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Scattering Relations for Two-Dimensional Electromagnetic Waves in Chiral Media

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Abstract

We consider the scattering of time-harmonic plane waves by an infinitely long penetrable chiral cylinder. The electromagnetic scattering problem is reduced to a transmission problem for a system of two-dimensional Helmholtz equations. We prove the classical reciprocity principle, a general scattering theorem and an optical theorem in $\mathbb{R}^2$. Using Herglotz wave functions we define the corresponding far field operator. Applying the general scattering theorem useful relations are proved for the reconstruction of the scatterer. We also prove that for real chirality measure of the penetrable scatterer the far field operator has a countable number of eigenvalues which lie on a circle.

Keywords

Reciprocity Principle, Chiral Media, Two-Dimensional Scattering, Scattering Relations

1. Introduction

In the present work, we deal with electromagnetic scattering by a chiral obstacle in an achiral environment. We focus our study on the two-dimensional case and we formulate the scattering problem for the x_3-components of the electromagnetic field.

Chiral materials are those which exhibit the phenomenon of optical activity, by means of the plane of vibration of linearly polarized light is rotated upon passing through an optically active medium [1] [2]. These materials are characterized by a physical parameter which is called chirality and it connects the electric and magnetic fields. Concerning the electromagnetic scattering, chiral media often appear in applications such as biomedicine, telecommunications, even in archeology and so forth [1] [3]. Therefore, it is a major issue in the study of electromagnetic waves in chiral media. In literature, there are many works which are dealing with electromagnetic scattering problems involving chiral media in $\mathbb{R}^3$. Indicatively, we refer to [4] [5] [6]. Especially, in [6] the solvability of a scattering problem for a chiral dielectric which lies in an achiral environment in three-dimensional case is studied, which is in correspondence to the two-dimensional scattering problem that we study in the present work.

Electromagnetic scattering by a chiral object has been intensively studied in three dimensions. However, there are a few works for the corresponding two-dimensional scattering problems. Gerlach in [7] studied the direct and an inverse electromagnetic scattering problem for a chiral dielectric in the two-dimensional case. In [8] a two-dimensional direct electromagnetic scattering by a perfectly conducting obstacle in a homogeneous chiral environment was studied by using an integral equation approach. Also, a uniqueness result for an inverse scattering problem was proved. In [9] the 2D scattering of electromagnetic waves at oblique incidence in a chiral medium was studied and in [10] an inverse electromagnetic scattering problem for chiral structures was presented. Moreover, in the case of achiral media the two-dimensional problem has been widely studied. In book [11] two-dimensional scattering problems were studied for an imperfect conductor, a partially coated perfect conductor and an orthotropic medium. The scattering of an electromagnetic wave by an imperfectly conducting infinite cylinder at oblique incidence is considered in [12]. Recently in [13], it was studied the inverse scattering problem of obliquely incident electromagnetic waves by a penetrable homogeneous cylinder in 2D. Moreover, in reference [14] the method of auxiliary sources for 2D electromagnetic scattering problems in achiral media was applied.

Scattering relations for electromagnetic waves have been proposed in the literature. In particular, reciprocity principle and other scattering relations have been proved in [15] for acoustic waves and in [16] for elastic waves. Scattering theorems of time-harmonic electromagnetic waves have been stated and proved in [17]. In [5] scattering relations for plane electromagnetic waves are studied in a 3D chiral environment. In reference [18] scattering relations and the far field operator for electromagnetic waves scattered by a 3D chiral obstacle in an achiral environment have been studied. In [19] scattering relations for spherical electromagnetic waves in a chiral environment have been presented. Other related works are [20] which deals with some oscillation criteria for the two-dimensional neutral delay difference systems and [21] which addresses the scattering theory for the Laplacian spectrum on the manifold with bounded curvature comparison dynamics.

The paper is organized as follows. In Section 2 we formulate a two-dimensional scattering problem. Specifically, we establish scalar Helmholtz equations for the x_3-components of the electromagnetic field and we express the transmission conditions in terms of x_3-components. In the interior of the chiral scatterer we use Beltrami fields. The x_3-components of the electric and the magnetic field cannot be decoupled on the boundary of the scatterer since they are connected via the Beltrami fields. In Section 3, we develop some scattering relations for the problem. Specifically, we state and prove the reciprocity principle (Theorem 1), a general scattering theorem (Theorem 2) and an optical theorem (Theorem 3) in two dimensions. Some useful corollaries of these theorems are presented for special cases of incidence. In Section 4, theoretical results for the far field operator are derived by applying the general scattering theorem and by using Herglotz functions. A reduction to the achiral case is presented in Section 5. We end up this work with concluding remarks given in Section 6.

2. Formulation of the Problem

We consider a bounded region $\tilde{D}$ in $\mathbb{R}^3$ with a C^2-boundary, occupied by a homogeneous, isotropic, chiral medium of chirality measure β, electric permittivity ε and magnetic permeability μ. The exterior $\mathbb{R}^3 \setminus \bar{\tilde{D}}$ of the scatterer is an achiral infinite free space with electric permittivity ε_0 and magnetic permeability μ_0.

The total exterior electric field $\boldsymbol{E}^{tot}$ and the total exterior magnetic field $\boldsymbol{H}^{tot}$ satisfy the time-harmonic Maxwell equations [22],

$$curl\boldsymbol{E}^{tot} = ik_0\boldsymbol{H}^{tot}, \quad curl\boldsymbol{H}^{tot} = -ik_0\boldsymbol{E}^{tot} \text{ in } \mathbb{R}^3 \setminus \bar{\tilde{D}}, \tag{1}$$

where $k_0 = \omega\sqrt{\varepsilon_0\mu_0}$ is the wave number and ω is the angular frequency. We assume that the incident electromagnetic wave is plane and it is given by

$$\boldsymbol{E}^{inc} = \frac{1}{k_0^2} curlcurl\boldsymbol{p}\mathrm{e}^{ik_0\boldsymbol{x}\cdot\hat{\boldsymbol{d}}} = \left(\hat{\boldsymbol{d}} \times \boldsymbol{p}\right) \times \hat{\boldsymbol{d}}\mathrm{e}^{ik_0\boldsymbol{x}\cdot\hat{\boldsymbol{d}}}, \tag{2}$$

$$\boldsymbol{H}^{inc} = \frac{1}{ik_0} curl\boldsymbol{p}\mathrm{e}^{ik_0\boldsymbol{x}\cdot\hat{\boldsymbol{d}}} = \left(\hat{\boldsymbol{d}} \times \boldsymbol{p}\right)\mathrm{e}^{ik_0\boldsymbol{x}\cdot\hat{\boldsymbol{d}}}, \tag{3}$$

where $\boldsymbol{p}$ is the polarization and $\hat{\boldsymbol{d}}$ is the incident direction with $\left|\hat{\boldsymbol{d}}\right| = 1$ ([11], p. 46).

The total exterior electromagnetic field $\boldsymbol{E}^{tot}$, $\boldsymbol{H}^{tot}$ is given by

$$\boldsymbol{E}^{tot} = \boldsymbol{E}^{inc} + \boldsymbol{E}^{sc}, \quad \boldsymbol{H}^{tot} = \boldsymbol{H}^{inc} + \boldsymbol{H}^{sc} \text{ in } \mathbb{R}^3 \setminus \tilde{D}, \tag{4}$$

where $\boldsymbol{E}^{sc}$, $\boldsymbol{H}^{sc}$ is the scattered electromagnetic field which satisfies the Silver-Müller radiation condition

$$\lim_{r\to\infty}\left(\boldsymbol{H}^{sc} \times \boldsymbol{x} - r\boldsymbol{E}^{sc}\right) = 0, \quad r = |\boldsymbol{x}|, \tag{5}$$

uniformly in all directions $\hat{\boldsymbol{x}} = \dfrac{\boldsymbol{x}}{|\boldsymbol{x}|}$.

In the interior, as we have mentioned above, the medium is chiral. By using the Drude-Born-Fedorov constitutive relations ([1], p. 153), and the source-free Maxwell curl postulates, the total electromagnetic field $\boldsymbol{E}, \boldsymbol{H}$ satisfies the equations

$$curl\boldsymbol{E} = \gamma^2\beta\boldsymbol{E} + i\frac{\gamma^2}{k}\boldsymbol{H} \text{ in } \tilde{D}, \tag{6}$$

$$curl\boldsymbol{H} = \gamma^2 \beta \boldsymbol{H} - i\frac{\gamma^2}{k}\boldsymbol{E} \text{ in } \tilde{D}, \tag{7}$$

where $k = \omega\sqrt{\varepsilon\mu}$ and $\gamma^2 = \dfrac{k^2}{1-k^2\beta^2}$, $|k\beta| < 1$. The Equations (6) and (7) have been derived from the classical Maxwell equations in chiral media [2], by applying the same transformation as in the achiral case, *i.e.* $\boldsymbol{E}$ has been replaced by $\dfrac{1}{\sqrt{\varepsilon}}\boldsymbol{E}$ and $\boldsymbol{H}$ by $\dfrac{1}{\sqrt{\mu}}\boldsymbol{H}$ ([22], p. 154).

In isotropic homogeneous chiral media, we can use the Bohren decomposition [1], for the electric $\boldsymbol{E}$ and magnetic $\boldsymbol{H}$ fields via Beltrami fields $\boldsymbol{Q}_L$, $\boldsymbol{Q}_R$, considering the above transformation, as follows

$$\boldsymbol{E} = \sqrt{\varepsilon}\left(\boldsymbol{Q}_L + \boldsymbol{Q}_R\right) \text{ in } \tilde{D}, \tag{8}$$

$$\boldsymbol{H} = -i\sqrt{\varepsilon}\left(\boldsymbol{Q}_L - \boldsymbol{Q}_R\right) \text{ in } \tilde{D}, \tag{9}$$

where $\boldsymbol{Q}_L$ and $\boldsymbol{Q}_R$ satisfy the equations, [1],

$$curl\boldsymbol{Q}_L = \gamma_L \boldsymbol{Q}_L,\ curl\boldsymbol{Q}_R = -\gamma_R \boldsymbol{Q}_R \text{ in } \tilde{D}, \tag{10}$$

where $\gamma_L = k(1-k\beta)^{-1}$, $\gamma_R = k(1+k\beta)^{-1}$ are the wave numbers of $\boldsymbol{Q}_L$, $\boldsymbol{Q}_R$, respectively. Moreover, all the fields $\boldsymbol{E}^{tot}$, $\boldsymbol{H}^{tot}$, $\boldsymbol{E}$, $\boldsymbol{H}$, $\boldsymbol{Q}_L$, $\boldsymbol{Q}_R$ are divergence-free.

The transmission conditions for a dielectric will be posed on the total fields as

$$\sqrt{\varepsilon}\,\hat{\boldsymbol{\nu}} \times \boldsymbol{E}^{tot} = \sqrt{\varepsilon_0}\,\hat{\boldsymbol{\nu}} \times \boldsymbol{E} \text{ on } \partial\tilde{D}, \tag{11}$$

$$\sqrt{\mu}\,\hat{\boldsymbol{\nu}} \times \boldsymbol{H}^{tot} = \sqrt{\mu_0}\,\hat{\boldsymbol{\nu}} \times \boldsymbol{H} \text{ on } \partial\tilde{D}, \tag{12}$$

where $\hat{\boldsymbol{\nu}}$ is the unit outward normal vector to $\partial\tilde{D}$.

We now assume that $\tilde{D} = \left\{\boldsymbol{x} = (x_1, x_2, x_3) \in \mathbb{R}^3 : (x_1, x_2) \in D, x_3 \in \mathbb{R}\right\}$ is an infinitely long cylinder which is oriented parallel to the x_3-axis. The cross-section D of $\tilde{D}$ in the x_1x_2-plane has C^2-boundary $\partial D = S$ and will be referred to as the scatterer in $\mathbb{R}^2$, see **Figure 1**.

We also suppose that all the fields are independent of the x_3 variable. Hence, the total exterior electromagnetic field is expressed as

$$\boldsymbol{E}^{tot}(x_1, x_2) = \left(E_1^{tot}(x_1, x_2), E_2^{tot}(x_1, x_2), \mathcal{U}_E^{tot}(x_1, x_2)\right), \tag{13}$$

$$\boldsymbol{H}^{tot}(x_1, x_2) = \left(H_1^{tot}(x_1, x_2), H_2^{tot}(x_1, x_2), \mathcal{U}_H^{tot}(x_1, x_2)\right). \tag{14}$$

The Maxwell Equation (1) lead to the following relations in $\mathbb{R}^2 \setminus \bar{D}$

$$\frac{\partial \mathcal{U}_E^{tot}}{\partial x_2} = ik_0 H_1^{tot}, \tag{15}$$

$$\frac{\partial \mathcal{U}_E^{tot}}{\partial x_1} = -ik_0 H_2^{tot}, \tag{16}$$

$$\frac{\partial E_2^{tot}}{\partial x_1} - \frac{\partial E_1^{tot}}{\partial x_2} = ik_0 \mathcal{U}_H^{tot}, \tag{17}$$

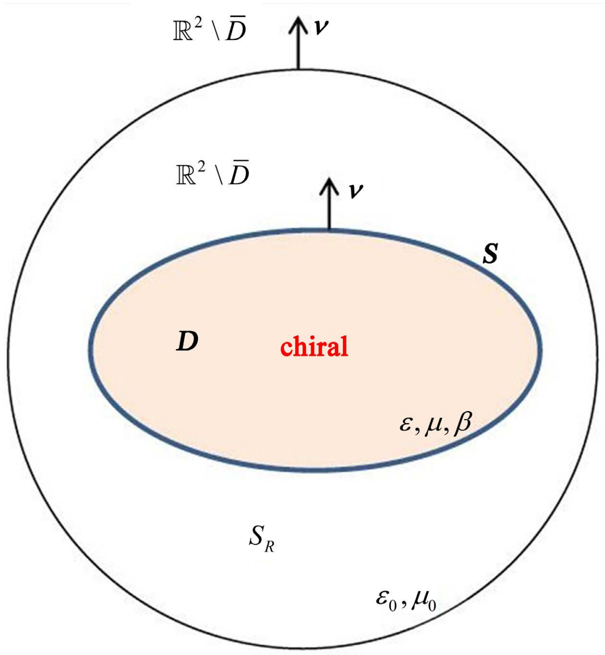

Figure 1. The cross-section in the x_1x_2-plane and the notation used in the two-dimensional problem.

$$\frac{\partial \mathcal{U}_H^{tot}}{\partial x_2} = -ik_0 E_1^{tot}, \tag{18}$$

$$\frac{\partial \mathcal{U}_H^{tot}}{\partial x_1} = ik_0 E_2^{tot}, \tag{19}$$

$$\frac{\partial H_2^{tot}}{\partial x_1} - \frac{\partial H_1^{tot}}{\partial x_2} = -ik_0 \mathcal{U}_E^{tot}. \tag{20}$$

Therefore, the x_3-components of $\boldsymbol{E}^{tot}$, $\boldsymbol{H}^{tot}$, *i.e.* $\mathcal{U}_E^{tot}$ and $\mathcal{U}_H^{tot}$, satisfy the following Helmholtz equation

$$\Delta \mathcal{U}_B^{tot} + k_0^2 \mathcal{U}_B^{tot} = 0 \text{ in } \mathbb{R}^2 \setminus \bar{D}, B = E, H, \tag{21}$$

where $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$ is the two-dimensional Laplace operator.

In order to eliminate the x_3 variable of the incident wave $\boldsymbol{E}^{inc}$, $\boldsymbol{H}^{inc}$, the direction of propagation has the form $\hat{\boldsymbol{d}} = (d_1, d_2, 0)$. For polarization vector $\boldsymbol{p} = (p_1, p_2, p_3)$, the x_3-component of the incident wave in two dimensions has the form

$$\mathcal{U}_E^{inc} = p_3 \mathrm{e}^{ik_0 \boldsymbol{x}\cdot\hat{\boldsymbol{d}}}, \quad \mathcal{U}_H^{inc} = (d_1 p_2 - d_2 p_1) \mathrm{e}^{ik_0 \boldsymbol{x}\cdot\hat{\boldsymbol{d}}}. \tag{22}$$

According to the form of the polarization vector $\boldsymbol{p}$, the incident wave can be reformulated. In particular, if $\boldsymbol{p} = (0,0,1)$, then $(\mathcal{U}_E^{inc}, \mathcal{U}_H^{inc}) = (\mathrm{e}^{ik_0 \boldsymbol{x}\cdot\hat{\boldsymbol{d}}}, 0)$ and $\mathcal{U}_E^{inc}$ will be referred to as E-wave. If $\boldsymbol{p} = (-d_2, d_1, 0)$, then $(\mathcal{U}_E^{inc}, \mathcal{U}_H^{inc}) = (0, \mathrm{e}^{ik_0 \boldsymbol{x}\cdot\hat{\boldsymbol{d}}})$ and $\mathcal{U}_H^{inc}$ will be referred to as H-wave. Also, in the case of $\boldsymbol{p} = (-d_2, d_1, 1)$, then $(\mathcal{U}_E^{inc}, \mathcal{U}_H^{inc}) = (\mathrm{e}^{ik_0 \boldsymbol{x}\cdot\hat{\boldsymbol{d}}}, \mathrm{e}^{ik_0 \boldsymbol{x}\cdot\hat{\boldsymbol{d}}})$ and it will be referred to as (E,H)-wave.

The total exterior fields $\mathcal{U}_E^{tot}$ and $\mathcal{U}_H^{tot}$ are given by

$$\mathcal{U}_B^{tot} = \mathcal{U}_B^{inc} + \mathcal{U}_B^{sc} \text{ in } \mathbb{R}^2 \setminus D, \; B = E, H, \tag{23}$$

where $\mathcal{U}_E^{sc}$ and $\mathcal{U}_H^{sc}$ are the x_3-components of the scattered fields $\boldsymbol{E}^{sc}$ and $\boldsymbol{H}^{sc}$, respectively and satisfy the Sommerfeld radiation condition

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial \mathcal{U}_B^{sc}}{\partial r} - ik_0 \mathcal{U}_B^{sc}\right) = 0, \; B = E, H. \tag{24}$$

Also, it holds the following asymptotic behaviour ([11], p. 62),

$$\mathcal{U}_B^{sc}(\boldsymbol{x}) = \frac{\mathrm{e}^{ik_0 r}}{\sqrt{r}} \mathcal{U}_B^{\infty}(\hat{\boldsymbol{x}}) + \mathcal{O}\left(r^{-3/2}\right), \; r \to \infty, \tag{25}$$

where $\mathcal{U}_B^{\infty}(\hat{\boldsymbol{x}})$ is the far field pattern which is given by

$$\mathcal{U}_B^{\infty}(\hat{\boldsymbol{x}}) = \frac{\mathrm{e}^{i\pi/4}}{\sqrt{8\pi k_0}} \int_S \left[\mathcal{U}_B^{sc}(\boldsymbol{y}) \frac{\partial}{\partial \nu} \mathrm{e}^{-ik_0 \hat{x}\cdot y} - \mathrm{e}^{-ik_0 \hat{x}\cdot y} \frac{\partial}{\partial \nu} \mathcal{U}_B^{sc}(\boldsymbol{y})\right] \mathrm{d}s(\boldsymbol{y}). \tag{26}$$

We also consider the two-dimensional Beltrami fields

$$\boldsymbol{Q}_L(x_1, x_2) = \left(Q_{L1}(x_1, x_2), Q_{L2}(x_1, x_2), U_L(x_1, x_2)\right), \tag{27}$$

$$\boldsymbol{Q}_R(x_1, x_2) = \left(Q_{R1}(x_1, x_2), Q_{R2}(x_1, x_2), U_R(x_1, x_2)\right). \tag{28}$$

By using the Equation (10) and the above representations of $\boldsymbol{Q}_L$ and $\boldsymbol{Q}_R$, we obtain the following equations in D

$$\frac{\partial U_L}{\partial x_2} = \gamma_L Q_{L1}, \tag{29}$$

$$\frac{\partial U_L}{\partial x_1} = -\gamma_L Q_{L2}, \tag{30}$$

$$\frac{\partial Q_{L2}}{\partial x_1} - \frac{\partial Q_{L1}}{\partial x_2} = \gamma_L U_L, \tag{31}$$

$$\frac{\partial U_R}{\partial x_2} = -\gamma_R Q_{R1}, \tag{32}$$

$$\frac{\partial U_R}{\partial x_1} = \gamma_R Q_{R2}, \tag{33}$$

$$\frac{\partial Q_{R2}}{\partial x_1} - \frac{\partial Q_{R1}}{\partial x_2} = -\gamma_R U_R, \tag{34}$$

which imply the following two-dimensional Helmholtz equations for the x_3-components of $\boldsymbol{Q}_L$, $\boldsymbol{Q}_R$

$$\Delta U_A + \gamma_A^2 U_A = 0 \text{ in } D, A = L, R. \tag{35}$$

In a similar sense, we express the transmission conditions (11), (12) in terms of the x_3-components of the corresponding fields. Specifically, we derive the following transmission conditions for the two-dimensional scattering problem

$$\mathcal{U}_E^{tot} = \sqrt{\varepsilon_0}\left(U_L + U_R\right) \text{ on } S, \tag{36}$$

$$\frac{\partial \mathcal{U}_E^{tot}}{\partial \nu} = \omega\mu_0\sqrt{\frac{\varepsilon\varepsilon_0}{\mu}}\left(\frac{1}{\gamma_L}\frac{\partial U_L}{\partial \nu}+\frac{1}{\gamma_R}\frac{\partial U_R}{\partial \nu}\right) \text{ on } S, \tag{37}$$

$$\mathcal{U}_H^{tot} = -i\sqrt{\frac{\varepsilon\mu_0}{\mu}}\left(U_L - U_R\right) \text{ on } S, \tag{38}$$

$$\frac{\partial \mathcal{U}_H^{tot}}{\partial \nu} = -i\omega\varepsilon_0\sqrt{\mu_0}\left(\frac{1}{\gamma_L}\frac{\partial U_L}{\partial \nu}-\frac{1}{\gamma_R}\frac{\partial U_R}{\partial \nu}\right) \text{ on } S. \tag{39}$$

Summarizing the above analysis, we are able to define a two-dimensional scattering problem. In particular, we will denote by (P) the two-dimensional scattering problem which is defined by the Equations (21)-(24) and (35)-(39). The existence and uniqueness of solutions of the scattering problem (P) have been proved by Gerlach in [7].

3. Scattering Relations

Let $\mathcal{U}_B^{inc}\left(\boldsymbol{x},\hat{\boldsymbol{d}}\right) = \mathrm{e}^{ik_0\boldsymbol{x}\cdot\hat{\boldsymbol{d}}}$ be an incident plane electric ($B = E$) or magnetic wave ($B = H$). We will denote the dependence of the total fields in $\mathbb{R}^2 \setminus \bar{D}$, of the scattered field and of the far field pattern for the scattering problem (P) on the incident direction $\hat{\boldsymbol{d}}$, by writing $\mathcal{U}_B^{tot}\left(\boldsymbol{x},\hat{\boldsymbol{d}}\right)$, $\mathcal{U}_B^{sc}\left(\boldsymbol{x},\hat{\boldsymbol{d}}\right)$, $\mathcal{U}_B^{\infty}\left(\hat{\boldsymbol{x}},\hat{\boldsymbol{d}}\right)$, $B = E, H$, respectively. Moreover, we will denote the dependence of the total Beltrami fields in D on the incident direction $\hat{\boldsymbol{d}}$, by writing $U_A\left(\boldsymbol{x},\hat{\boldsymbol{d}}\right)$, $A = L, R$.

Using polar coordinates $\boldsymbol{x} = \left(r\cos\theta, r\sin\theta\right)$, $r = |\boldsymbol{x}|$, $\hat{\boldsymbol{d}} = \left(\cos\phi,\sin\phi\right)$, $\theta,\phi \in [0,2\pi]$ and writing for convenience $\mathcal{U}_B^{\infty}\left(\hat{\boldsymbol{x}},\hat{\boldsymbol{d}}\right) = \mathcal{U}_B^{\infty}(\theta,\phi)$ the asymptotic behaviour (25) of the scattered field can be written as

$$\mathcal{U}_B^{sc}\left(\boldsymbol{x},\hat{\boldsymbol{d}}\right) = \frac{\mathrm{e}^{ik_0 r}}{\sqrt{r}}\mathcal{U}_B^{\infty}(\theta,\phi) + \mathcal{O}\left(r^{-3/2}\right), \quad r \to \infty. \tag{40}$$

Also, the far field pattern $\mathcal{U}_B^{\infty}(\theta,\phi)$ is expressed as

$$\mathcal{U}_B^{\infty}(\theta,\phi) = \frac{\mathrm{e}^{i\pi/4}}{\sqrt{8\pi k_0}}\int_S\left[\mathcal{U}_B^{sc}\frac{\partial}{\partial \nu_y}\mathrm{e}^{-ik_0 r_y\cos\left(\theta-\theta_y\right)} - \frac{\partial \mathcal{U}_B^{sc}}{\partial \nu_y}\mathrm{e}^{-ik_0 r_y\cos\left(\theta-\theta_y\right)}\right]\mathrm{d}s(\boldsymbol{y}), \tag{41}$$

where $\left(r_y,\theta_y\right)$ are the polar coordinates of $\boldsymbol{y}$.

In what follows, we will use Twersky's notation [15]

$$\{u,v\}_S = \int_S\left(u\frac{\partial v}{\partial \nu} - v\frac{\partial u}{\partial \nu}\right)\mathrm{d}s. \tag{42}$$

We note that the far field pattern can be expressed via the Twersky's notation as

$$\mathcal{U}_B^{\infty}(\theta,\phi) = \mathcal{U}_B^{\infty}\left(\hat{\boldsymbol{x}},\hat{\boldsymbol{d}}\right) = \frac{\mathrm{e}^{i\pi/4}}{\sqrt{8\pi k_0}}\left\{\mathcal{U}_B^{sc}\left(\cdot,\hat{\boldsymbol{d}}\right),\mathcal{U}_B^{inc}\left(\cdot,-\hat{\boldsymbol{x}}\right)\right\}_S. \tag{43}$$

We now are in the position to state and prove the following reciprocity theorem.

Theorem 1 (Reciprocity). Let $\left(\mathcal{U}_E^{inc}\left(\cdot,\hat{\boldsymbol{d}}\right),\mathcal{U}_H^{inc}\left(\cdot,\hat{\boldsymbol{d}}\right)\right)$ and $\left(\mathcal{U}_E^{inc}\left(\cdot,-\hat{\boldsymbol{x}}\right),\mathcal{U}_H^{inc}\left(\cdot,-\hat{\boldsymbol{x}}\right)\right)$, with $\hat{\boldsymbol{d}} = (\cos\phi,\sin\phi)$ and $\hat{\boldsymbol{x}} = (\cos\theta,\sin\theta)$, be two incident plane (E,H)-waves. Then the far field patterns $\mathcal{U}_E^{\infty}(\theta,\phi)$ and $\mathcal{U}_H^{\infty}(\theta,\phi)$

corresponding to the scattering problem (P) satisfy the reciprocity principle

$$\mathcal{U}_E^{\infty}(\theta,\phi)-\mathcal{U}_E^{\infty}(\phi+\pi,\theta+\pi)=\mathcal{U}_H^{\infty}(\theta,\phi)-\mathcal{U}_H^{\infty}(\phi+\pi,\theta+\pi), \tag{44}$$

for all $\theta,\phi\in[0,2\pi]$.

Proof. Let $\mathcal{U}_B^{sc}(\cdot,\hat{\boldsymbol{d}})$, $\mathcal{U}_B^{sc}(\cdot,-\hat{\boldsymbol{x}})$ and $\mathcal{U}_B^{tot}(\cdot,\hat{\boldsymbol{d}})$, $\mathcal{U}_B^{tot}(\cdot,-\hat{\boldsymbol{x}})$ be the corresponding scattered and total fields in $\mathbb{R}^2\setminus\bar{D}$ respectively. Considering the relation (23) and due to the bilinearity of the form (42), we obtain

$$\begin{aligned}&\left\{\mathcal{U}_B^{tot}(\cdot,\hat{\boldsymbol{d}}),\mathcal{U}_B^{tot}(\cdot,-\hat{\boldsymbol{x}})\right\}_S\\&=\left\{\mathcal{U}_B^{inc}(\cdot,\hat{\boldsymbol{d}}),\mathcal{U}_B^{inc}(\cdot,-\hat{\boldsymbol{x}})\right\}_S+\left\{\mathcal{U}_B^{inc}(\cdot,\hat{\boldsymbol{d}}),\mathcal{U}_B^{sc}(\cdot,-\hat{\boldsymbol{x}})\right\}_S\\&\quad+\left\{\mathcal{U}_B^{sc}(\cdot,\hat{\boldsymbol{d}}),\mathcal{U}_B^{inc}(\cdot,-\hat{\boldsymbol{x}})\right\}_S+\left\{\mathcal{U}_B^{sc}(\cdot,\hat{\boldsymbol{d}}),\mathcal{U}_B^{sc}(\cdot,-\hat{\boldsymbol{x}})\right\}_S.\end{aligned} \tag{45}$$

For the integral in the left-hand side of the above relation and for $B=E$, we use the transmission conditions (36), (37) and we have

$$\begin{aligned}&\left\{\mathcal{U}_E^{tot}(\cdot,\hat{\boldsymbol{d}}),\mathcal{U}_E^{tot}(\cdot,-\hat{\boldsymbol{x}})\right\}_S\\&=a\Bigg[\frac{1}{\gamma_L}\int_S\left(U_L(\cdot,\hat{\boldsymbol{d}})\frac{\partial U_L(\cdot,-\hat{\boldsymbol{x}})}{\partial\nu}-U_L(\cdot,-\hat{\boldsymbol{x}})\frac{\partial U_L(\cdot,\hat{\boldsymbol{d}})}{\partial\nu}\right)\mathrm{ds}\\&\quad+\frac{1}{\gamma_R}\int_S\left(U_R(\cdot,\hat{\boldsymbol{d}})\frac{\partial U_R(\cdot,-\hat{\boldsymbol{x}})}{\partial\nu}-U_R(\cdot,-\hat{\boldsymbol{x}})\frac{\partial U_R(\cdot,\hat{\boldsymbol{d}})}{\partial\nu}\right)\mathrm{ds}\\&\quad+\int_S\left(\frac{1}{\gamma_R}U_L(\cdot,\hat{\boldsymbol{d}})\frac{\partial U_R(\cdot,-\hat{\boldsymbol{x}})}{\partial\nu}-\frac{1}{\gamma_L}U_R(\cdot,-\hat{\boldsymbol{x}})\frac{\partial U_L(\cdot,\hat{\boldsymbol{d}})}{\partial\nu}\right)\mathrm{ds}\\&\quad+\int_S\left(\frac{1}{\gamma_L}U_R(\cdot,\hat{\boldsymbol{d}})\frac{\partial U_L(\cdot,-\hat{\boldsymbol{x}})}{\partial\nu}-\frac{1}{\gamma_R}U_L(\cdot,-\hat{\boldsymbol{x}})\frac{\partial U_R(\cdot,\hat{\boldsymbol{d}})}{\partial\nu}\right)\mathrm{ds}\Bigg],\end{aligned} \tag{46}$$

where $a=\omega\varepsilon_0\mu_0\sqrt{\dfrac{\varepsilon}{\mu}}$.

Applying the scalar Green's second theorem on $U_A(\cdot,\hat{\boldsymbol{d}})$ and $U_A(\cdot,-\hat{\boldsymbol{x}})$, $A=L,R$, in D and taking into account that $U_A(\cdot,\hat{\boldsymbol{d}})$, $U_A(\cdot,-\hat{\boldsymbol{x}})$ are solutions of the Helmholtz equation (35), the first two integrals of the right-hand side of relation (46) are equal to zero. In a similar way by using (38) and (39), we obtain

$$\begin{aligned}&\left\{\mathcal{U}_H^{tot}(\cdot,\hat{\boldsymbol{d}}),\mathcal{U}_H^{tot}(\cdot,-\hat{\boldsymbol{x}})\right\}_S\\&=a\Bigg[\int_S\left(\frac{1}{\gamma_R}U_L(\cdot,\hat{\boldsymbol{d}})\frac{\partial U_R(\cdot,-\hat{\boldsymbol{x}})}{\partial\nu}-\frac{1}{\gamma_L}U_R(\cdot,-\hat{\boldsymbol{x}})\frac{\partial U_L(\cdot,\hat{\boldsymbol{d}})}{\partial\nu}\right)\mathrm{ds}\\&\quad+\int_S\left(\frac{1}{\gamma_L}U_R(\cdot,\hat{\boldsymbol{d}})\frac{\partial U_L(\cdot,-\hat{\boldsymbol{x}})}{\partial\nu}-\frac{1}{\gamma_R}U_L(\cdot,-\hat{\boldsymbol{x}})\frac{\partial U_R(\cdot,\hat{\boldsymbol{d}})}{\partial\nu}\right)\mathrm{ds}\Bigg].\end{aligned} \tag{47}$$

Therefore, we have

$$\left\{\mathcal{U}_E^{tot}(\cdot,\hat{\boldsymbol{d}}),\mathcal{U}_E^{tot}(\cdot,-\hat{\boldsymbol{x}})\right\}_S-\left\{\mathcal{U}_H^{tot}(\cdot,\hat{\boldsymbol{d}}),\mathcal{U}_H^{tot}(\cdot,-\hat{\boldsymbol{x}})\right\}_S=0. \tag{48}$$

For the first integral in the right-hand side of the relation (45), due to the fact that $\mathcal{U}_B^{inc}(\cdot,\hat{\boldsymbol{d}})$ and $\mathcal{U}_B^{inc}(\cdot,-\hat{\boldsymbol{x}})$ are regular solutions of Helmholtz Equation (21) in *D*, applying the scalar Green's second theorem, we get

$$\left\{\mathcal{U}_B^{inc}(\cdot,\hat{\boldsymbol{d}}),\mathcal{U}_B^{inc}(\cdot,-\hat{\boldsymbol{x}})\right\}_S=0. \tag{49}$$

In order to compute the last integral in the right-hand side of (45), we consider a disc S_R centred at the origin with radius *R* large enough to include $\bar{D}$ in its interior.

Applying again the scalar Green's second theorem on $\mathcal{U}_B^{sc}(\cdot,\hat{\boldsymbol{d}})$ and $\mathcal{U}_B^{sc}(\cdot,-\hat{\boldsymbol{x}})$ in the region exterior to *S* and interior to ∂S_R, we can get that the desired integral is equal to the line integral on ∂S_R. Letting $R\to\infty$ and using (25), we have

$$\left\{\mathcal{U}_B^{sc}(\cdot,\hat{\boldsymbol{d}}),\mathcal{U}_B^{sc}(\cdot,-\hat{\boldsymbol{x}})\right\}_S=0. \tag{50}$$

From the relation (43), we have

$$\left\{\mathcal{U}_B^{sc}(\cdot,\hat{\boldsymbol{d}}),\mathcal{U}_B^{inc}(\cdot,-\hat{\boldsymbol{x}})\right\}_S=\sqrt{8\pi k_0}\,\mathrm{e}^{-i\pi/4}\,\mathcal{U}_B^{\infty}(\hat{\boldsymbol{x}},\hat{\boldsymbol{d}}), \tag{51}$$

$$\left\{\mathcal{U}_B^{inc}(\cdot,\hat{\boldsymbol{d}}),\mathcal{U}_B^{sc}(\cdot,-\hat{\boldsymbol{x}})\right\}_S=-\sqrt{8\pi k_0}\,\mathrm{e}^{-i\pi/4}\,\mathcal{U}_B^{\infty}(-\hat{\boldsymbol{d}},-\hat{\boldsymbol{x}}). \tag{52}$$

Applying (45) for $B=E,H$ and subtracting the two formulae, considering the relations (48)-(52), we obtain

$$\mathcal{U}_E^{\infty}(\hat{\boldsymbol{x}},\hat{\boldsymbol{d}})-\mathcal{U}_H^{\infty}(\hat{\boldsymbol{x}},\hat{\boldsymbol{d}})=\mathcal{U}_E^{\infty}(-\hat{\boldsymbol{d}},-\hat{\boldsymbol{x}})-\mathcal{U}_H^{\infty}(-\hat{\boldsymbol{d}},-\hat{\boldsymbol{x}}), \tag{53}$$

□

which proves the theorem.

We now consider incident E-waves, *i.e.*

$$\mathcal{U}_E^{tot}=\mathcal{U}_E^{inc}+\mathcal{U}_E^{sc},\quad \mathcal{U}_H^{tot}=\mathcal{U}_H^{sc}, \tag{54}$$

or incident H-waves, *i.e.*

$$\mathcal{U}_E^{tot}=\mathcal{U}_E^{sc},\quad \mathcal{U}_H^{tot}=\mathcal{U}_H^{inc}+\mathcal{U}_H^{sc}. \tag{55}$$

For these cases the reciprocity principle (44) takes the following form.

Corollary 1. Let $\mathcal{U}_B^{inc}(\cdot,\hat{\boldsymbol{d}})$ and $\mathcal{U}_B^{inc}(\cdot,-\hat{\boldsymbol{x}})$, with $\hat{\boldsymbol{d}}=(\cos\phi,\sin\phi)$, $\hat{\boldsymbol{x}}=(\cos\theta,\sin\theta)$, be two incident plane B-waves, $B=E,H$. Then the far field pattern $\mathcal{U}_B^{\infty}(\theta,\phi)$ corresponding to the scattering problem (*P*) satisfies the reciprocity principle

$$\mathcal{U}_B^{\infty}(\theta,\phi)=\mathcal{U}_B^{\infty}(\phi+\pi,\theta+\pi), \tag{56}$$

for all $\theta,\phi\in[0,2\pi]$.

Proof. In case of incident E-waves, we have $\mathcal{U}_H^{inc}=0$ and by applying the relation (45) for $B=H$ we get $\left\{\mathcal{U}_H^{tot}(\cdot,\hat{\boldsymbol{d}}),\mathcal{U}_H^{tot}(\cdot,-\hat{\boldsymbol{x}})\right\}_S=0$. Therefore, from (48) we have $\left\{\mathcal{U}_E^{tot}(\cdot,\hat{\boldsymbol{d}}),\mathcal{U}_E^{tot}(\cdot,-\hat{\boldsymbol{x}})\right\}_S=0$ and through the relation (45) for $B=E$ we obtain $\mathcal{U}_E^{\infty}(\theta,\phi)=\mathcal{U}_E^{\infty}(\phi+\pi,\theta+\pi)$. Similarly, for incident H-waves we obtain the corresponding result. □

In the sequel, $\overline{w}$ denotes the complex conjugate of w.

Theorem 2 (General scattering). Let $\left(\mathcal{U}_E^{inc}\left(\cdot,\hat{\boldsymbol{d}}\right),\mathcal{U}_H^{inc}\left(\cdot,\hat{\boldsymbol{d}}\right)\right)$ and $\left(\mathcal{U}_E^{inc}\left(\cdot,\hat{\boldsymbol{x}}\right),\mathcal{U}_H^{inc}\left(\cdot,\hat{\boldsymbol{x}}\right)\right)$, with $\hat{\boldsymbol{d}}=\left(\cos\phi,\sin\phi\right)$ and $\hat{\boldsymbol{x}}=\left(\cos\theta,\sin\theta\right)$, be two incident plane (E,H)-waves. Then the far field patterns $\mathcal{U}_B^{\infty}\left(\theta,\phi\right)$, $B=E,H$, corresponding to the scattering problem (P) satisfy

$$\begin{aligned}&\mathrm{e}^{-i\pi/4}\mathcal{U}_E^{\infty}\left(\theta,\phi\right)-\mathrm{e}^{i\pi/4}\overline{\mathcal{U}_E^{\infty}\left(\theta,\phi\right)}-i\sqrt{\frac{k_0}{2\pi}}\int_0^{2\pi}\overline{\mathcal{U}_E^{\infty}\left(\theta_y,\phi\right)}\,\mathcal{U}_E^{\infty}\left(\theta_y,\theta\right)\mathrm{d}s\left(\theta_y\right)\\&=-e^{-i\pi/4}\mathcal{U}_H^{\infty}\left(\theta,\phi\right)+\mathrm{e}^{i\pi/4}\overline{\mathcal{U}_H^{\infty}\left(\theta,\phi\right)}+i\sqrt{\frac{k_0}{2\pi}}\int_0^{2\pi}\overline{\mathcal{U}_H^{\infty}\left(\theta_y,\phi\right)}\,\mathcal{U}_H^{\infty}\left(\theta_y,\theta\right)\mathrm{d}s\left(\theta_y\right).\end{aligned}\tag{57}$$

Proof. Let $\mathcal{U}_B^{sc}\left(\cdot,\hat{\boldsymbol{d}}\right)$, $\mathcal{U}_B^{sc}\left(\cdot,\hat{\boldsymbol{x}}\right)$ and $\mathcal{U}_B^{tot}\left(\cdot,\hat{\boldsymbol{d}}\right)$, $\mathcal{U}_B^{tot}\left(\cdot,\hat{\boldsymbol{x}}\right)$, $B=E,H$, be the corresponding scattered and total fields in $\mathbb{R}^2\setminus\overline{D}$ respectively. This theorem is proved following a similar analysis with the proof of Theorem 1. We briefly describe the basic steps of the proof. It holds

$$\begin{aligned}&\left\{\overline{\mathcal{U}_B^{tot}\left(\cdot,\hat{\boldsymbol{d}}\right)},\mathcal{U}_B^{tot}\left(\cdot,\hat{\boldsymbol{x}}\right)\right\}_S\\&=\left\{\overline{\mathcal{U}_B^{inc}\left(\cdot,\hat{\boldsymbol{d}}\right)},\mathcal{U}_B^{inc}\left(\cdot,\hat{\boldsymbol{x}}\right)\right\}_S+\left\{\overline{\mathcal{U}_B^{inc}\left(\cdot,\hat{\boldsymbol{d}}\right)},\mathcal{U}_B^{sc}\left(\cdot,\hat{\boldsymbol{x}}\right)\right\}_S\\&\quad+\left\{\overline{\mathcal{U}_B^{sc}\left(\cdot,\hat{\boldsymbol{d}}\right)},\mathcal{U}_B^{inc}\left(\cdot,\hat{\boldsymbol{x}}\right)\right\}_S+\left\{\overline{\mathcal{U}_B^{sc}\left(\cdot,\hat{\boldsymbol{d}}\right)},\mathcal{U}_B^{sc}\left(\cdot,\hat{\boldsymbol{x}}\right)\right\}_S.\end{aligned}\tag{58}$$

For the first integral, which concerns the total fields, we have

$$\begin{aligned}&\left\{\overline{\mathcal{U}_E^{tot}\left(\cdot,\hat{\boldsymbol{d}}\right)},\mathcal{U}_E^{tot}\left(\cdot,\hat{\boldsymbol{x}}\right)\right\}_S\\&=a\left[\int_S\left(\frac{1}{\gamma_R}\overline{U_L\left(\cdot,\hat{\boldsymbol{d}}\right)}\frac{\partial U_R\left(\cdot,\hat{\boldsymbol{x}}\right)}{\partial\nu}-\frac{1}{\gamma_L}U_R\left(\cdot,\hat{\boldsymbol{x}}\right)\frac{\partial\overline{U_L\left(\cdot,\hat{\boldsymbol{d}}\right)}}{\partial\nu}\right)\mathrm{d}s\right.\\&\quad\left.+\int_S\left(\frac{1}{\gamma_L}\overline{U_R\left(\cdot,\hat{\boldsymbol{d}}\right)}\frac{\partial U_L\left(\cdot,\hat{\boldsymbol{x}}\right)}{\partial\nu}-\frac{1}{\gamma_R}U_L\left(\cdot,\hat{\boldsymbol{x}}\right)\frac{\partial\overline{U_R\left(\cdot,\hat{\boldsymbol{d}}\right)}}{\partial\nu}\right)\mathrm{d}s\right]\\&=-\left\{\overline{\mathcal{U}_H^{tot}\left(\cdot,\hat{\boldsymbol{d}}\right)},\mathcal{U}_H^{tot}\left(\cdot,\hat{\boldsymbol{x}}\right)\right\}_S\end{aligned}\tag{59}$$

and hence we get

$$\left\{\overline{\mathcal{U}_E^{tot}\left(\cdot,\hat{\boldsymbol{d}}\right)},\mathcal{U}_E^{tot}\left(\cdot,\hat{\boldsymbol{x}}\right)\right\}_S+\left\{\overline{\mathcal{U}_H^{tot}\left(\cdot,\hat{\boldsymbol{d}}\right)},\mathcal{U}_H^{tot}\left(\cdot,\hat{\boldsymbol{x}}\right)\right\}_S=0.\tag{60}$$

For the integral which involves the two incident waves of (58), we apply the scalar Green's second theorem and taking into account that $\mathcal{U}_B^{inc}\left(\cdot,\hat{\boldsymbol{d}}\right)$, $\mathcal{U}_B^{inc}\left(\cdot,\hat{\boldsymbol{x}}\right)$ are regular solutions of the Helmholtz equation (21) in D, we directly obtain

$$\left\{\overline{\mathcal{U}_B^{inc}\left(\cdot,\hat{\boldsymbol{d}}\right)},\mathcal{U}_B^{inc}\left(\cdot,\hat{\boldsymbol{x}}\right)\right\}_S=0.\tag{61}$$

Moreover, we derive the following relations

$$\left\{\overline{\mathcal{U}_B^{inc}\left(\cdot,\hat{\boldsymbol{d}}\right)},\mathcal{U}_B^{sc}\left(\cdot,\hat{\boldsymbol{x}}\right)\right\}_S=-\sqrt{8\pi k_0}\,\mathrm{e}^{-i\pi/4}\mathcal{U}_B^{\infty}\left(\hat{\boldsymbol{d}},\hat{\boldsymbol{x}}\right),\tag{62}$$

$$\left\{\overline{\mathcal{U}_B^{sc}\left(\cdot,\hat{\boldsymbol{d}}\right)},\mathcal{U}_B^{inc}\left(\cdot,\hat{\boldsymbol{x}}\right)\right\}_S=\sqrt{8\pi k_0}\,\mathrm{e}^{i\pi/4}\overline{\mathcal{U}_B^{\infty}\left(\hat{\boldsymbol{x}},\hat{\boldsymbol{d}}\right)}, \tag{63}$$

$$\left\{\overline{\mathcal{U}_B^{sc}\left(\cdot,\hat{\boldsymbol{d}}\right)},\mathcal{U}_B^{sc}\left(\cdot,\hat{\boldsymbol{x}}\right)\right\}_S=2ik_0\int_0^{2\pi}\overline{\mathcal{U}_B^{\infty}\left(\hat{\boldsymbol{y}},\hat{\boldsymbol{d}}\right)}\mathcal{U}_B^{\infty}\left(\hat{\boldsymbol{y}},\hat{\boldsymbol{x}}\right)\mathrm{d}s\left(\hat{\boldsymbol{y}}\right). \tag{64}$$

Adding the two generated relations (58) for $B=E,H$, and considering the relations (60)-(64), we derive

$$\begin{aligned}&\mathrm{e}^{-i\pi/4}\mathcal{U}_E^{\infty}\left(\hat{\boldsymbol{d}},\hat{\boldsymbol{x}}\right)-\mathrm{e}^{i\pi/4}\overline{\mathcal{U}_E^{\infty}\left(\hat{\boldsymbol{x}},\hat{\boldsymbol{d}}\right)}-i\sqrt{\frac{k_0}{2\pi}}\int_0^{2\pi}\overline{\mathcal{U}_E^{\infty}\left(\hat{\boldsymbol{y}},\hat{\boldsymbol{d}}\right)}\mathcal{U}_E^{\infty}\left(\hat{\boldsymbol{y}},\hat{\boldsymbol{x}}\right)\mathrm{d}s\left(\hat{\boldsymbol{y}}\right)\\&=-\mathrm{e}^{-i\pi/4}\mathcal{U}_H^{\infty}\left(\hat{\boldsymbol{d}},\hat{\boldsymbol{x}}\right)+\mathrm{e}^{i\pi/4}\overline{\mathcal{U}_H^{\infty}\left(\hat{\boldsymbol{x}},\hat{\boldsymbol{d}}\right)}+i\sqrt{\frac{k_0}{2\pi}}\int_0^{2\pi}\overline{\mathcal{U}_H^{\infty}\left(\hat{\boldsymbol{y}},\hat{\boldsymbol{d}}\right)}\mathcal{U}_H^{\infty}\left(\hat{\boldsymbol{y}},\hat{\boldsymbol{x}}\right)\mathrm{d}s\left(\hat{\boldsymbol{y}}\right),\end{aligned} \tag{65}$$

which proves the theorem.

□

Corollary 2. Let $\mathcal{U}_B^{inc}\left(\cdot,\hat{\boldsymbol{d}}\right)$ and $\mathcal{U}_B^{inc}\left(\cdot,\hat{\boldsymbol{x}}\right)$, with $\hat{\boldsymbol{d}}=\left(\cos\phi,\sin\phi\right)$ and $\hat{\boldsymbol{x}}=\left(\cos\theta,\sin\theta\right)$, be two incident plane B-waves, B = E,H. Then the far field pattern $\mathcal{U}_B^{\infty}\left(\theta,\phi\right)$ corresponding to the scattering problem (P) with the additional assumption that $B^{sc}=\left(0,0,\mathcal{U}_B^{sc}\right)$ satisfies

$$\mathrm{e}^{-i\pi/4}\mathcal{U}_B^{\infty}\left(\theta,\phi\right)-\mathrm{e}^{i\pi/4}\overline{\mathcal{U}_B^{\infty}\left(\theta,\phi\right)}=i\sqrt{\frac{k_0}{2\pi}}\int_0^{2\pi}\overline{\mathcal{U}_B^{\infty}\left(\theta_y,\phi\right)}\mathcal{U}_B^{\infty}\left(\theta_y,\theta\right)\mathrm{d}s\left(\theta_y\right), \tag{66}$$

for all $\theta,\phi\in\left[0,2\pi\right]$.

Proof. For $B=E$ we have that $\boldsymbol{E}^{sc}=\left(0,0,\mathcal{U}_E^{sc}\right)$, $\boldsymbol{E}^{inc}=\left(0,0,\mathcal{U}_E^{inc}\right)$ and hence $\boldsymbol{E}^{tot}=\left(0,0,\mathcal{U}_E^{tot}\right)$. Considering the Maxwell equation $curl\boldsymbol{E}^{tot}=ik_0\boldsymbol{H}^{tot}$, we get $\boldsymbol{H}^{tot}=\left(H_1^{tot},H_2^{tot},0\right)$, *i.e.* $\mathcal{U}_H^{tot}=0$ in $\mathbb{R}^2\setminus\bar{D}$.

By using the transmission conditions (38), (39) and due to the continuity, we obtain the equalities

$$U_L=U_R,\ \frac{1}{\gamma_L}\frac{\partial U_L}{\partial\nu}=\frac{1}{\gamma_R}\frac{\partial U_R}{\partial\nu}\ \text{ on } S. \tag{67}$$

From relation (59) and using (67), we have $\left\{\overline{\mathcal{U}_E^{tot}\left(\cdot,\hat{\boldsymbol{d}}\right)},\mathcal{U}_E^{tot}\left(\cdot,\hat{\boldsymbol{x}}\right)\right\}_S=0$, which proves the corollary for $B=E$.

Similarly, if $B=H$ we have $\mathcal{U}_E^{tot}=0$, due to the Maxwell equation $curl\boldsymbol{H}^{tot}=-ik_0\boldsymbol{E}^{tot}$. In this case we utilize the transmission conditions (36) and (37), which lead to the relations

$$U_L=-U_R,\ \frac{1}{\gamma_L}\frac{\partial U_L}{\partial\nu}=-\frac{1}{\gamma_R}\frac{\partial U_R}{\partial\nu}\ \text{ on } S, \tag{68}$$

and from (59) we have $\left\{\overline{\mathcal{U}_H^{tot}\left(\cdot,\hat{\boldsymbol{d}}\right)},\mathcal{U}_H^{tot}\left(\cdot,\hat{\boldsymbol{x}}\right)\right\}_S=0$, which completes the proof of the corollary.

□

The scattering cross-section σ^{sc} is a measure of the disturbance caused by the chiral scatterer to the incident wave. It is defined by $\sigma^{sc}=\frac{1}{2\pi}\int_0^{2\pi}\sigma\left(\theta\right)\mathrm{d}\theta$, where $\sigma\left(\theta\right)$ is the differential scattering cross-section given by $\sigma\left(\theta\right)=2\pi\left|\mathcal{U}^{\infty}\left(\theta\right)\right|^2$ [23]. Then we have

$$\sigma^{sc} = \int_0^{2\pi} \left| \mathcal{U}^{\infty}(\theta) \right|^2 \mathrm{d}\theta. \tag{69}$$

Theorem 3 (Optical). Let $\left(\mathcal{U}_E^{inc}(\cdot, \hat{\boldsymbol{d}}), \mathcal{U}_H^{inc}(\cdot, \hat{\boldsymbol{d}}) \right)$ with $\hat{\boldsymbol{d}} = (\cos\phi, \sin\phi)$ be an incident plane (E,H)-wave and $\mathcal{U}_E^{\infty}$, $\mathcal{U}_H^{\infty}$ the far field patterns corresponding to the scattering problem (P). Then we have

$$\sigma_E^{sc} + \sigma_H^{sc} = \sqrt{\frac{8\pi}{k_0}} Im\left[\mathrm{e}^{-i\pi/4} \left(\mathcal{U}_E^{\infty}(\phi,\phi) + \mathcal{U}_H^{\infty}(\phi,\phi) \right) \right]. \tag{70}$$

Proof. Applying Theorem 2 for $\theta = \phi$ we derive

$$\begin{aligned} & Im\left[\mathrm{e}^{-i\pi/4} \mathcal{U}_E^{\infty}(\phi,\phi) \right] - \sqrt{\frac{k_0}{8\pi}} \int_0^{2\pi} \left| \mathcal{U}_E^{\infty}(\theta) \right|^2 \mathrm{d}s(\theta) \\ & = -Im\left[\mathrm{e}^{-i\pi/4} \mathcal{U}_H^{\infty}(\phi,\phi) \right] + \sqrt{\frac{k_0}{8\pi}} \int_0^{2\pi} \left| \mathcal{U}_H^{\infty}(\theta) \right|^2 \mathrm{d}s(\theta). \end{aligned} \tag{71}$$

From (69) and (71), we obtain the relation (70).

□

Corollary 3. Let $\mathcal{U}_B^{inc}(\boldsymbol{x}, \hat{\boldsymbol{d}})$ with $\hat{\boldsymbol{d}} = (\cos\phi, \sin\phi)$ be an incident B-wave and $\mathcal{U}_B^{\infty}$ the corresponding far field pattern of the scattering problem (P) with the additional assumption that $B^{sc} = (0, 0, \mathcal{U}_B^{sc})$, $B = E, H$. Then the scattering cross-section satisfies

$$\sigma_B^{sc} = \sqrt{\frac{8\pi}{k_0}} Im\left[\mathrm{e}^{-i\pi/4} \mathcal{U}_B^{\infty}(\phi,\phi) \right], \quad B = E, H. \tag{72}$$

Proof. We apply Corollary 2 for $\theta = \phi$ and we derive

$$Im\left[\mathrm{e}^{-i\pi/4} \mathcal{U}_B^{\infty}(\phi,\phi) \right] = \sqrt{\frac{k_0}{8\pi}} \int_0^{2\pi} \left| \mathcal{U}_B^{\infty}(\theta) \right|^2 \mathrm{d}s(\theta). \tag{73}$$

From relations (69) and (73), we obtain (72).

□

4. The Far Field Operator

Two meaningful notions, that play a central role in studying inverse scattering problems, are the Herglotz wave functions and the far field operators. In the present scattering problem, the Herglotz wave function is an entire solution of the Helmholtz Equation (21) and it is given by

$$\mathcal{U}_g(\theta) = \int_0^{2\pi} g(\phi) \mathrm{e}^{ik_0 r \cos(\theta - \phi)} \mathrm{d}\phi, \tag{74}$$

with kernel $g \in L^2[0, 2\pi]$.

The far field operator $F_B : L^2[0, 2\pi] \to L^2[0, 2\pi]$ corresponding to the far field pattern $\mathcal{U}_B^{\infty}$ is defined by

$$(F_B g)(\theta) := \int_0^{2\pi} \mathcal{U}_B^{\infty}(\theta, \phi) g(\phi) \mathrm{d}\phi. \tag{75}$$

We now consider as incident field a Herglotz wave function of the form (74), which will be denoted by $\mathcal{U}_{B,g}^{inc}$ with $B = E, H$, and we prove the formulae given in the following corollary.

Corollary 4. We consider two incident Herglotz wave functions $\mathcal{U}_{B,g}^{inc}$ and

$\mathcal{U}_{B,h}^{inc}$, $B=E,H$. Let $\mathcal{U}_{B,g}^{sc}$, $\mathcal{U}_{B,h}^{sc}$ and $\mathcal{U}_{B,g}^{\infty}$, $\mathcal{U}_{B,h}^{\infty}$ be the corresponding scattered fields and far field patterns for the scattering problem (P), respectively. Then we have

$$\left\{\mathcal{U}_{B,g}^{sc}, \overline{\mathcal{U}_{B,h}^{inc}}\right\}_S = \sqrt{8\pi k_0}\,\mathrm{e}^{-i\pi/4}\int_0^{2\pi}\overline{h(\phi)}\,\mathcal{U}_{B,g}^{\infty}(\phi)\,\mathrm{d}\phi, \tag{76}$$

$$\left\{\overline{\mathcal{U}_{B,g}^{sc}}, \mathcal{U}_{B,h}^{sc}\right\}_S = 2ik_0\int_0^{2\pi}\overline{\mathcal{U}_{B,g}^{\infty}(\phi)}\,\mathcal{U}_{B,h}^{\infty}(\phi)\,\mathrm{d}\phi. \tag{77}$$

Proof. From the definition of Herglotz wave function (74) and by using the relation (43), we have

$$\begin{aligned}\left\{\mathcal{U}_{B,g}^{sc}, \overline{\mathcal{U}_{B,h}^{inc}}\right\}_S &= \int_0^{2\pi}\overline{h(\phi)}\left\{\mathcal{U}_{B,g}^{sc}, \mathrm{e}^{-ik_0\boldsymbol{x}\cdot\hat{\boldsymbol{d}}}\right\}_S\mathrm{d}\phi\\ &= \sqrt{8\pi k_0}\,\mathrm{e}^{-i\pi/4}\int_0^{2\pi}\overline{h(\phi)}\,\mathcal{U}_{B,g}^{\infty}(\phi)\,\mathrm{d}\phi,\end{aligned} \tag{78}$$

where $\hat{\boldsymbol{d}} = (\cos\phi, \sin\phi)$. The relation (64) leads to the generation of (77). □

Let us denote the inner product on $L^2[0,2\pi]$ by $\langle\cdot,\cdot\rangle$. Then we can formulate the following theorem.

Theorem 4. Let $\left(\mathcal{U}_{E,g}^{inc}, \mathcal{U}_{H,g}^{inc}\right)$ and $\left(\mathcal{U}_{E,h}^{inc}, \mathcal{U}_{H,h}^{inc}\right)$ be two incident Herglotz wave functions. Then the far field operator $F_B : L^2[0,2\pi] \to L^2[0,2\pi]$, $B=E,H$, corresponding to the scattering problem (P) satisfies the relation

$$\begin{aligned}&\mathrm{e}^{-i\pi/4}\langle F_E g, h\rangle - \mathrm{e}^{i\pi/4}\langle g, F_E h\rangle - i\sqrt{\frac{k_0}{2\pi}}\langle F_E g, F_E h\rangle\\ &= -\mathrm{e}^{-i\pi/4}\langle F_H g, h\rangle + \mathrm{e}^{i\pi/4}\langle g, F_H h\rangle + i\sqrt{\frac{k_0}{2\pi}}\langle F_H g, F_H h\rangle,\end{aligned} \tag{79}$$

for every $g,h \in L^2[0,2\pi]$.

Proof. As it is well-known, the far field operator $F_B g$ is the far field pattern corresponding to the incident field $\mathcal{U}_{B,g}^{inc}$ ([22], p. 231). Applying Theorem 2 by considering as incident fields Herglotz wave functions $\mathcal{U}_{B,g}^{inc}$ and $\mathcal{U}_{B,h}^{inc}$ and taking into consideration the relations (76), (77), Theorem 2 is reformulated and leads directly to the proof of the present theorem. □

Corollary 5. Let $\mathcal{U}_{B,g}^{inc}$ and $\mathcal{U}_{B,h}^{inc}$ be two incident Herglotz wave functions. Then the far field operator $F_B : L^2[0,2\pi] \to L^2[0,2\pi]$ corresponding to the scattering problem (P) with the additional assumption that $B^{sc} = \left(0,0,\mathcal{U}_B^{sc}\right)$, $B=E,H$, satisfies the relation

$$\mathrm{e}^{-i\pi/4}\langle F_B g, h\rangle - \mathrm{e}^{i\pi/4}\langle g, F_B h\rangle = i\sqrt{\frac{k_0}{2\pi}}\langle F_B g, F_B h\rangle. \tag{80}$$

Corollary 6. The far field operator $F_B : L^2[0,2\pi] \to L^2[0,2\pi]$ corresponding to the scattering problem (P) with the additional assumption that

$B^{sc} = \left(0,0,\mathcal{U}_B^{sc}\right)$ is normal and its eigenvalues lie on the circle of radius $\sqrt{\frac{2\pi}{k_0}}$ with center at $\mathrm{e}^{3\pi i/4}\sqrt{\frac{2\pi}{k_0}}$.

Proof. This corollary is proved as Theorem 7.15 presented in ([11], p. 144).

□

When the incident field is a Herglotz type wave $\left(\mathcal{U}_{E,g}^{inc},\mathcal{U}_{H,g}^{inc}\right)$, then two far field patterns $\mathcal{U}_E^{\infty}$, $\mathcal{U}_H^{\infty}$ are derived and hence two far field operators $F_E g$, $F_H g$, for $g \in L^2[0,2\pi]$, are defined. In this case for the two-dimensional scattering problem (P) we can define an operator $\mathbb{F}: L^2[0,2\pi] \to L^2[0,2\pi]$ with

$$(\mathbb{F}g)(\theta) = \int_0^{2\pi}\left[\mathcal{U}_E^{\infty}(\theta,\phi) + \mathcal{U}_H^{\infty}(\theta,\phi)\right] g(\phi)\,\mathrm{d}\phi. \tag{81}$$

It is clear that $\mathbb{F} = F_E + F_H$ and the following theorem is valid.

Theorem 5. Let $\left(\mathcal{U}_{E,g}^{inc},\mathcal{U}_{H,g}^{inc}\right)$ and $\left(\mathcal{U}_{E,h}^{inc},\mathcal{U}_{H,h}^{inc}\right)$ be two incident Herglotz type waves. The operator $\mathbb{F}: L^2[0,2\pi] \to L^2[0,2\pi]$ corresponding to the scattering problem (P) satisfies the relation

$$\mathrm{e}^{-i\pi/4}\langle \mathbb{F}g, h\rangle - \mathrm{e}^{i\pi/4}\langle g, \mathbb{F}h\rangle = i\sqrt{\frac{k_0}{2\pi}}\langle \mathbb{F}g, \mathbb{F}h\rangle. \tag{82}$$

Proof. From relation (79) we obtain the relation (82). □

5. Reduction to Achiral Case

When the scatterer is an achiral dielectric, *i.e.* $\beta = 0$, then it holds $\gamma_L = \gamma_R = \gamma = k = \omega\sqrt{\varepsilon\mu}$. In this case, the relations which are presented in Section 2 can be simplified. Specifically, the Equations (6) and (7) are rewritten as

$$curl\boldsymbol{E} = ik\boldsymbol{H},\ curl\boldsymbol{H} = -ik\boldsymbol{E} \ \text{ in } \tilde{D}. \tag{83}$$

The x_3-components U_E, U_H of $\boldsymbol{E}$, $\boldsymbol{H}$ respectively satisfy the following Helmholtz equation

$$\Delta U_B + k^2 U_B = 0 \ \text{ in } D,\ B = E, H. \tag{84}$$

Moreover, the transmission conditions (36)-(39) are reformulated as follows

$$\mathcal{U}_E^{tot} = \frac{\sqrt{\varepsilon_0}}{\sqrt{\varepsilon}} U_E \ \text{ on } S, \tag{85}$$

$$\frac{\partial \mathcal{U}_E^{tot}}{\partial \nu} = \frac{\mu_0\sqrt{\varepsilon_0}}{\mu\sqrt{\varepsilon}}\frac{\partial U_E}{\partial \nu} \ \text{ on } S, \tag{86}$$

$$\mathcal{U}_H^{tot} = \frac{\sqrt{\mu_0}}{\sqrt{\mu}} U_H \ \text{ on } S, \tag{87}$$

$$\frac{\partial \mathcal{U}_H^{tot}}{\partial \nu} = \frac{\varepsilon_0\sqrt{\mu_0}}{\varepsilon\sqrt{\mu}}\frac{\partial U_H}{\partial \nu} \ \text{ on } S. \tag{88}$$

The Equations (83) and (84), the transmission conditions (85)-(88) are included in [24] and hence the corresponding scattering relations can be obtained in a similar way.

6. Conclusion

In the present work, we formulated a two-dimensional electromagnetic scattering problem for a chiral scatterer in an achiral environment. Although the interior of the scatterer is a chiral material, we proved that it holds the reciprocity principle (56) in Corollary 1 and a general scattering relation (66) in Corollary 2. The scattering relations that have been proved as well as the far field operator play a central role in the solution of inverse scattering problems. In particular, using the reciprocity principle and appropriate conditions for our scattering problem we can prove, as in the achiral case [11], that the far field operator is injective and its range is dense. These properties are used to apply the linear sampling method for studying inverse scattering problems. From the process of proof we see that the reciprocity principle can be proved for more general scattering models (multi-layered scatterer, mixed boundary value problem, etc., [11]). If $\beta = 0$, then it holds $\gamma_L = \gamma_R = \gamma = k = \omega\sqrt{\varepsilon\mu}$ and the achiral scattering problem can be written as two decoupled scattering problems for the electric and the magnetic field. In this case the presented relations cover the corresponding ones which already have been proposed in the literature [11] [24]. Specifically, the reciprocity principle given by (56) and the general scattering theorem given by (66) are separately stated for electric and magnetic fields. Moreover, the operators F_B and $\mathbb{F}$ can be used for studying inverse scattering problems. We aim to study this problem in a future work. Specifically, we will prove a mixed reciprocity theorem which will be used to prove the uniqueness of an inverse scattering problem. Also, we will use the factorization method in order to specify the shape of the scatterer and its chirality.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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