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Existence and Multiplicity of Positive Solutions for a Singular Third-Order Three-Point Boundary Value Problem with a Parameter

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Abstract

In this paper, we investigate the existence of positive solutions for a singular third-order three-point boundary value problem with a parameter. By using fixed point index theory, some existence, multiplicity and nonexistence results for positive solutions are derived in terms of different values of λ.

Keywords

Three-Point Boundary Value Problem, Fixed Point Index, Positive Solution, Existence, Multiplicity

1. Introduction

In this paper, we are concerned with the existence, multiplicity and nonexistence of positive solutions for the following third-order boundary value problem (BVP for short):

$$u'''(t) = \lambda q(t) f(t, u(t)),\ t \in (0,1), \tag{1}$$

$$u(0) = \alpha u(\eta),\ u'(\eta) = 0,\ u''(1) = 0, \tag{2}$$

where $\alpha \in (0,1)$, $\eta \in \left[\frac{1}{2}, 1\right)$ are constants, λ is a positive parameter, $q:(0,1) \to [0,\infty)$, $f:[0,1]\times[0,\infty) \to [0,\infty)$ are continuous and $q(t)$ may be singular at $t = 0$ and 1.

Third-order differential equations arise in a variety of different areas of applied mathematics and physics, e.g., in the deflection of curved beam having a constant or varying cross section, a three-layer beam, electromagnetic waves or gravity driven flows and so on [1]. In recent years, third-order boundary value problems have been studied by many methods [2]-[10], such as upper and lower solutions method, monotone iterative method and the different fixed point theory, etc.

In [11], Sun proved the existence of triple positive solutions to the following BVP by using a fixed-point theorem due to Avery and Peterson:

$$u'''(t) = a(t) f(t, u(t), u'(t), u''(t)), \ t \in (0,1),$$
$$u(0) = \delta u(\eta), \ u'(\eta) = 0, \ u''(1) = 0,$$

where $\delta \in (0,1)$, $\eta \in \left[\frac{1}{2}, 1\right)$ are constants. $a:(0,1) \to [0,\infty)$ and $f:[0,1] \times [0,\infty) \times R \times R \to [0,\infty)$ are continuous.

By applying the Krasnoselskii's fixed point theorem, Sun [12] established the existence of infinitely many solutions to the following BVP, which is the special case for $\alpha = 0$ in BVP (1) and (2):

$$u'''(t) = \lambda a(t) F(t, u(t)), \ t \in (0,1),$$
$$u(0) = u'(\eta) = u''(1) = 0,$$

with $\lambda > 0$, $\eta \in \left[\frac{1}{2}, 1\right)$, where $a(t)$ is nonnegative continuous function defined on $(0,1)$ and $F:[0,1] \times [0,\infty) \to [0,\infty)$ is continuous, $a(t)$ may be singular at $t = 0$ and/or $t = 1$.

Motivated by the above works, here we study the third order BVP (1) and (2). Under certain suitable conditions, we establish the results of existence, multiplicity and nonexistence of positive solutions for BVP (1) and (2) via the fixed point index theory.

2. Preliminaries

In this section, we present some notation and Lemmas that will be used in subsequent sections.

Lemma 2.1. [11] Let $\alpha \neq 1$, $h \in C[0,1]$, then the BVP

$$u'''(t) = h(t), \ t \in (0,1), \tag{3}$$

$$u(0) = \alpha u(\eta), \ u'(\eta) = 0, \ u''(1) = 0, \tag{4}$$

has a unique solution

$$u(t) = \int_0^1 G(t,s) h(s) \mathrm{d}s,$$

where

$$G(t,s) = \begin{cases} \dfrac{s^2}{2(1-\alpha)}, & s \le t, \ s \le \eta, \\ -\dfrac{1}{2}t^2 + ts + \dfrac{\alpha s^2}{2(1-\alpha)}, & t \le s \le \eta, \\ \dfrac{1}{2}s^2 - ts + \eta t + \dfrac{\alpha \eta^2}{2(1-\alpha)}, & \eta \le s \le t, \\ -\dfrac{1}{2}t^2 + \eta t + \dfrac{\alpha \eta^2}{2(1-\alpha)}, & \eta \le s, \ t \le s. \end{cases} \tag{5}$$

Lemma 2.2. [11] Suppose $0<\alpha<1$, $\frac{1}{2}\le\eta<1$, $g(s)=\frac{1}{2(1-\alpha)}\min\{s^2,\eta^2\}$, then

$$\alpha g(s)\le G(t,s)\le g(s),\ t,s\in[0,1].$$

Let $E=C[0,1]$ be equipped with norm $\|u\|=\max_{t\in[0,1]}|u(t)|$, then $(E,\|\cdot\|)$ is a real Banach space.

From Lemma 2.2, we know that if $0<\alpha<1$, $\frac{1}{2}\le\eta<1$, then for $h\in C^+[0,1]=\{x\in C[0,1]:x(t)\ge 0,t\in[0,1]\}$, the unique solution $u(t)$ of BVP (2.1) and (2.2) is nonnegative and satisfies

$$\min_{t\in[0,1]}u(t)\ge\alpha\|u\|.$$

Define the cone P by

$$P=\left\{u\in C[0,1]:\min_{t\in[0,1]}u(t)\ge\alpha\|u\|\right\},$$

then P is a non-empty closed and convex subset of *E*.

For $u,v\in E$, we write $u\le v$ if $u(t)\le v(t)$ for any $t\in[0,1]$. For any $r>0$, let $K_r=\{u\in E:\|u\|<r\}$ and $\partial K_r=\{u\in E:\|u\|=r\}$.

Define the operator $T:P\to E$ by

$$(Tu)(t)=\int_0^1 G(t,s)q(s)f(s,u(s))\mathrm{d}s. \qquad (6)$$

In view of the Lemma 2.1, it is easy to see that u is a positive solution BVP (1) and (2) if and only if u is a fixed point of the operator λT.

In the following, we assume that:

(H_1) $q(t)\ge 0$, $q(t)\not\equiv 0$ and $\int_0^1 q(s)\mathrm{d}s<\infty$.

(H_2) $f\in C([0,1]\times[0,\infty),[0,\infty))$, $f(t,u)$ is non-decreasing in u and $f(t,u)>0$ for any $t\in[0,1]$, $u>0$.

Lemma 2.3. Assume (H_1)-(H_2) hold, then the operator $T:P\to P$ is completely continuous.

Proof. For $u\in P$, according to the definition of T and Lemma 2.2, it is easy to prove that $T(P)\subset P$. By the Ascoli-Arzela theorem, it is easy to show $T:P\to P$ is completely continuous.

The proofs of our main theorems are based on the fixed index theory. The following three well-known Lemmas in [13] [14].

Lemma 2.4. Let E be a Banach space and $P\subset E$ be a cone in *E*. Assume that Ω is a bounded open subset of *E*. Suppose that $T:P\cap\bar{\Omega}\to P$ is a completely continuous operator. If there exists $x_0\in P\backslash\{\theta\}$ such that $x-Tx\ne\mu x_0$, for all $x\in P\cap\partial\Omega$ and $\mu\ge 0$, then the fixed point index $i(T,P\cap\Omega,P)=0$.

Lemma 2.5. Let E be a Banach space and $P\subset E$ be a cone in *E*. Assume that Ω is a bounded open subset of *E*. Suppose that $T:P\cap\bar{\Omega}\to P$ is a completely continuous operator. If $\inf_{x\in P\cap\partial\Omega}\|Tx\|>0$ and $\mu Tx\ne x$ for $x\in P\cap\partial\Omega$ and

$\mu \ge 1$, then the fixed point index $i(T, P \cap \Omega, P) = 0$.

Lemma 2.6. Let E be a Banach space and $P \subset E$ be a cone in E. Assume that Ω is a bounded open subset of E with $\theta \in \Omega$. Suppose that $T : P \cap \bar{\Omega} \to P$ is a completely continuous operator. If $Tx \ne \mu x$ for all $x \in P \cap \partial\Omega$ and $\mu \ge 1$, then the fixed point index $i(T, P \cap \Omega, P) = 1$.

Now for convenience we use the following notations. Let

$$f^0 = \lim_{x \to 0^+} \max_{t \in [0,1]} \frac{f(t,x)}{x}, \quad f^\infty = \lim_{x \to +\infty} \max_{t \in [0,1]} \frac{f(t,x)}{x},$$

$$f_0 = \lim_{x \to 0^+} \min_{t \in [0,1]} \frac{f(t,x)}{x}, \quad f_\infty = \lim_{x \to +\infty} \min_{t \in [0,1]} \frac{f(t,x)}{x},$$

$$A = \int_0^1 g(s) q(s) \mathrm{d}s,$$

$$\Phi = \{(\lambda, u) : \lambda > 0, u \in P \text{ is a positive solution of BVP } (1.1) \text{ and } (1.2)\};$$

$$\Lambda = \{\lambda > 0 : \text{there exists } u \in P \text{ such that } (\lambda, u) \in \Phi\};$$

$$\lambda^* = \sup \Lambda, \quad \lambda_* = \inf \Lambda.$$

3. The Main Results and Proofs

Lemma 3.1. Suppose (H_1) holds and $f_0 = \infty$, then $\Phi \ne \varnothing$.

Proof. Let $R > 0$ be fixed, then we can choose $\lambda_0 > 0$ small enough such that $\lambda_0 \sup_{u \in P \cap \bar{K}_R} \|Tu\| < R$. It is easy to see that

$$\lambda_0 Tu \ne \mu u, \quad \forall u \in P \cap \partial K_R, \mu \ge 1.$$

By Lemma 2.6, it follows that

$$i(\lambda_0 T, P \cap K_R, P) = 1. \tag{7}$$

From $f_0 = \infty$, it follows that there exists $r \in (0, R)$ such that

$$f(t,x) \ge \frac{1}{\lambda_0 \alpha^2 A} x, \quad \forall x \in [0, r], \ t \in [0,1]. \tag{8}$$

We may suppose that $\lambda_0 T$ has no fixed point on $P \cap \partial K_r$. Otherwise, the proof is finished. Let $e(t) \equiv 1$ for $t \in [0,1]$, Then $e \in \partial K_1$. We claim that

$$u \ne \lambda_0 Tu + \mu e, \quad \forall u \in P \cap \partial K_r, \ \mu \ge 0. \tag{9}$$

In fact, if not, there exist $u_1 \in P \cap \partial K_r$ and $\mu_1 \ge 0$ such that $u_1 = \lambda_0 Tu_1 + \mu_1 e$, then $\mu_1 > 0$. For $u_1 \in P \cap \partial K_r$ and $\mu_1 > 0$, by Lemma 2.2 and (8), we have

$$\begin{aligned}
\|u_1\| \ge u_1(t) &= (\lambda_0 Tu_1)(t) + \mu_1 e(t) \\
&= \lambda_0 \int_0^1 G(t,s) q(s) f(s, u_1(s)) \mathrm{d}s + \mu_1 \\
&\ge \alpha \lambda_0 \int_0^1 g(s) q(s) f(s, u_1(s)) \mathrm{d}s + \mu_1 \\
&\ge \alpha \lambda_0 \frac{1}{\alpha^2 \lambda_0 A} \int_0^1 g(s) q(s) u_1(s) \mathrm{d}s + \mu_1 \\
&\ge \frac{1}{A} \|u_1\| \int_0^1 g(s) q(s) \mathrm{d}s + \mu_1 = \|u_1\| + \mu_1 = r + \mu_1,
\end{aligned}$$

we get $r \ge r + \mu_1$, which is a contradiction. Hence by Lemma 2.4, it follows that

$$i\left(\lambda_0 T, P \cap K_r, P\right) = 0. \quad (10)$$

By virtue of the additivity of the fixed point index, by (7) and (10), we have

$$i\left(\lambda_0 T, P \cap \left(K_R \setminus \bar{K}_r\right), P\right) = i\left(\lambda_0 T, P \cap K_R, P\right) - i\left(\lambda_0 T, P \cap K_r, P\right) = 1,$$

which implies that the nonlinear operator $\lambda_0 T$ has one fixed point $u_0 \in P \cap \left(K_R \setminus \bar{K}_r\right)$. Therefore, $\left(\lambda_0, u_0\right) \in \Phi$. The proof is complete.

Lemma 3.2. Suppose (H_1) and (H_2) hold, $f^\infty = 0$, then $\Phi \neq \varnothing$.

Proof. Let $r > 0$ be fixed. From (H_2) and the definition of cone *P*, it follows that there exists $C > 0$ such that $f\left(t, u(t)\right) \geq C$ for all $t \in [0,1]$ and $u \in P \cap \partial K_r$. Then for sufficiently large λ with $\lambda > \frac{r}{AC\alpha}$ and $u \in P \cap \partial K_r$, we have

$$\begin{aligned}(\lambda Tu)(t) &= \lambda \int_0^1 G(t,s) q(s) f\left(s, u(s)\right) \mathrm{d}s \\ &\geq \lambda \alpha C \int_0^1 g(s) q(s) \mathrm{d}s > r, \ t \in [0,1].\end{aligned}$$

This implies that $\inf_{u \in P \cap \partial K_r} \|\lambda Tu\| > 0$ and $\mu \lambda Tu \neq u$ for $u \in P \cap \partial K_r$, $\mu \geq 1$. By Lemma 2.5, it follows that

$$i\left(\lambda T, P \cap K_r, P\right) = 0. \quad (11)$$

From $f^\infty = 0$, there exists $R > r$ such that

$$f(t,u) \leq \frac{1}{2A\lambda} u, \quad \forall u \in [\alpha R, \infty), \ t \in [0,1].$$

Then for $u \in P \cap \partial K_R$, by the definition of cone *P*, we get $\min_{t \in [0,1]} u(t) \geq \alpha \|u\| = \alpha R$, and so

$$\begin{aligned}(\lambda Tu)(t) &= \lambda \int_0^1 G(t,s) q(s) f\left(s, u(s)\right) \mathrm{d}s \\ &\leq \lambda \frac{1}{2\lambda A} \int_0^1 g(s) q(s) u(s) \mathrm{d}s < R, \ \ t \in [0,1].\end{aligned}$$

We obtain $\lambda Tu \neq \mu u$ for $u \in P \cap \partial K_R$, $\mu \geq 1$. It follows from Lemma 2.6 that

$$i\left(\lambda T, P \cap K_R, P\right) = 1. \quad (12)$$

According to the additivity of the fixed point index, by (11) and (12), we have

$$i\left(\lambda T, P \cap \left(K_R \setminus \bar{K}_r\right), P\right) = i\left(\lambda T, P \cap K_R, P\right) - i\left(\lambda T, P \cap K_r, P\right) = 1,$$

which implies that the nonlinear operator λT has at least one fixed point $u \in P \cap \left(K_R \setminus \bar{K}_r\right)$. Therefore, $(\lambda, u) \in \Phi$. The proof is complete.

Lemma 3.3. Suppose (H_1) and (H_2) hold, $f_0 = f_\infty = 0$, then $0 < \lambda^* < \infty$.

Proof. By Lemma 3.1, it is easy to see that $\lambda^* > 0$. It follows from (H_2) and $f_0 = f_\infty = \infty$ that there exists $C > 0$ such that $f(t,u) \geq Cu$ for all $u \geq 0$ and $t \in [0,1]$. Let $(\lambda, u) \in \Phi$, by the definition of cone *P* and Lemma 2.1, we obtain that

$$\begin{aligned}u(t)=(\lambda Tu)(t)&=\lambda\int_0^1 G(t,s)q(s)f(s,u(s))\mathrm{d}s\\&\ge\lambda\alpha C\int_0^1 g(s)q(s)u(s)\mathrm{d}s\\&\ge\lambda\alpha^2 C\|u\|\int_0^1 g(s)q(s)\mathrm{d}s\\&=\lambda\alpha^2 AC\|u\|,\end{aligned}$$

so $\|u\|\ge\lambda\alpha^2 AC\|u\|$, thus $\lambda\le(\alpha^2 AC)^{-1}$. This completes the proof of Lemma 3.3.

Lemma 3.4. Suppose (H_1) and (H_2) hold, hold and $f^0=f^\infty=0$, then $0<\lambda_*<\infty$.

Proof. By Lemma 3.2, it is easy to see that $\lambda_*<\infty$. It follows from (H_2) and $f^0=f^\infty=0$ that there exists $C_1>0$ such that $f(t,u)\le C_1 u$ for all $u\ge 0$ and $t\in[0,1]$. Let $(\lambda,u)\in\Phi$, from the definition of cone P and Lemma 2.2, we have

$$\begin{aligned}u(t)=(\lambda Tu)(t)&=\lambda\int_0^1 G(t,s)q(s)f(s,u(s))\mathrm{d}s\\&\le\lambda C_1\int_0^1 g(s)q(s)u(s)\mathrm{d}s\\&\le\lambda C_1\|u\|\int_0^1 g(s)q(s)\mathrm{d}s\\&=\lambda AC_1\|u\|,\ t\in[0,1],\end{aligned}$$

so $\|u\|\le\lambda AC_1\|u\|$, thus $\lambda\ge\dfrac{1}{AC_1}$. This completes the proof of Lemma 3.4.

Lemma 3.5. Suppose (H_1) and (H_2) hold, $f_0=f_\infty=\infty$, then $(0,\lambda^*)\subset\Lambda$. Moreover, for any $\lambda\in(0,\lambda^*)$, BVP (1) and (2) has at least two positive solutions.

Proof. For any fixed $\lambda\in(0,\lambda^*)$, we prove that $\lambda\in\Lambda$. By the definition of λ^*, there exists $\lambda_2\in\Lambda$, such that $\lambda<\lambda_2\le\lambda^*$ and $(\lambda_2,u_2)\in\Phi$. Let $R<\min_{t\in[0,1]}u_2(t)$ be fixed. From the proof of Lemma 3.1, we see that there exist $\lambda_1<\lambda, r<R$ and $u_1(t)\in P\cap(K_R\setminus\bar{K}_r)$ such that $(\lambda_1,u_1)\in\Phi$. It is easy to see that $0<u_1(t)<u_2(t)$ for all $t\in[0,1]$. Then we have

$$u_1'''(t)=\lambda_1 q(t)f(t,u_1(t)),\ t\in(0,1),$$

and

$$u_2'''(t)=\lambda_2 q(t)f(t,u_2(t)),\ t\in(0,1).$$

Consider now the modified BVP:

$$u'''(t)=\lambda q(t)f_1(t,u(t)),\ t\in(0,1), \tag{13}$$

$$u(0)=\alpha u(\eta),\ u'(\eta)=0,\ u''(1)=0, \tag{14}$$

where

$$f_1(t,u(t))=\begin{cases}f(t,u_1(t)),\ u(t)\le u_1(t),\\ f(t,u(t)),\ u_1(t)<u(t)<u_2(t),\\ f(t,u_2(t)),\ u(t)\ge u_2(t).\end{cases}$$

Clearly, the function λf_1 is bounded for $t\in[0,1]$, $u\in P$ and is continuous in u. Define the operator $T_1 : E \to E$ by

$$\left(T_1 u\right)(t) = \int_0^1 G(t,s) q(s) f_1\left(s, u(s)\right) \mathrm{d}s,\ u \in E,\ t \in [0,1].$$

Then $T_1 : P \to P$ is completely continuous and all the fixed points of operator λT_1 are the solutions for BVP (13) and (14). It is easy to see that there exists $r_0 > \|u_2\|$ such that $\|\lambda T_1 u\| < r_0$ for any $u \in P$. From Lemma 2.6, we have

$$i\left(\lambda T_1, P \cap K_{r_0}, P\right) = 1. \tag{15}$$

Let

$$U = \left\{u \in P : u_1(t) < u(t) < u_2(t), \forall t \in [0,1]\right\}.$$

We claim that if $u \in P$ is a fixed point of operator λT_1, then $u \in U$. In fact, if $u = \lambda T_1 u$, then

$$\begin{aligned} u(t) &= \left(\lambda T_1 u\right)(t) = \lambda \int_0^1 G(t,s) q(s) f_1\left(s, u(s)\right) \mathrm{d}s \\ &< \lambda_2 \int_0^1 G(t,s) q(s) f\left(s, u_2(s)\right) \mathrm{d}s = \left(\lambda_2 T u_2\right)(t) = u_2(t), \end{aligned}$$

and

$$\begin{aligned} u(t) &= \left(\lambda T_1 u\right)(t) = \lambda \int_0^1 G(t,s) q(s) f_1\left(s, u(s)\right) \mathrm{d}s \\ &> \lambda_1 \int_0^1 G(t,s) q(s) f\left(s, u_1(s)\right) \mathrm{d}s = \left(\lambda_1 T u_1\right)(t) = u_1(t). \end{aligned}$$

From the excision property of the fixed point index and (15), we obtain that

$$i\left(\lambda T_1, U, P\right) = i\left(\lambda T_1, P \cap K_{r_0}, P\right) = 1.$$

From the definition of T_1, we know that $T_1 = T$ on $\bar{U}$, then

$$i\left(\lambda T, U, P\right) = 1. \tag{16}$$

Hence, the nonlinear operator λT has at least fixed point $v_1 \in U$. Then v_1 is one positive solution of BVP (1) and (2). This gives $\lambda \in \Lambda$, $\left(\lambda, v_1\right) \in \Phi$ and $(0, \lambda) \subset \Lambda$.

We now find the second positive solution of BVP (1) and (2). By $f_\infty = \infty$ and the continuity of $f(t,u)$ with respect to u, there exists $C > 0$ such that

$$f(t,u) \ge \frac{2u}{\lambda \alpha^2 A} - \frac{C}{\alpha A},\quad \forall u \ge 0, t \in [0,1]. \tag{17}$$

For $e(t) \equiv 1$, let

$$\Omega = \left\{u \in P : \text{there exists } \tau \ge 0 \text{ such that } u = \lambda T u + \tau e\right\}.$$

We claim that Ω is bounded in *E*. In fact, for any $u \in \Omega$, it follows from Lemma 2.2 and (17) that

$$\begin{aligned} u(t) &= \left(\lambda T u\right)(t) + \tau e(t) = \left(\lambda T u\right)(t) + \tau \\ &\ge \lambda \int_0^1 G(t,s) q(s) f\left(s, u(s)\right) \mathrm{d}s \\ &\ge \lambda \alpha \int_0^1 g(s) q(s) \left[\frac{2u(t)}{\lambda \alpha^2 A} - \frac{C}{\alpha A}\right] \mathrm{d}s \end{aligned}$$

$$\geq \lambda\alpha\int_0^1 g(s)q(s)\left[\frac{2\alpha\|u\|}{\lambda\alpha^2 A}-\frac{C}{\alpha A}\right]\mathrm{d}s$$
$$= 2\|u\|-\lambda C.$$

This implies $\|u\|\leq\lambda C$. Thus Ω is bounded in *E*. Therefore there exists $R_1>\|u_2\|$ such that

$$u\neq\lambda Tu+\tau e,\ \forall u\in P\cap\partial K_{R_1},\tau\geq 0.$$

By Lemma 2.4, we get that

$$i\left(\lambda T,P\cap K_{R_1},P\right)=0. \tag{18}$$

Using a similar argument as in deriving (10), we have that

$$i\left(\lambda T,P\cap K_{r_1},P\right)=0, \tag{19}$$

where $0<r_1<\min_{t\in[0,1]}u_1(t)$. According to the additivity of the fixed point index, by (16), (18) and (19), we have

$$i\left(\lambda T,P\cap\left(K_{R_1}\setminus\left(\bar{U}\cup\bar{K}_{r_1}\right)\right),P\right)$$
$$=i\left(\lambda T,P\cap K_{R_1},P\right)-i\left(\lambda T,U,P\right)-i\left(\lambda T,P\cap K_{r_1},P\right)=-1,$$

which implies that the nonlinear operator λT has at least one fixed point $v_2\in P\cap\left(K_{R_1}\setminus\left(\bar{U}\cup\bar{K}_{r_1}\right)\right)$. Thus, BVP (1)-(2) has another positive solution. The proof is complete.

Lemma 3.6. Suppose (H_1) and (H_2) hold, $f^0=f^\infty=0$, then $(\lambda_*,+\infty)\subset\Lambda$. Moreover, for any $\lambda\in(\lambda_*,+\infty)$, BVP (1)-(2) has at least two positive solutions.

Proof. For any fixed $\lambda\in(\lambda_*,+\infty)$, we prove that $\lambda\in\Lambda$. By the definition of λ_*, there exists $\lambda_1\in\Lambda$, such that $\lambda_*\leq\lambda_1<\lambda$ and $(\lambda_1,u_1)\in\Phi$. Let $r>\frac{1}{\alpha}\|u_1\|$ be fixed. From the proof of Lemma 3.2, we see that there exist $\lambda_2>\lambda$, $R>r$ and $u_2(t)\in P\cap\left(K_R\setminus\bar{K}_r\right)$ such that $(\lambda_2,u_2)\in\Phi$. By the definition of cone *P*, it is easy to see that $0<u_1(t)<u_2(t)$ for all $t\in[0,1]$. Define

$$V=\left\{u\in P:u_1(t)<u(t)<u_2(t),\forall t\in[0,1]\right\}.$$

Using the method similar to get (16), we yield

$$i\left(\lambda T,V,P\right)=1, \tag{20}$$

Hence, the nonlinear operator λT has at least fixed point $v_1\in V$. Then v_1 is one positive solution of BVP (1) and (2). This gives $\lambda\in\Lambda$, $(\lambda,v_1)\in\Phi$ and $(\lambda_*,+\infty)\subset\Lambda$.

We now find the second positive solution of BVP (1) and (2). From $f^0=0$, there exists $0<r_0<\min_{t\in[0,1]}u_1(t)$ such that

$$f(t,u)\leq\frac{u}{2\lambda A},\ \forall u\in[0,r_0],t\in[0,1].$$

Then for $u\in P\cap\partial K_{r_0}$, we have

$$(\lambda Tu)(t)=\lambda\int_0^1 G(t,s)q(s)f(s,u(s))\mathrm{d}s$$
$$\le \lambda\int_0^1 g(s)q(s)\frac{\|u\|}{2\lambda A}\mathrm{d}s=\frac{\|u\|}{2}<r_0,\ t\in[0,1].$$

This implies $\lambda Tu\ne\mu u$ for $u\in P\cap\partial K_{r_0}$, $\mu\ge 1$. It follows from Lemma 2.6 that

$$i\left(\lambda T,P\cap K_{r_0},P\right)=1. \tag{21}$$

Using a similar argument as in deriving (12), we have that

$$i\left(\lambda T,P\cap K_{R_0},P\right)=1. \tag{22}$$

where $R_0>\|u_2\|$. According to the additivity of the fixed point index, by (20), (21) and (22), we have

$$i\left(\lambda T,P\cap\left(K_{R_0}\setminus\left(\bar V\cup\bar K_{r_0}\right)\right),P\right)$$
$$=i\left(\lambda T,P\cap K_{R_0},P\right)-i\left(\lambda T,V,P\right)-i\left(\lambda T,P\cap K_{r_0},P\right)=-1,$$

which implies that the nonlinear operator λT has at least one fixed point $v_2\in P\cap\left(K_{R_0}\setminus\left(\bar V\cup\bar K_{r_0}\right)\right)$. Thus, BVP (1)-(2) has another positive solution. The proof is complete.

Lemma 3.7. Suppose (H_1) and (H_2) hold, $f_0=f_\infty=\infty$, then $\Lambda=\left(0,\lambda^*\right]$.

Proof. In view of Lemma 3.5, it suffices to prove that $\lambda^*\in\Lambda$. By the definition of λ^*, we can choose $\{\lambda_n\}\subset\Lambda$ with $\lambda_n\ge\frac{\lambda^*}{2}(n=1,2,\cdots)$ such that $\lambda_n\to\lambda^*$ as $n\to\infty$. By the definition of Λ, there exists $\{u_n\}\subset P\setminus\{\theta\}$ such that $(\lambda_n,u_n)\in\Phi$. We now show that $\{u_n\}$ is bounded. Suppose the contrary, then there exists a subsequence of $\{u_n\}$ (still denoted by $\{u_n\}$) such that $\|u_n\|\to\infty$ as $n\to\infty$. It follows from $\{u_n\}\subset P\setminus\{\theta\}$ that $u_n\ge\alpha\|u_n\|$ for all $t\in[0,1]$. Choose sufficiently large τ such that

$$\frac{\lambda^*\alpha^2A\tau}{2}>1.$$

By $f_\infty=\infty$, there exists $R>0$ such that $f(t,u)\ge\tau u$ for all $u>\alpha R$ and $t\in[0,1]$. Since $\|u_n\|\to\infty$ as $n\to\infty$, there exists sufficiently large n_0 such that $\|u_{n_0}\|\ge R$. Thus, we have

$$\|u_{n_0}\|\ge u_{n_0}(t)=\left(\lambda_{n_0}Tu_{n_0}\right)(t)=\lambda_{n_0}\int_0^1 G(t,s)q(s)f\left(s,u_{n_0}(s)\right)\mathrm{d}s$$
$$\ge\frac{\lambda^*}{2}\alpha\tau\int_0^1 g(s)q(s)u_{n_0}(s)\mathrm{d}s$$
$$\ge\frac{\lambda^*}{2}\alpha^2\tau\|u_{n_0}\|\int_0^1 g(s)q(s)\mathrm{d}s=\frac{\lambda^*}{2}\alpha^2\tau A\|u_{n_0}\|.$$

This gives

$$\frac{\lambda^*\alpha^2A\tau}{2}\le 1, \tag{23}$$

which contradicts the choice of τ. Hence, $\{u_n\}$ is bounded. It follows from the completely continuous of T that $\{Tu_n\}$ is equicontinuous, *i.e.*, for each $\varepsilon > 0$, there is a $\delta > 0$ such that

$$\left|u_n(t_1) - u_n(t_2)\right| = \lambda_n \left|(Tu_n)(t_1) - (Tu_n)(t_2)\right| < \lambda_n \varepsilon \le \lambda^* \varepsilon,$$

where $n = 1, 2, \cdots$, $t_1, t_2 \in [0,1]$ and $|t_1 - t_2| < \delta$. Then $\{u_n\}$ is equicontinuous. According to the Ascoli-Arzela theorem, $\{u_n\}$ is relatively compact. Hence, there exists a subsequence of $\{u_n\}$ (still denoted by $\{u_n\}$) and $u^* \in P$ such that $u_n \to u^*$ as $n \to \infty$. By $u_n = \lambda_n T u_n$, letting $n \to \infty$, we obtain that $u^* = \lambda^* T u^*$. If $u^* = \theta$, using a similar argument as in deriving (23), by $f_0 = \infty$, we also get a contradiction. Then $u^* \in P \setminus \{\theta\}$, and so $\lambda^* \in \Lambda$. This completes the proof.

Lemma 3.8. Suppose (H$_1$) and (H$_2$) hold, $f^0 = f^\infty = 0$, then $\Lambda = [\lambda_*, +\infty)$.

Proof. In view of Lemma 3.6, it suffices to prove that $\lambda_* \in \Lambda$. By the definition of λ_*, we can choose $\{\lambda_n\} \subset \Lambda$ with $\lambda_n \le 2\lambda_* (n = 1, 2, \cdots)$ such that $\lambda_n \to \lambda_*$ as $n \to \infty$. By the definition of Λ, there exists $\{u_n\} \subset P \setminus \{\theta\}$ such that $(\lambda_n, u_n) \in \Phi$. We now show that $\{u_n\}$ is bounded. Suppose the contrary, then there exists a subsequence of $\{u_n\}$ (still denoted by $\{u_n\}$) such that $\|u_n\| \to \infty$ as $n \to \infty$. It follows from $\{u_n\} \subset P \setminus \{\theta\}$ that $u_n \ge \alpha \|u_n\|$ for all $t \in [0,1]$. Choose τ small enough such that

$$2\lambda_* A \tau < 1.$$

By $f^\infty = 0$, there exists $R > 0$ such that $f(t,u) \le \tau u$ for all $u > \alpha R$ and $t \in [0,1]$. Since $\|u_n\| \to \infty$ as $n \to \infty$, there exists sufficiently large n_0 such that $\|u_{n_0}\| \ge R$. Thus, we have

$$\begin{aligned} u_{n_0}(t) &= \left(\lambda_{n_0} T u_{n_0}\right)(t) = \lambda_{n_0} \int_0^1 G(t,s) q(s) f\left(s, u_{n_0}(s)\right) \mathrm{d}s \\ &\le 2\lambda_* \tau \int_0^1 g(s) q(s) u_{n_0}(s) \mathrm{d}s \\ &\le 2\lambda_* \tau \left\|u_{n_0}\right\| \int_0^1 g(s) q(s) \mathrm{d}s = 2\lambda_* \tau A \left\|u_{n_0}\right\|. \end{aligned}$$

This gives

$$2\lambda_* \tau A \ge 1, \tag{24}$$

which contradicts the choice of τ. Hence, $\{u_n\}$ is bounded. It follows from the completely continuous of T that $\{Tu_n\}$ is equicontinuous, *i.e.*, for each $\varepsilon > 0$, there is a $\delta > 0$ such that

$$\left|u_n(t_1) - u_n(t_2)\right| = \lambda_n \left|(Tu_n)(t_1) - (Tu_n)(t_2)\right| < \lambda_n \varepsilon \le 2\lambda_* \varepsilon,$$

where $n = 1, 2, \cdots$, $t_1, t_2 \in [0,1]$ and $|t_1 - t_2| < \delta$. Then $\{u_n\}$ is equicontinuous. According to the Ascoli-Arzela theorem, $\{u_n\}$ is relatively compact. Hence, there exists a subsequence of $\{u_n\}$ (still denoted by $\{u_n\}$) and $u_* \in P$ such that $u_n \to u_*$ as $n \to \infty$. By $u_n = \lambda_n T u_n$, letting $n \to \infty$, we obtain that $u_* = \lambda_* T u_*$. If $u_* = \theta$, using a similar argument as in deriving (24), by $f^0 = 0$, we also get a contradiction. Then $u_* \in P \setminus \{\theta\}$, and so $\lambda_* \in \Lambda$. This completes the proof.

From Lemmas 3.1, 3.3, 3.5 and 3.7, we get the main result as follows.

Theorem 3.1. Let (H_1), (H_2) be fulfilled and suppose that $f_0 = f_\infty = \infty$, then there exists $\lambda^* > 0$ such that BVP (1)-(2) has at least two positive solutions for $\lambda \in (0, \lambda^*)$, at least one positive solution for $\lambda = \lambda^*$ and no positive solution for $\lambda > \lambda^*$.

By Lemmas 3.2, 3.4, 3.6 and 3.8, we obtain the main result as follows.

Theorem 3.2. Let (H_1), (H_2) be fulfilled and suppose that $f^0 = f^\infty = 0$, then there exists $\lambda_* > 0$ such that BVP (1)-(2) has at least two positive solutions for $\lambda > \lambda_*$, at least one positive solution for $\lambda = \lambda_*$ and no positive solution for $\lambda \in (0, \lambda_*)$.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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