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The Non-Equivalence of Pyramids and Their Pseudo-Cones: Important New Insights

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Abstract

The simulation of indentations with so called "equivalent" pseudo-cones for decreasing computer time is challenged. The mimicry of pseudo-cones having equal basal surface and depth with pyramidal indenters is excluded by basic arithmetic and trigonometric calculations. The commonly accepted angles of so called "equivalent" pseudo-cones must not also claim equal depth. Such bias (answers put into the questions to be solved) in the historical values of the generally used half-opening angles of pseudo-cones is revealed. It falsifies all simulations or conclusions on that basis. The enormous errors in the resulting hardness H_{ISO} and elastic modulus $E_{r\text{-}ISO}$ values are disastrous not only for the artificial intelligence. The straightforward deduction for possibly ψ-cones (ψ for pseudo) without biased depths' errors for equal basal surface and equal volume is reported. These ψ-cones would of course penetrate much more deeply than the three-sided Berkovich and cube corner pyramids ($r < a/2$), and their half-opening angles would be smaller than those of the respective pyramids (reverse with $r > a/2$ for four-sided Vickers). Also the unlike forces' direction angles are reported for the more sideward and the resulting downward directions. They are reflected by the diameter of the parallelograms with length and off-angle from the vertical axis. Experimental loading curves before and after the phase-transition onsets are indispensable. Mimicry of ψ-cones and pyramids is also quantitatively excluded. All simulations on their bases would also be dangerously invalid for industrial and solid pharmaceutical materials.

Keywords

Basic Mathematics, Extreme Errors, False Cone Angles, Indentation, Unphysical Cone Models for Pyramids, Undue Simulations

1. Introduction

The common indentations with equilateral three-sided pyramids facilitate simulations when pyramids are treated as equivalent cones. The formulas for so-called "equivalent" cone models are easier for iterative simulations and require considerably less computer time when mimicking the pyramids (e.g. [1] [2] etc.). The questionable "assumption" is that equal basal surfaces would lead to "equal penetration depths". Thus [3], ISO 14577, so guided textbooks and indentation simulation reports choose between pyramids and "equivalent cones". But how is the so-called "equivalency" of pyramid models founded? The mathematical formulas for the areas $A_{\text{triangle}} = A_{\text{circle}}$ of equilateral triangle $(3^{0.5}/4)a^2$ and circle πr^2 (**Figure 1**) are rightfully equated. But we showed in [4] that no "equivalent" behavior results in such commonly believed conditions. Pyramids and pseudo-cones behaved unlike for various reasons. But we could not yet quantify the sizes of the differences. We look now for correct deductions.

The equal area radius r of the ψ-cone (ψ for pseudo) is transformed into the a-unit from the pyramid with correct mathematics. This will remain the handle for the calculation of equal volumes and unequal heights. These heights are equal to the penetration depths in the absence of pile-up and hidden internal migrations along cleavage planes or channels [4]. When these are present, the penetration depth is smaller than the calculated height [4]. The generally used angles of 70.2996˚ for Berkovich pseudo-cone, 42.28˚ for the cubecorner pseudo-cone, and 70.32˚ or 70.2996˚ for Vickers-pseudo cone in the literature are used for all time saving simulations. This has been challenged in [4] for various geometric reasons and will now be quantified.

2. Methods

All calculations used a common scientific pocket calculator Rebell® SC2030 with 10 digits. All of them were used and results are reasonable rounded only when necessary. The worldwide unchangeable angles of diamond Berkovich and cube corner indenters were taken as fixed crystallographically approved quantities. Tip rounding is always removed in physically analyzed indentation curves as part of initial effects and did not interfere. Only undeniable trigonometry and mathematical formulas for the basal areas and volumes were used for the mathematical deductions without prejudice and without data-fitting.

3. Results

The literature values of half opening angles of the pseudo-cones for Berkovich, cube corner, and Vickers indenters (of 70.2996˚, 42.28˚, and 70.32 or 70.2996˚ respectively) are incorrectly made to have the same basal areas, volumes, and heights as the pyramids. Historically, heights equivalence ("$h_{\text{pyramid}} = h_{\text{cone}}$") might have been a "necessity", because the indentation depths are used for ISO-hardness and ISO-indentation modulus. We deduce here equal basal area and equal volume for ψ-cones but unequal depths. The used terms are indicated in **Figure 1(a)** (taken from [4]) and in **Figure 1(b)**. The three-times flat and totally different all around circular force fields alone should have halted using

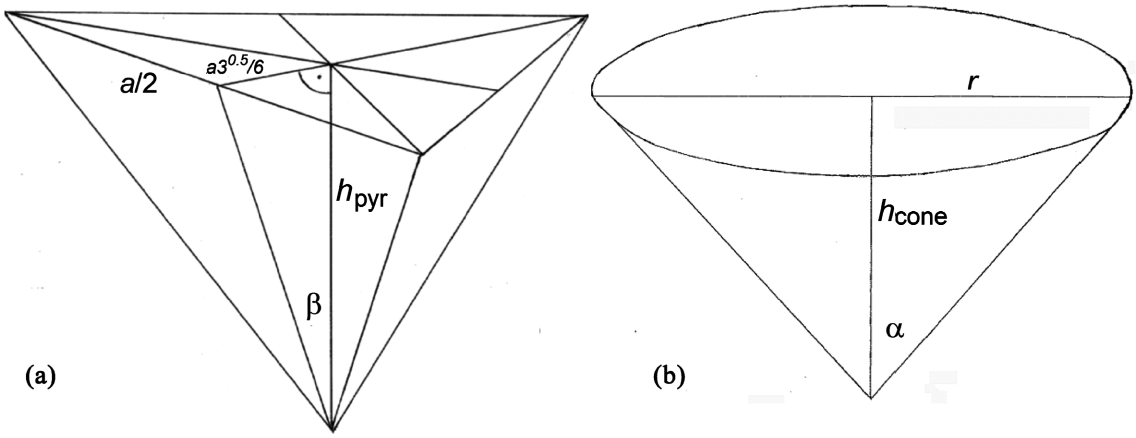

Figure 1. Perspective images for (a) a three-sided pyramid and (b) a cone (not true to scale).

pseudo cones!

3.1. Error Discovery on the Deduction Ways of the Common "Equivalent" Pseudo-Cones

The comparison of the so-called "equivalent cones" with the corresponding three-sided normal pyramids requires straightforward basic algebra and trigonometry with always 10 significant figures, due to numerous irrational numbers with numerous equations. We test on the basis of the known formulas for equal-sided triangle, circle, pyramid and cone by equating the triangle $A_{\text{triangle}} = a^2 3^{0.5}/4$ and circle areas $A_{\text{circle}} = \pi r^2$. Such equality is the basis for pseudo-cones. One obtains from the 2/1 ratio at the central cut of the equal-sided triangle heights $r^2 = 3^{0.5} a^2 / 4\pi = 0.1378322 a^2$ and $r = 0.371257624a$ (**Figure 1**). The pyramidal angle $\tan\beta = 3^{0.5} a / 6 h_{\text{pyr}}$ and the pyramidal depth $h_{\text{pyr}} = 3^{0.5} a / 6 \tan\beta$, where β is the well-known half-angle of the diamond Berkovich ($\beta = 65.27^\circ$) or of cubecorner ($\beta = 35.264^\circ$). For the pseudo-cone we have

$h_{\text{pseudo-cone}} = r/\tan\alpha = 3^{0.25} a / 2\pi^{0.5} \tan\alpha$.

One obtains $h_{\text{pseudo-cone}} / h_{\text{pyr}} = 3^{0.25} a 6 \tan\beta / 2\pi^{0.5} a 3^{0.5} \tan\alpha$. For Berkovich with $\beta_{\text{-B}} = 65.27^\circ$ and $\tan\beta = 2.171160716$ results $h_{\text{pseudo-cone}}/h_{\text{pyr}} = 2.792413659/\tan\alpha$. Here comes the historical error: Only by setting $h_{\text{pseudo-cone}}/h_{\text{pyr}}$ to 1, which is the same as dividing V_{cone} over V_{cone}, was the divisor h_{pyr} equal to the dividend h_{cone}. Such setting is absolutely cheating: It is putting a desired answer into the question. The dividend 2.792413659 is taken as an unbelievably biased "tanα" from the cone to give "$\alpha_{\text{cone}} = 70.29688723^\circ$" (undistinguishable from the less precisely calculated common 70.2996˚; maybe historical equalization with Vickers?) in the case of Berkovich. It was falsely created, spread, and believed. Unimaginably, despite the correctly calculated equal basal circle-surface area, where r is smaller than 0.5a and also smaller than half of the basal triangle height? It directly indicates, without any further calculation effort, that the pseudo-cone must be sharper but not blunter than the pyramid with $\beta = 65.27^\circ$.

Surprisingly, the corresponding bias was repeated for cubecorner$_{\text{-c}}$ with $\beta_{\text{-c}} = 35.264^\circ$ and $h_{\text{pseudocone-c}}/h_{\text{pyr-c}} = 0.909378623/\tan\alpha$ where the divisor was falsely made to "$\tan\alpha_{\text{-c}}$ and thus $\alpha_{\text{-c}}$ to 42.282713˚". This bias is the commonly used false "value of 42.28˚".

The four-sided Vickers indenter is more often used in industries. The biased published angle values for the "equivalent" pseudo-cone$_{\text{-V}}$ are 70.32˚ or 70.2996˚: As above, the corresponding unbelievable trick was used for precisely obtaining the second of these values.

All these false pseudo-cone angles withstood for more than 30 years until the apparently first challenge started with [4]. Involved scientists, authors, reviewers, funding providers, textbook writers, academic teachers, and industrial users did not check and complain. But apparently, all of them liked a "same depth" for pyramids and their pseudo-cone heights at the same force. Any "equality" of these pseudo-cones and pyramids with the faulty biased angles is now strictly excluded.

Our error discovery clearly reveals the disastrous historical "deductions", more than about 30 years ago. Every hardness measurement (e.g. $H_{\text{ISO}} = F_{\text{N}} / h_{\text{contact}}^{2}$ or $E_{\text{r-ISO}}$) by simulations with iterating data-fitting that used this type of pseudo-cones (notwithstanding the unphysical exponent on h that should be 3/2 instead [5] [6]) is also obsolete for that reason. Unfortunately, these very frequent unphysical simulations create severe risks with the technical materials' characterizations. An unbiased deduction of pseudo-cone geometries is thus very important for the quantification of the huge involved errors. It will become evident in Section 3.2.

3.2. Deduction Test for Unbiased ψ-Cones with Correct Volumes and Heights

We start with the equalized basal areas for the expression of radius r in units of the three-sided pyramidal side length a (**Figure 1**). For the correct deduction of unbiased ψ-cones (now ψ for pseudo) with the equal volume (as required by the energy law) the unequal heights of the pyramids and ψ-cones ensue. The requirement of $A_{\text{triangle-pyr}} = A_{\text{circle-cone}}$ gives Equation (1).

$$a^2 3^{0.5} / 4 = \pi r^2 \quad \text{and} \quad r^2 = a^2 3^{0.5} / 4\pi \quad \text{with} \quad r = a 3^{0.25} / 2\pi^{0.5} \tag{1}$$

With $V_{\text{pyr}} = Ah_{\text{pyr}}/3$ and $V_{\text{cone}} = Ah_{\text{cone}}/3$ the respective heights are $h_{\text{pyr}} = 3^{0.5}a/6\tan\beta$ and $h_{\text{cone}} = r/\tan\alpha = 3^{0.25}a/2\pi^{0.5}\tan\alpha$. The respective volumes are $V_{\text{pyr}} = a^3/24\tan\beta_{\text{pyr}}$ and $V_{\psi\text{-cone}} = a^3 3^{0.125}/24\pi^{0.5}\tan\alpha_{\text{cone}}$ after substitutions and simplifications. For Berkovich ($\beta_{\text{-B}} = 65.27˚$) we calculate $V_{\text{pyr-B}} = 0.019190963a^3$ at $h_{\text{pyr-B}} = 0.132958897a$ and for its ψ-cone $V_{\psi\text{-cone-B}} = 0.019190963a^3$ at $h_{\psi\text{-cone-B}} = 0.264191103a$. For cubecorner ($\beta_{\text{-c}} = 35.264˚$) we calculate $V_{\text{pyr-c}} = 0.058926415a^3$ at $h_{\text{pyr-c}} = 0.408254180a$ and for $V_{\psi\text{-cone-c}} = 0.058926415a^3$ at $h_{\psi\text{-cone-c}} = 0.811206506a$. The height values calculate unequal for pyramid and its ψ-cone. We use the energy law that requires equalizing the volumes at equal force to obtain Equation (2), and there from Equation (3). This allows for the calculation of $\tan\alpha_{\psi\text{-cone}}$.

$$a^3 / 24\tan\beta_{\text{pyr}} = a^3 3^{0.125} / 24\pi^{0.5}\tan\alpha_{\psi\text{-cone}} \tag{2}$$

$$\tan\alpha_{\psi\text{-cone}} = 3^{0.125}\tan\beta_{\text{pyr}}/\pi^{0.5} = 0.647239808\tan\beta_{\text{pyr}} \tag{3}$$

The angles α_{cone} are thus 54.563917˚ for the Berkovich-ψ-cone (as compared with commonly 70.2996˚) and 24.591634 for the cubecorner-ψ-cone (as compared with commonly 42.28˚).

For four-sided Vickers ($\beta_{\text{-v}}$ = 68˚) is $r = a/\pi^{0.5}$ larger than $a/2$ for the ψ-cone$_{\text{-v}}$. Thus, $h_{\text{pyr-v}}$ = 0.20201a is now larger than $h_{\psi\text{-cone-v}}$ = 0.19741a and also the angle $\alpha_{\text{cone-v}}$ = 70.71521˚ (here larger than $\beta_{\text{-v}}$) for the Vickers-ψ-cone (as compared with commonly 70.32˚ or 70.2996˚). Mimicry is also here excluded.

The different bracketed values from Section 3.1 compare the still stubbornly used common pseudo-cone angle values. These huge angle faults of the biased common values of Section 3.1 are enormous for the biased simulations of (nano)indentations. They make them completely worthless.

Our results with so many decimals demonstrate the precision of the used arithmetic. They have to be rounded to the precision of the β-angles. We must stress that they represent the height of the indenters. The penetration depths are only equal to the heights in the absence of pile-up and internal migrations upon indentation. These cases require corrections for their depth decreases, as reported in [4]. The sideward influences had been exhaustively exemplified in [7].

We do not encourage using the non-biased ψ-cones for simulations. On the contrary: Pyramids and their ψ-cones are also not equivalent due to their different sloping angles. The unbiased Berkovich- and cubecorner-ψ-cones would penetrate about two times deeper (49.67%) than the pyramids. The now completed challenge of [4] was therein already evident but required this final quantifying deduction. When the unbiased ψ-cones would be used for simulations their outputs would also be incorrect for the unequal directions. Such simulations with whatever mimicking cones must never be tried again; the existing ones must be deleted. Phase-transition onsets under load must be experimentally detected and for technical objects strictly avoided upon operation, because polymorph-interfaces promote disastrous cracking (e.g. at airliners) [4] [7] [8]. Phase-transitions play also their important role in pharmaceutical solids (e.g. two polymorphs of crystallized *cis*-platinum [9]).

3.3. The Depth Directions for the Forces in Pyramids as Compared with Their ψ-Cones

It is our duty now to calculate the differences between the pyramids and their unbiased ψ-cones without data-fitting. The calculated sideward force component angles vertical to the indenter slopes of the pyramid are 90˚ − 65.27˚ = 24.73˚ for Berkovich and 90˚ − 54.564˚ = 35.436˚ for its ψ-cone. In the case of cube corner we have correspondingly 90˚ − 35.264˚ = 54.736˚ and 90˚ − 24.5916˚ = 65.4084˚. These directional angles with respect to the central vertical axis are now 15.73˚ and 18.03˚ respectively steeper than in [4] where the biased false common pseudo-cone α_{cone}-angles had been used. **Figure 2** exemplifies it with the cube corner angles. It

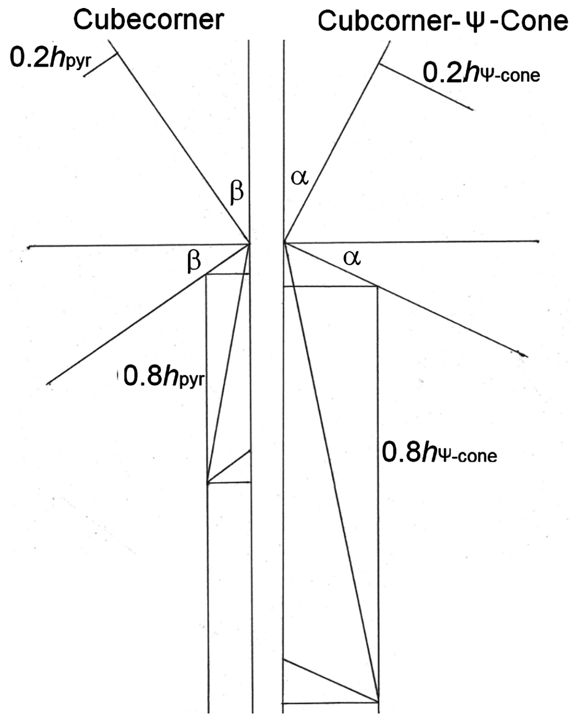

Figure 2. The depth directions diagram with the angles of the cube corner (left side) in relation to its vertical axis and of its ψ-cone (right side) in relation to its vertical axis.

depicts the enlarged pyramidal cross-section of one from the flat triangular force-fields and for its ψ-cone the enlarged cross-section of the circular force-field all around (cf **Figure 1**). They are set close to each other for immediately observing the enormous differences, e.g. their depth differences. The geometric questions (including off-angle and length of the diameter in the parallelograms) are trigonometrically evident.

The sidewise angles (lesser down) with respect to the horizontal axis are equal to the β-angles of the pyramids and the α-angles of the ψ-cones (cf **Figure 1**). They indicate the flatness of the sidewise force component. It is much flatter for cube corner than for Berkovich and it had already been told in [4] that this qualifies the cube corner for fracture toughness determinations. Here, the ψ-cone models with the unbiased α-angles would be flatter than the pyramids. But that excludes their mimicry power completely. Also simulations with the new mimicking models could again not take care of the slope-angle influence in relation to materials' cleavage planes or channels. There is no pass by 1) at the use of pyramidal geometry and 2) at the prior experimental detection of the phase-transition onset with depth and force [5] [6].

For the calculation of the resulting downward direction we distinguish the downward and sideward depths with their long known undeniable 80:20 ratio [6] to obtain the directional parallelogram from the pyramidal apex at both sides of **Figure 2**. It is calculated with the respective sine, tangent, and cosine functions. The parallelograms are characterized by their smaller angle ($90 - \beta_{pyr}$) for the pyramid or ($90 - \alpha_{\psi\text{-cone}}$) for the ψ-cone. Their sides are the respective fractions of h_{pyr} or $h_{\psi\text{-cone}}$: 0.2 times for sideward and 0.8 times for downward direction. For the calculation of the resulting diameter length and off-angle we add the small top triangles to the bottom of the parallelogram. The so obtained right angle triangle gives the resulting downward depth direction and its off-angle with the vertical axis. **Table 1** compares the pyramidal and ψ-conical angle and lengths to show how much they differ from each other. From there we can calculate the forces by using the experimental indentation of individual materials with $F_N = kh^{3/2}$ [5] in their calculated directions up to the (by simulations unavailable) phase-transition onset. From such onset force we start with a physically and chemically different polymorph. We can also calculate the different directions of the not mimicking ψ-cone for comparison to see how much the error of ψ-cones would further increase when using these.

Table 1 shows the calculated slightly rounded depth directions and angles of the more sideward and the resulting downward directions. The forces at these directions are obtained by using the physically deduced [5] formula $F_N = kh^{3/2}$ after determination of the physical hardness k (mN/µm$^{3/2}$) (F_N is the normal indentation force) from the slope of the indented material's loading curve. All values in **Table 1** are larger for the sharper ψ-cones that do not mimic.

We must stress, that the resulting vertical force direction departs significantly from the vertical applied axis of the indentation.

The commonly disregarded differences between the pyramids and their biased pseudo-cones (with equal heights) or unbiased ψ-cones (with enormous height differences) are very large. But both are in fact not mimicking the pyramids.

Our quantification of the huge differences between pyramids and their ψ-cones makes obsolete any use for simulations of (nano)indentations. Their false claimed results are extremely dangerous for the use of technical including solid pharmaceutical materials [9], the mechanical properties of which must be very precisely known.

4. Conclusions

The purpose of this paper is to discourage any use of simulations by using faster

Table 1. Depth direction angles, heights and lengths for Berkovich and cube corner indentations and for the respective ψ-cone models.

Indenter with β-Angle or ψ-cone[a)]	Sideward[b)] DeepAngle	Sideward[c)] Flat Angle	Vertical off-Angle	0.8 Indenter Height[d)]	Diagonal Length
Berkovich (65.27˚)	24.73˚	65.27˚	4.8894˚	0.106367a	0.130997a
Berkovich-ψ-Cone	35.436˚	54.564˚	6.9163˚	0.211353a[a)]	0.256268a
Cube corner (35.264˚	54.754˚	35.246˚	10.2979˚	0.326603a	0.379948a
Cube corner-ψ-Cone	65.4083˚	24.5917˚	11.8992˚	0.648965a[a)]	0.731194a

[a)]Instead of r the a-fractions from the pyramid is used for the equal basal area calculation; [b)]in relation to the center axis; [c)]in relation to the horizontal axis; [d)]the height and length values represent the mathematical 8/2 ratio of the force distribution directions downward and sideward [6] in the absence of pile-up and hidden internal migration apart from the created half volume diameter [4].

calculated cones. Physically sound undeniable mathematic calculations are reliable and much easier. The false common angles of the widely used pseudo-cones are severely biased. Their use for simulations to save computer time is strongly falsifying. All such simulations are obsolete and dangerous. They cannot simulate phase-transition onsets and they violate the energy law by excluding the 20% loss of normal force for not-penetrating events [6]. These simulations try to help themselves with a multitude of further iterative "work-hardening" simulations. Such published "results" cannot be repaired and must be fully extinguished. Also our tentatively deducted unbiased ψ-cones are not mimicking the pyramids. Three-sided pyramid-pseudo-cones are sharper and would go deeper than the pyramids with unlike force directions. Advanced simulations with the new unbiased ψ-cones are also impossible, because the force direction influences respond to specific materials properties. These must be experimentally determined (phase-transition onsets, cleavage planes' or channels' or holes' orientations and widths) [7].

Computer time is only saved by physical analysis using basic mathematical calculations, avoiding simulations and data-fittings. That requires characterization with properly analyzed pyramidal (or with real cones) indentations. Only these reveal the previously ignored sharp phase-transition onsets and energies under load. One needs crystallographic investigations for the pile-up questions in case of materials' anisotropy. Indentations are most important for the rapid optimization of materials' properties with respect to their safety, when exposed to unavoidable forces. It is important to always stay below any materials' phase-transition onset force to avoid the cracking-risk. Simulations and data-fitting iterations produce dangerous risks with false H_{ISO} or false $E_{\mathrm{r\text{-}ISO}}$ and therefrom derived mechanical properties of technical materials and by denying phase-transitions. Beware of using simulated and fitted indentation data for artificial intelligence applications. They are on the Internet but they must be urgently disregarded and stopped.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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