interpolation of Banach spaces ([33], Section 1.3), we have

$$E_{02} = \left(W^{s,p}\left(\mathbb{R}^n; H^2(0,1), L^2(0,1)\right), L^p\left(\mathbb{R}^n; L^2(0,1)\right)\right)_{\frac{1}{2p},p}$$
$$= B_{p,2}^{s\left(1-\frac{1}{2p}\right)}\left(\mathbb{R}^n; H^{2l\left(1-\frac{1}{2p}\right)}(0,1), L^2(0,1)\right).$$

Then, by using the properties of spaces $Y^{s,p,2}$, $Y_\infty^{s,p,2}$, E_{02} we get that all conditions of Theorem 3.1 are hold, *i.e.*, we obtain the conclusion.

5. Conclusion

Here, assuming enough smoothness on the initial data in terms of interpolation spaces $H(A)$, H and the sectorial operators, the existence, uniqueness, regularity properties of solutions are established. By choosing the space H and A, the regularity properties of solutions of a wide class of wave equations in the field of physics are obtained.

Data Availability

Data sharing is not applicable to this article as no datasets were generated or analysed during the current study.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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