

Some Identities Involving the High-Order Cauchy Polynomials

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How to cite this paper: Liu, L.W. and Wuyungaowa (2022) Some Identities Involving the High-Order Cauchy Polynomials. *Journal of Applied Mathematics and Physics*, 10, 1126-1145.
<https://doi.org/10.4236/jamp.2022.104079>

Received: March 2, 2022

Accepted: April 9, 2022

Published: April 12, 2022

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Abstract

In this paper, we consider the Cauchy numbers and polynomials of order k and give some relation between Cauchy polynomials of order k and special polynomials by using generating functions and the Riordan matrix methods. In addition, we establish some new equalities and relations involving high-order Cauchy numbers and polynomials, high-order Daehee numbers and polynomials, the generalized Bell polynomials, the Bernoulli numbers and polynomials, high-order Changhee polynomials, high-order Changhee-Genocchi polynomials, the combinatorial numbers, Lah numbers and Stirling numbers, etc.

Keywords

High-Order Daehee Numbers and Polynomials, The Bernoulli Numbers and Polynomials, High-Order Changhee Polynomials, Stirling Numbers, The Lah Numbers

1. Introduction

Combinatorial constants are widely used in many disciplines such as probabilistic calculations, theoretical physics problem solving, computer algorithm analysis, etc. Cauchy numbers are special sequences that are widely used in number theory, numerical analysis, etc. In recent years, many papers in [1] [2] [3] [4] have been devoted to the study of Cauchy numbers and polynomials identities by various methods. High-order Cauchy numbers and polynomials are introduced by Taekyun Kim, Dae San Kim, Hyuck In Kwon and Jongjin Seo in [1]. Higher-order Cauchy of the first kind and poly-Cauchy of the first kind mixed type polynomials are introduced by D. S. Kim, T in [2]. About Cauchy numbers and polynomials and other polynomials, more combinatorial identities are de-

rived. In this paper, we establish some new identities and properties by using High-order Cauchy polynomials.

The High-Order Cauchy polynomials of the first kind and the second are defined by the following generating function in [1] [2].

The High-Order Cauchy polynomials of the first kind are defined by the following generating function

$$\left(\frac{t}{\ln(1+t)}\right)^k (1+t)^x = \sum_{n=0}^{\infty} C_n^{(k)}(x) \frac{t^n}{n!}. \quad (1)$$

When $x=0$, $k=1$, $C_n = C_n(0)$ are called the Cauchy numbers.

The High-Order Cauchy polynomials of the second kind are given by the generating function

$$\left(\frac{t}{(1+t)\ln(1+t)}\right)^k (1+t)^x = \sum_{n=0}^{\infty} \hat{C}_n^{(k)}(x) \frac{t^n}{n!}. \quad (2)$$

When $x=0$, $k=1$, $\hat{C}_n = \hat{C}_n(0)$ are called the Cauchy numbers of second kind.

The generating functions of the relevant special combinatorial sequences involved in this paper are as follows [3]-[17]:

The α -Cauchy numbers of the first kind are given by the generating function to be

$$\frac{t(1+t)^\alpha}{\ln(1+t)} = \sum_{n=0}^{\infty} C_n^\alpha \frac{t^n}{n!}. \quad (3)$$

The α -Cauchy numbers of the second kind are given by the generating function to be

$$\frac{t(1+t)^{\alpha-1}}{\ln(1+t)} = \sum_{n=0}^{\infty} \hat{C}_n^\alpha \frac{t^n}{n!}. \quad (4)$$

For $\alpha \in \mathbb{N}^+$, the high-order generalized Cauchy numbers are given by the generating function to be

$$\left(\frac{\alpha t}{(1+t)^\alpha - 1}\right)^k = \sum_{n=0}^{\infty} c_n^{(k)}(\alpha) \frac{t^n}{n!}. \quad (5)$$

The High-order Daehee polynomials of the second kind are given by the generating function to be

$$\left(\frac{(1+t)\ln(1+t)}{t}\right)^k (1+t)^x = \sum_{n=0}^{\infty} \hat{D}_n^{(k)}(x) \frac{t^n}{n!}. \quad (6)$$

When $x=0$, $k=1$, $D_n(x) = D_n(0)$ are called the Daehee numbers of second kind.

For $\alpha \in \mathbb{N}^+$, the α -Daehee polynomials of the first kind are given by the generating function to be

$$\left(\frac{\alpha \ln(1+t)}{(1+t)^\alpha - 1}\right)^k (1+t)^x = \sum_{n=0}^{\infty} D_{n,\alpha}^{(k)}(x) \frac{t^n}{n!}. \quad (7)$$

The generalized Bell polynomials of the first kind are given by the generating function

$$\frac{(e^{e^t-1} - 1)^k}{k!} = \sum_{n=1}^{\infty} B(n, k) \frac{t^n}{n!}. \quad (8)$$

The generalized Bell polynomials of the second kind are given by the generating function

$$\frac{(\ln(1 + \ln(1+t)))^k}{k!} = \sum_{n=1}^{\infty} \beta(n, k) \frac{t^n}{n!}. \quad (9)$$

The generalized Bernoulli polynomials are given by the generating function to be

$$\left(\frac{t}{e^t - 1}\right)^\alpha e^{xt} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x) \frac{t^n}{n!}. \quad (10)$$

When $\alpha = 1$, $\left(\frac{t}{e^t - 1}\right) e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}$ are called Bernoulli polynomials.

For $\alpha \in \mathbb{N}^+$, the high-order degenerate Bernoulli numbers of the second are given by the generating function

$$\left(\frac{\alpha t}{(1+t)^\alpha - 1}\right)^k (1+t)^x = \sum_{n=0}^{\infty} b_{n,\alpha}^{(k)}(x) \frac{t^n}{n!}. \quad (11)$$

The high-order Changhee polynomials are defined by

$$\left(\frac{2}{2+t}\right)^k (1+t)^x = \sum_{n=0}^{\infty} Ch_n^{(k)}(x) \frac{t^n}{n!}. \quad (12)$$

When $x = 0$, $Ch_n^{(k)} = Ch_n^{(k)}(0)$ are called the Changhee-Genocchi numbers.

When $k = 0$, we get the following generating function

$$(1+t)^x = \sum_{n=0}^{\infty} Ch_n^{(0)}(x) \frac{t^n}{n!}. \quad (13)$$

The negative order Changhee polynomials are defined by

$$\left(\frac{2+t}{2}\right)^k (1+t)^x = \sum_{n=0}^{\infty} Ch_n^{(-k)}(x) \frac{t^n}{n!}. \quad (14)$$

The high-order Changhee-Genocchi polynomials are defined by

$$\left(\frac{2 \ln(1+t)}{2+t}\right)^k (1+t)^x = \sum_{n=0}^{\infty} CG_n^{(k)}(x) \frac{t^n}{n!}. \quad (15)$$

When $x = 0$, $CG_n^{(k)} = CG_n^{(k)}(0)$ are called the high-order Changhee-Genocchi numbers.

The Lah numbers are given by the generating function

$$\frac{\left(\frac{-t}{1+t}\right)^k}{k!} = \sum_{n \geq k} L(n, k) \frac{t^n}{n!}. \quad (16)$$

The generalized Lah numbers are given by the generating function

$$\frac{\left(\frac{-t}{1+t}\right)^k}{k!} (1+t)^r = \sum_{n \geq k} L(n, k; r) \frac{t^n}{n!}. \quad (17)$$

When $r = 0$, $L(n, k; 0) = L(n, k)$.

The classical Harmonic numbers are given by the generating function

$$\frac{-\ln(1-t)}{1-t} = \sum_{n=1}^{\infty} H_n t^n. \quad (18)$$

The generalized Harmonic numbers are given by the generating function

$$\frac{(-\ln(1-t))^{r+1}}{t^{r+1}(1-t)} = \sum_{n=0}^{\infty} H(n+r+1, r) t^n. \quad (19)$$

The Stirling numbers of the first kind and the second kind are defined by

$$\frac{\ln^k(1+t)}{k!} \sum_{n \geq k} S_1(n, k) \frac{t^n}{n!}. \quad (20)$$

$$\frac{(e^t - 1)^k}{k!} = \sum_{n \geq k} S_2(n, k) \frac{t^n}{n!}. \quad (21)$$

The generalized Stirling numbers of the first kind and the second kind are defined by

$$\frac{\ln^k(1+t)}{(1+t)^r k!} = \sum_{n \geq k} S_1(n, k; r) \frac{t^n}{n!}. \quad (22)$$

$$\frac{(e^t - 1)^k}{k!} e^{rt} = \sum_{n \geq k} S_2(n, k; r) \frac{t^n}{n!}. \quad (23)$$

For integer $n, r \geq 1$, the combinatorial numbers are defined by

$$\frac{(-\ln(1-t))^r}{(1-t)^{k+1}} = \sum_{n \geq k} \binom{n}{k} P(r, n, k) t^{n-k}. \quad (24)$$

then Equation is equivalent to

$$\frac{(-\ln(1-t))^r}{(1-t)^{k+1}} = \sum_{n=0}^{\infty} \binom{n+k}{k} P(r, n+k, k) t^n. \quad (25)$$

Lemma 1 (Inversion Formula) [15] Let f, g be functions defined on the set of positive integers, then

$$g_n = \sum_{k=0}^n S_1(n, k) f_k \Leftrightarrow f_n = \sum_{k=0}^n S_2(n, k) g_k. \quad (26)$$

$$g_n = \sum_{k=0}^n S_1(n, k; r) f_k \Leftrightarrow f_n = \sum_{k=0}^n S_2(n, k; r) g_k. \quad (27)$$

A Riordan array is a pair $(g(t), f(t))$ of formal power series with $f_0 = f(0) = 0$. It defines an infinite lower triangular array $(d_{n,k})_{n,k \in \mathbb{N}}$ according to the rule:

$$d_{n,k} = [t^n] g(t)(f(t))^k. \tag{28}$$

Hence we write $R(d_{n,k}) = (g(t), f(t))$.

Lemma 2 [16] [17] If $D = (g(t), f(t)) = (d_{n,k})_{n,k \in \mathbb{N}}$ is a Riordan array and $h(t)$ is the generating function of the sequence $(h_k)_{k \in \mathbb{N}}$, i.e., $f(t) = \sum_{k=0}^{\infty} f_k t^k$ or $h(t) = G(h_k)$. Then we have

$$\sum_{k=0}^n d_{n,k} h_k = [t^n] g(t)h(f(t)). \tag{29}$$

2. Properties about Cauchy Polynomials

In this section, we establish some identities and give some properties of high-order Cauchy polynomials by using generating functions.

Theorem 2.1. For nonnegative integer n , we obtain

$$\begin{aligned} & \sum_{n_1+n_2+\dots+n_m=n} \binom{n}{n_1, n_2, \dots, n_m} C_{n_1}^{(r_1)}(x_1) C_{n_2}^{(r_2)}(x_2) \dots C_{n_m}^{(r_m)}(x_m) \\ &= C_n^{(r_1+\dots+r_m)}(x_1+\dots+x_m). \end{aligned} \tag{30}$$

Proof By (1), we get

$$\begin{aligned} & \sum_{n=0}^{\infty} C_n^{(r_1+\dots+r_m)}(x_1+\dots+x_m) \frac{t^n}{n!} \\ &= \left(\frac{t}{\ln(1+t)} \right)^{r_1+r_2+\dots+r_m} (1+t)^{x_1+x_2+\dots+x_m} \\ &= \sum_{n_1=0}^{\infty} C_{n_1}^{(r_1)}(x_1) \frac{t^{n_1}}{n_1!} \dots \sum_{n_m=0}^{\infty} C_{n_m}^{(r_m)}(x_m) \frac{t^{n_m}}{n_m!} \\ &= \sum_{n=0}^{\infty} \sum_{n_1+n_2+\dots+n_m=n} \binom{n}{n_1, n_2, \dots, n_m} C_{n_1}^{(r_1)}(x_1) \dots C_{n_m}^{(r_m)}(x_m) \frac{t^n}{n!}. \end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ in both sides, we get the identities.

Corollary 2.1. For $x_i = 0 (i = 1, 2, \dots)$ in (30), we obtain

$$\sum_{n_1+n_2+\dots+n_m=n} \binom{n}{n_1, n_2, \dots, n_m} C_{n_1}^{(r_1)} C_{n_2}^{(r_2)} \dots C_{n_m}^{(r_m)} = C_n^{(r_1+r_2+\dots+r_m)}. \tag{31}$$

Corollary 2.2. For $m = 2$ in (30), we obtain

$$\sum_{k=0}^n \binom{n}{k} C_k^{(r)}(x) C_{n-k}^{(s)}(y) = C_n^{(r+s)}(x+y). \tag{32}$$

Corollary 2.3. For $s = 0$ in (32), we obtain

$$\sum_{k=0}^n \binom{n}{k} C_k^{(r)}(x) (y)_{n-k} = C_n^{(r)}(x+y). \tag{33}$$

For $y = 0$ in (33), we obtain

$$\sum_{k=0}^n \binom{n}{k} C_k^{(r)}(x)_{n-k} = C_n^{(r)}(x). \quad (34)$$

For $r = 0$ in (33), we obtain

$$\sum_{k=0}^n \binom{n}{k} C_{n-k}(x+y)_k = C_n(x+y). \quad (35)$$

Similarly, we can obtain

Theorem 2.2. For nonnegative integer n , we obtain

$$\begin{aligned} & \sum_{n_1+n_2+\dots+n_m=n} \binom{n}{n_1, n_2, \dots, n_m} \hat{C}_{n_1}^{(r_1)}(x_1) \hat{C}_{n_2}^{(r_2)}(x_2) \dots \hat{C}_{n_m}^{(r_m)}(x_m) \\ &= \hat{C}_n^{(r_1+\dots+r_m)}(x_1+\dots+x_m). \end{aligned} \quad (36)$$

Theorem 2.3. For integer $n \geq k \geq 1$, we have

$$\frac{d^k \left(\frac{1}{k!} C_n^{(l)}(x) \right)}{dx^k} = \sum_{m=0}^{n-k} \binom{n}{m} C_m^{(l)}(x) S_1(n-m, k). \quad (37)$$

Proof From generating function (1), we have

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{d^k \left(\frac{1}{k!} C_n^{(l)}(x) \right) t^n}{dx^k n!} \\ &= \left(\frac{t}{\ln(1+t)} \right)^l (1+t)^x \frac{\ln^k(1+t)}{k!} \\ &= \sum_{n=0}^{\infty} C_n^{(l)}(x) \frac{t^n}{n!} \sum_{n \geq k} S_1(n, k) \frac{t^n}{n!} \\ &= \sum_{n \geq k} \sum_{m=0}^{n-k} \binom{n}{m} C_m^{(l)}(x) S_1(n-m, k) \frac{t^n}{n!}. \end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ in both sides, we can easily get the identities.

Similarly, we can obtain

Theorem 2.4. For integer $n \geq k \geq 1$, we have

$$\frac{d^k \left(\frac{1}{k!} \hat{C}_n^{(l)}(x) \right)}{dx^k} = \sum_{m=0}^{n-k} \binom{n}{m} \hat{C}_m^{(l)}(x) S_1(n-m, k). \quad (38)$$

Theorem 2.5. For nonnegative integer n , we obtain

$$\sum_{m=0}^n \binom{n}{m} \hat{D}_m^{(r)}(x) C_{n-m}^{(s)}(x) = \begin{cases} C_n^{(s-r)}(2x+r), & r < s, \\ (2x+r)_n, & r = s, \\ \hat{D}_n^{(r-s)}(2x+s), & r > s. \end{cases} \quad (39)$$

Proof From generating function (1), (6), we get

$$\begin{aligned}
 & \sum_{n=0}^{\infty} \sum_{m=0}^n \binom{n}{m} \hat{D}_m^{(r)}(x) C_{n-m}^{(s)}(x) \frac{t^n}{n!} \\
 &= \sum_{n=0}^{\infty} \hat{D}_n^{(r)}(x) \frac{t^n}{n!} \sum_{n=0}^{\infty} C_n^{(s)}(x) \frac{t^n}{n!} \\
 &= \left(\frac{(1+t)\ln(1+t)}{t} \right)^r (1+t)^x \left(\frac{t}{\ln(1+t)} \right)^s (1+t)^x \\
 &= \begin{cases} \left(\frac{t}{\ln(1+x)} \right)^{(s-r)} (1+t)^{2x+r}, & r < s, \\ (1+t)^{2x+r}, & r = s, \\ \left(\frac{\ln(1+x)}{t} \right)^{(r-s)} (1+t)^{2x+s}, & r > s. \end{cases} \\
 &= \begin{cases} \sum_{n=0}^{\infty} C_n^{(s-r)}(2x+r) \frac{t^n}{n!}, & r < s, \\ \sum_{n=0}^{\infty} (2x+r)_n \frac{t^n}{n!}, & r = s, \\ \sum_{n=0}^{\infty} \hat{D}_n^{(r-s)}(2x+s) \frac{t^n}{n!}, & r > s. \end{cases}
 \end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ in both sides, we can easily get the identities.

Corollary 2.4. For nonnegative integer n , $x = 0$ in (30), we have

$$\sum_{m=0}^n \binom{n}{m} \hat{D}_m^{(r)} C_{n-m}^{(s)} = \begin{cases} C_n^{(s-r)}(r), & r < s, \\ \binom{r}{n}, & r = s, \\ \hat{D}_n^{(r-s)}(s), & r > s. \end{cases} \tag{40}$$

Similarly, we can obtain

Theorem 2.6. For nonnegative integer n , we have

$$\sum_{m=0}^n \binom{n}{m} \hat{D}_m^{(r)}(x) \hat{C}_{n-m}^{(s)}(x) = \begin{cases} \hat{C}_n^{(s-r)}(x+r), & r < s, \\ (2x)_n, & r = s, \\ \hat{D}_n^{(r-s)}(x+s), & r > s. \end{cases} \tag{41}$$

Corollary 2.5. For nonnegative integer n , $x = 0$ in (41), we have

$$\sum_{m=0}^n \binom{n}{m} \hat{D}_m^{(r)} \hat{C}_{n-m}^{(s)} = \begin{cases} \hat{C}_n^{(s-r)}(r), & r < s, \\ 1, & r = s, \\ \hat{D}_n^{(r-s)}(s), & r > s. \end{cases} \tag{42}$$

Theorem 2.7. For nonnegative integer n , $\alpha \in \mathbb{N}^+$, we have

$$\sum_{m=0}^n \binom{n}{m} D_{m,\alpha}^{(r)}(x) C_{n-m}^{(s)}(x) = \begin{cases} \sum_{m=0}^n \binom{n}{m} c_{n-m}^{(r)}(\alpha) C_m^{(s-r)}(2x), & r < s, \\ \sum_{m=0}^n \binom{n}{m} c_m^{(r)}(\alpha) (2x)_{n-m}, & r = s, \\ \sum_{m=0}^n \binom{n}{m} c_{n-m}^{(r)}(\alpha) D_m^{(r-s)}(2x), & r > s. \end{cases} \tag{43}$$

Proof From generating function (1), (7), we get

$$\begin{aligned}
 & \sum_{n=0}^{\infty} \sum_{m=0}^n \binom{n}{m} D_{m,\alpha}^{(r)}(x) C_{n-m}^{(s)}(x) \frac{t^n}{n!} \\
 &= \sum_{n=0}^{\infty} D_{n,\alpha}^{(r)}(x) \frac{t^n}{n!} \sum_{n=0}^{\infty} C_n^{(s)}(x) \frac{t^n}{n!} \\
 &= \left(\frac{\alpha \ln(1+t)}{(1+t)^\alpha - 1} \right)^r (1+t)^x \left(\frac{t}{\ln(1+t)} \right)^s (1+t)^x \\
 &= \begin{cases} \left(\frac{\alpha t}{(1+t)^\alpha - 1} \right)^r (1+t)^{2x} \left(\frac{t}{\ln(1+t)} \right)^{s-r}, & r < s, \\ \left(\frac{\alpha t}{(1+t)^\alpha - 1} \right)^r (1+t)^{2x}, & r = s, \\ \left(\frac{\alpha t}{(1+t)^\alpha - 1} \right)^r (1+t)^{2x} \left(\frac{\ln(1+t)}{t} \right)^{r-s}, & r > s. \end{cases} \\
 &= \begin{cases} \sum_{n=0}^{\infty} c_n^{(r)}(\alpha) \frac{t^n}{n!} \sum_{n=0}^{\infty} C_n^{(s-r)}(2x) \frac{t^n}{n!}, & r < s, \\ \sum_{n=0}^{\infty} c_n^{(r)}(\alpha) \frac{t^n}{n!} \sum_{n=0}^{\infty} (2x)_n \frac{t^n}{n!}, & r = s, \\ \sum_{n=0}^{\infty} c_n^{(r)}(\alpha) \frac{t^n}{n!} \sum_{n=0}^{\infty} D_n^{(r-s)}(2x) \frac{t^n}{n!}, & r > s. \end{cases} \\
 &= \begin{cases} \sum_{n=0}^{\infty} \sum_{m=0}^n \binom{n}{m} c_{n-m}^{(r)}(\alpha) C_m^{(s-r)}(2x) \frac{t^n}{n!}, & r < s, \\ \sum_{n=0}^{\infty} \sum_{m=0}^n \binom{n}{m} c_m^{(r)}(\alpha) (2x)_{n-m} \frac{t^n}{n!}, & r = s, \\ \sum_{n=0}^{\infty} \sum_{m=0}^n \binom{n}{m} c_{n-m}^{(r)}(\alpha) D_m^{(r-s)}(2x) \frac{t^n}{n!}, & r > s. \end{cases}
 \end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ in both sides, we can easily get the identities.

Corollary 2.6. For nonnegative integer n , $x=0$ in (43), we have

$$\sum_{m=0}^n \binom{n}{m} D_{m,\alpha}^{(r)} C_{n-m}^{(s)} = \begin{cases} \sum_{m=0}^n \binom{n}{m} c_{n-m}^{(r)}(\alpha) C_m^{(s-r)}, & r < s, \\ \sum_{m=0}^n \binom{n}{m} c_m^{(r)}(\alpha), & r = s, \\ \sum_{m=0}^n \binom{n}{m} c_{n-m}^{(r)}(\alpha) D_m^{(r-s)}, & r > s. \end{cases} \quad (44)$$

Theorem 2.8. For nonnegative integer n , we have

$$\sum_{m=0}^n \binom{n}{m} C_m^{(k)}(x) C_{n-m}^\alpha = C_n^{(k+1)}(x + \alpha). \quad (45)$$

Proof From generating function (1), (5), we get

$$\begin{aligned} & \sum_{n=0}^{\infty} \sum_{m=0}^n \binom{n}{m} C_m^{(k)}(x) C_{n-m}^{\alpha} \frac{t^n}{n!} = \sum_{n=0}^{\infty} C_n^{(k)}(x) \frac{t^n}{n!} \sum_{n=0}^{\infty} C_n^{\alpha} \frac{t^n}{n!} \\ & = \left(\frac{t}{\ln(1+t)} \right)^k (1+t)^x \frac{t(1+t)^{\alpha}}{\ln(1+t)} \\ & = \left(\frac{t}{\ln(1+t)} \right)^{k+1} (1+t)^{\alpha} (1+t)^x \\ & = \sum_{n=0}^{\infty} C_n^{(k+1)}(x+\alpha) \frac{t^n}{n!}. \end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ in both sides, we can easily get the identities.

The proof of the Theorem 2.9 is similar to the proof of the Theorem 2.8, we can obtain

Theorem 2.9. For nonnegative integer n , we have

$$\sum_{m=0}^n \binom{n}{m} \hat{C}_m^{(k)}(x) C_{n-m}^{\alpha} = \hat{C}_n^{(k+1)}(x+\alpha). \tag{46}$$

Corollary 2.7. For nonnegative integer n , $x=0$ in (45), (46), we have

$$\sum_{m=0}^n \binom{n}{m} C_m^{(k)} C_{n-m}^{\alpha} = C_n^{(k+1)}(\alpha). \tag{47}$$

$$\sum_{m=0}^n \binom{n}{m} \hat{C}_m^{(k)} C_{n-m}^{\alpha} = \hat{C}_n^{(k+1)}(\alpha). \tag{48}$$

Theorem 2.10. For nonnegative integer n , we have

$$\sum_{m=k}^n S_1(n-m, k; r) (-1)^k C_m^{(k)}(x-k) = L(n, k; x-r). \tag{49}$$

Proof From generating function (1), (22), we get

$$\begin{aligned} & \sum_{n=k}^{\infty} \sum_{m=0}^n S_1(n-m, k; r) (-1)^k C_m^{(k)}(x-k) \frac{t^n}{n!} \\ & = \sum_{n=k}^{\infty} S_1(n, k; r) \frac{t^n}{n!} \sum_{n=0}^{\infty} (-1)^k C_n^{(k)}(x-k) \frac{t^n}{n!} \\ & = (-1)^k \frac{(\ln(1+t))^k}{k!(1+t)^r} \left(\frac{t}{\ln(1+t)} \right)^k (1+t)^{x-k} \\ & = \frac{\left(\frac{-t}{1+t} \right)^k}{k!} (1+t)^{x-r} = \sum_{n=k}^{\infty} L(n, k; x-r) \frac{t^n}{n!}. \end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ in both sides, we can easily get the identities.

By means of Lemma 1, the inverse relation (27). This leads to the following conclusion,

Theorem 2.11. For nonnegative integer n , we have

$$\sum_{m=k}^n S_2(n-m, k; r) L(m, k; x-r) = (-1)^k C_n^{(k)}(x-k). \quad (50)$$

Similarly, we can obtain

Theorem 2.12. For nonnegative integer n , we have

$$\sum_{m=k}^n S_1(n-m, k; r) (-1)^k \hat{C}_m^{(k)}(x) = L(n, k; x-r). \quad (51)$$

$$\sum_{m=k}^n S_2(n-m, k; r) L(m, k; x-r) = (-1)^k \hat{C}_n^{(k)}(x-k). \quad (52)$$

Theorem 2.13. For integer $n, k \geq 1$, we have

$$\sum_{m=0}^n (-1)^{n-m} C_{n-m}^{(k)}(x) H_m = (-1)^n n C_{n-1}^{(k-1)}(x-1). \quad (53)$$

Proof From generating function (1), (18), we get

$$\begin{aligned} \sum_{n=1}^{\infty} \sum_{m=0}^n (-1)^{n-m} C_{n-m}^{(k)}(x) H_m \frac{t^n}{n!} &= \sum_{n=0}^{\infty} (-1)^n C_n^{(k)} \frac{t^n}{n!} \sum_{n=1}^{\infty} H_n t^n \\ &= \left(\frac{-t}{\ln(1-t)} \right)^k (1-t)^x \frac{-\ln(1-t)}{1-t} = \left(\frac{-t}{\ln(1-t)} \right)^{k-1} (1-t)^{x-1} t \\ &= \sum_{n=0}^{\infty} (-1)^n C_n^{(k-1)}(x-1) \frac{t^{n+1}}{n!} = \sum_{n=1}^{\infty} (-1)^n n C_{n-1}^{(k-1)}(x-1) \frac{t^n}{n!}. \end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ in both sides, we can easily get the identities.

Theorem 2.14. For nonnegative integer n , we have

$$\sum_{m=0}^n (-1)^{n-m} H(m+r+1, r) C_{n-m}^{(s)}(x) = \begin{cases} (-1)^n C_n^{(s-r-1)}(x-1), & r < s-1, \\ (n-x)_n, & r = s-1, \\ (-1)^n D_n^{(r+1-s)}(x-1), & r > s-1. \end{cases} \quad (54)$$

Proof From generating function (1), (19), we get

$$\begin{aligned} \sum_{n=0}^{\infty} \sum_{m=0}^n (-1)^{n-m} H(m+r+1, r) C_{n-m}^{(s)}(x) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} (-1)^n C_n^{(s)} \frac{t^n}{n!} \sum_{n=0}^{\infty} H(n+r+1, r) t^n \\ &= \left(\frac{-t}{\ln(1-t)} \right)^s (1-t)^x \frac{(-\ln(1-t))^{r+1}}{t^{r+1}(1-t)} \\ &= \begin{cases} (-1)^{r+s+1} \left(\frac{t}{\ln(1-t)} \right)^{s-r-1} (1-t)^{x-1}, & r < s-1, \\ (1-t)^{x-1}, & r = s-1, \\ (-1)^{r+s+1} \left(\frac{\ln(1-t)}{t} \right)^{r+1-s} (1-t)^{x-1}, & r > s-1. \end{cases} \end{aligned}$$

$$= \begin{cases} \sum_{n=0}^{\infty} (-1)^n C_n^{(s-r-1)} (x-1) \frac{t^n}{n!}, & r < s-1, \\ \sum_{n=0}^{\infty} (n-x)_n \frac{t^n}{n!}, & r = s-1, \\ \sum_{n=0}^{\infty} (-1)^n D_n^{(r+1-s)} (x-1) \frac{t^n}{n!}, & r > s. \end{cases}$$

Comparing the coefficients of $\frac{t^n}{n!}$ in both sides, we can easily get the identities.

Theorem 2.15. For integer $n \geq m \geq 1$, $\alpha \in \mathbb{N}^+$, we have

$$\sum_{i+j+k=n} \binom{n}{i, j, k} D_{i,\alpha}^{(m)}(x) C_j^{(m)}(x) L(n, m; r) = (-1)^m \binom{n}{m} b_{n,\alpha}^{(m)}(2x+r-m). \quad (55)$$

Proof From generating function (1), (7), (17), we get

$$\begin{aligned} & \sum_{n=m}^{\infty} \sum_{i+j+k=n} \binom{n}{i, j, k} D_{i,\alpha}^{(m)}(x) C_j^{(m)}(x) L(n, m; r) \frac{t^n}{n!} \\ &= \sum_{i=0}^{\infty} D_{i,\alpha}^{(m)}(x) \frac{t^i}{i!} \sum_{j=0}^{\infty} C_j^{(m)}(x) \frac{t^j}{j!} \sum_{k=m}^{\infty} L(k, m; r) \frac{t^k}{k!} \\ &= \left(\frac{\alpha \ln(1+t)}{(1+t)^\alpha - 1} \right)^m (1+t)^x \left(\frac{t}{\ln(1+t)} \right)^m (1+t)^x \frac{\left(\frac{-t}{1+t} \right)^m}{m!} (1+t)^r \\ &= \left(\frac{\alpha t}{(1+t)^\alpha - 1} \right)^m (1+t)^{2x} \frac{\left(\frac{-t}{1+t} \right)^m}{m!} (1+t)^r \\ &= \sum_{n=0}^{\infty} b_{n,\alpha}^{(m)}(2x+r-m) \frac{t^n}{n!} \frac{(-1)^m t^m}{m!} = \sum_{n=m}^{\infty} (-1)^m \binom{n}{m} b_{n,\alpha}^{(m)}(2x+r-m) \frac{t^n}{n!}. \end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ in both sides, we can easily get the identities.

Theorem 2.16. For nonnegative integer n , we have

$$\sum_{m=0}^n \binom{n}{m} Ch_{n-m}^{(-r)}(x) C_m^{(r)}(x) = \sum_{m=0}^n \binom{n}{m} \left(\frac{1}{2} \right)^m (r)_m C_{n-m}^{(r)}(2x). \quad (56)$$

Proof From generating function (1), (14), we get

$$\begin{aligned} \sum_{n=0}^{\infty} \sum_{m=0}^n \binom{n}{m} Ch_{n-m}^{(-r)}(x) C_m^{(r)}(x) \frac{t^n}{n!} &= \sum_{n=0}^{\infty} Ch_{n-m}^{(-r)}(x) \frac{t^n}{n!} \sum_{m=0}^n C_m^{(r)}(x) \frac{t^m}{m!} \\ &= \left(\frac{2+t}{2} \right)^r (1+t)^x \left(\frac{t}{\ln(1+t)} \right)^r (1+t)^x \\ &= \left(\frac{t}{\ln(1+t)} \right)^r (1+t)^{2x} \left(1 + \frac{1}{2}t \right)^r \\ &= \sum_{n=0}^{\infty} C_n^{(r)}(2x) \frac{t^n}{n!} \sum_{m=0}^n (r)_m \left(\frac{1}{2} \right)^m \frac{t^m}{m!} \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^n \binom{n}{m} \left(\frac{1}{2} \right)^m (r)_m C_{n-m}^{(r)}(2x) \frac{t^n}{n!}. \end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ in both sides, we can easily get the identities.

Theorem 2.17. For integer $n \geq r \geq 1$, we have

$$\sum_{m=0}^n \binom{n}{m} CG_{n-m}^{(r)}(x) C_m^{(s)}(x) = \begin{cases} \sum_{m=0}^n \frac{n!}{(n+r-m)!m!} Ch_m^{(r)} C_{n+r-m}^{(s)}(2x), & r < s, \\ \binom{n}{r} Ch_{n-r}^{(r)}(2x), & r = s, \\ \sum_{m=0}^{n-r} \frac{n!(r-s)!}{(n-s-m)!m!} Ch_m^{(r)}(2x) B_{n-s-m,r-s}(0!, (-1)!, \dots), & r > s. \end{cases} \quad (57)$$

Proof From generating function (1), (15), we get

$$\begin{aligned} & \sum_{n=0}^{\infty} \sum_{m=0}^n \binom{n}{m} CG_{n-m}^{(r)}(x) C_m^{(s)}(x) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} CG_n^{(r)}(x) \frac{t^n}{n!} \sum_{n=0}^{\infty} C_n^{(s)}(x) \frac{t^n}{n!} \\ &= \left(\frac{2 \ln(1+t)}{2+t} \right)^r (1+t)^x \left(\frac{t}{\ln(1+t)} \right)^s (1+t)^x \\ &= \begin{cases} \left(\frac{2}{2+t} \right)^r (1+t)^{2x} \left(\frac{t}{\ln(1+t)} \right)^s t^{-r}, & r < s, \\ \left(\frac{2t}{2+t} \right)^r (1+t)^{2x}, & r = s, \\ \left(\frac{2}{2+t} \right)^r (1+t)^{2x} [\ln(1+t)]^{r-s} t^s, & r > s. \end{cases} \\ &= \begin{cases} t^{-r} \sum_{n=0}^{\infty} Ch_n^{(r)} \frac{t^n}{n!} \sum_{n=0}^{\infty} C_n^{(s)}(2x) \frac{t^n}{n!}, & r < s, \\ t^r \sum_{n=0}^{\infty} Ch_n^{(r)}(2x) \frac{t^n}{n!}, & r = s, \\ t^s (r-s)! \sum_{n=0}^{\infty} Ch_n^{(r)}(2x) \frac{t^n}{n!} \frac{\left(\sum_{m=1}^{\infty} (m-1)! (-1)^{m-1} \frac{t^m}{m!} \right)^{r-s}}{(r-s)!}, & r > s. \end{cases} \\ &= \begin{cases} t^{-r} \sum_{n=0}^{\infty} \sum_{m=0}^n \binom{n}{m} Ch_m^{(r)} C_{n-m}^{(s)}(2x) \frac{t^n}{n!}, & r < s, \\ \sum_{n=0}^{\infty} Ch_n^{(r)}(2x) \frac{t^{n+r}}{n!}, & r = s, \\ t^s (r-s)! \sum_{n=0}^{\infty} Ch_n^{(r)}(2x) \frac{t^n}{n!} \sum_{n=r-s}^{\infty} B_{n,r-s}(0!, (-1)!, \dots) \frac{t^n}{n!}, & r > s. \end{cases} \\ &= \begin{cases} \sum_{n=r}^{\infty} \sum_{m=0}^n \frac{n!}{(n+r-m)!m!} Ch_m^{(r)} C_{n+r-m}^{(s)}(2x) \frac{t^n}{n!}, & r < s, \\ \sum_{n=r}^{\infty} \frac{n!}{(n-r)!} Ch_{n-r}^{(r)}(2x) \frac{t^n}{n!}, & r = s, \\ \sum_{n=r}^{\infty} \sum_{m=0}^{n-r} \frac{n!(r-s)!}{(n-s-m)!m!} Ch_m^{(r)}(2x) B_{n-s-m,r-s}(0!, (-1)!, \dots) \frac{t^n}{n!}, & r > s. \end{cases} \end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ in both sides, we can easily get the identities.

3. Identities about High-Order Cauchy Polynomials

In this section, by means of the Riordan matrix, we derive some new equalities between High-order Cauchy polynomials and Stirling numbers, Bell numbers, Bernoulli numbers, Lah numbers, Changhee numbers and so on.

Theorem 3.1. For nonnegative integer n , we have

$$\sum_{k=0}^n \sum_{j=0}^k L(j, k; r) C_k^{(m)}(x) S_2(n, k) = \frac{n!}{(n+m)!} S_2(n+m, m, r-m-x). \quad (58)$$

Proof By Lemma 2 (29), we get

$$\begin{aligned} R\left(\frac{k!}{n!} L(n, k; r)\right) &= \left((1+t)^r, \frac{-t}{1+t}\right). \quad (59) \\ \sum_{j=0}^k L(j, k; r) C_k^{(m)}(x) &= k! \sum_{j=0}^k \frac{j! L(j, k; r)}{k!} \frac{C_k^{(m)}(x)}{j!} \\ &= k! [t^k] (1+t)^r \left[\left(\frac{y}{\ln(1+y)} \right)^m (1+y)^x \mid y = \frac{-t}{1+t} \right] \\ &= k! [t^k] \left[\left(\frac{t}{\ln(1+t)} \right)^m (1+t)^{r-m-x} \right] = C_k^{(m)}(r-m-x). \\ \sum_{k=0}^n \sum_{j=0}^k L(j, k; r) C_k^{(m)}(x) S_2(n, k) &= n! \sum_{k=0}^n \frac{\sum_{j=0}^k L(j, k; r) C_k^{(m)}(x)}{k!} \frac{k! S_2(n, k)}{n!} \\ &= n! [t^n] \left[\left(\frac{y}{\ln(1+y)} \right)^m (1+y)^{r-m-x} \mid y = e^t - 1 \right] \\ &= n! [t^n] \left[\left(\frac{e^t - 1}{t} \right)^m (e^t)^{r-m-x} \right] \\ &= \frac{n!}{(n+m)!} S_2(n+m, m, r-m-x). \end{aligned}$$

Corollary 3.1. For $r=0$ in (58), we have

$$\sum_{k=0}^n \sum_{j=0}^k L(j, k) C_k^{(m)}(x) S_2(n, k) = \frac{n!}{(n+m)!} S_2(n+m, m, -m-x). \quad (60)$$

By means of Lemma 1, the inverse relation (27). This leads to the following conclusion.

Theorem 3.2. For nonnegative integer n , we have

$$\sum_{k=0}^n \frac{k!}{(k+m)!} S_2(k+m, m, r-m-x) S_1(n, k) = \sum_{j=0}^n L(j, n; r) C_j^{(m)}(x). \quad (61)$$

Theorem 3.3. For nonnegative integer n , we have

$$\sum_{k=0}^n S_2(n, k; r) C_k^{(m)}(x) = \binom{n+m}{m}^{-1} S_2(n+m, m; r+x). \quad (62)$$

Proof By (29), we get

$$R\left(\frac{k!}{n!} S_2(n, k; r)\right) = (e^r, e^t - 1). \quad (63)$$

$$\begin{aligned} \sum_{k=0}^n S_2(n, k; r) C_k^{(m)}(x) &= n! \sum_{k=0}^n \frac{k! S_2(n, k; r)}{n!} \frac{C_k^{(m)}(x)}{k!} \\ &= n! [t^n] e^{rt} \left[\left(\frac{y}{\ln(1+y)} \right)^m (1+y)^x \mid y = e^t - 1 \right] \\ &= n! [t^n] \left(\frac{e^t - 1}{t} \right)^m e^{(r+x)t} = n! [t^{n+m}] (e^t - 1)^m e^{(r+x)t} \\ &= \frac{n! m! S_2(n+m, m; r+x)}{(n+m)!} = \binom{n+m}{m}^{-1} S_2(n+m, m; r+x), \end{aligned}$$

which completes the proof.

By (27), we have

Theorem 3.4. For nonnegative integer n , we have

$$\sum_{k=0}^n S_1(n, k; r) \binom{k+m}{m}^{-1} S_2(k+m, m; r+x) = C_n^{(m)}(x), \quad (64)$$

which completes the proof.

Theorem 3.5. For nonnegative integer n , we have

$$\sum_{k=0}^n \sum_{j=1}^k B(k, j) C_j^{(m)}(x) S_1(n, k) = \binom{n+m}{m}^{-1} S_2(n+m, m; x). \quad (65)$$

Proof By (29), we get

$$R\left(\frac{k!}{n!} B(n, k)\right) = (1, e^{e^t-1} - 1), \quad R\left(\frac{k!}{n!} S_1(n, k)\right) = (1, \ln(1+t)). \quad (66)$$

$$\begin{aligned} \sum_{j=1}^k B(k, j) C_j^{(m)}(x) &= k! \sum_{j=1}^k \frac{j! B(k, j)}{k!} \frac{C_j^{(m)}(x)}{j!} \\ &= k! [t^k] \left[\left(\frac{y}{\ln(1+y)} \right)^m (1+y)^x \mid y = e^{e^t-1} - 1 \right] \\ &= k! [t^k] \left(\frac{e^{e^t-1} - 1}{e^t - 1} \right)^m e^{x(e^t-1)}. \end{aligned}$$

$$\begin{aligned} \sum_{k=0}^n \sum_{j=1}^k B(k, j) C_j^{(m)}(x) S_1(n, k) &= n! \sum_{k=0}^n \frac{\sum_{j=1}^k B(k, j) C_j^{(m)}(x) k! S_1(n, k)}{n!} \\ &= n! [t^n] \left[\left(\frac{e^{e^y-1} - 1}{e^y - 1} \right)^m e^{x(e^y-1)} \mid y = \ln(1+t) \right] \end{aligned}$$

$$\begin{aligned}
 &= n! [t^n] \left(\frac{e^t - 1}{t} \right)^m e^{xt} = n! [t^{n+m}] (e^t - 1)^m e^{xt} \\
 &= \frac{n! m! S_2(n+m, m; x)}{(n+m)!} = \binom{n+m}{m}^{-1} S_2(n+m, m; x),
 \end{aligned}$$

which completes the proof.

By (27), we have

Theorem 3.6. For integer $n \geq 1$, we have

$$\sum_{k=0}^n \binom{k+m}{m}^{-1} S_2(k+m, m; x) S_2(n, k) = \sum_{j=1}^n B(n, j) C_j^{(m)}(x). \tag{67}$$

Theorem 3.7. For nonnegative integer $n \geq 1$, we have

$$\begin{aligned}
 &\sum_{k=0}^n \sum_{j=0}^k \binom{n}{k} C_j^{(m)}(x) S_2(k, j) B_{n-k}^{(r)}(x) \\
 &= \begin{cases} B_n^{(r-m)}(2x), & m < r, \\ (2x)^n, & m = r, \\ \binom{n+m-r}{n}^{-1} S_2(n+m-r, m-r; 2x), & m > r. \end{cases} \tag{68}
 \end{aligned}$$

Proof By (29), we get

$$R\left(\frac{k!}{n!} S_2(n, k)\right) = (1, e^t - 1). \tag{69}$$

$$\begin{aligned}
 \sum_{j=0}^k C_j^{(m)}(x) S_2(k, j) &= k! \sum_{j=0}^k \frac{j! S_2(k, j)}{k!} \frac{C_j^{(m)}(x)}{j!} \\
 &= k! [t^k] \left[\left(\frac{y}{\ln(1+y)} \right)^m (1+y)^x \mid y = e^t - 1 \right] \\
 &= k! [t^k] \left(\frac{e^t - 1}{t} \right)^m e^{xt}.
 \end{aligned}$$

$$\begin{aligned}
 &\sum_{k=0}^n \sum_{j=0}^k \binom{n}{k} C_j^{(m)}(x) S_2(k, j) B_{n-k}^{(r)}(x) \\
 &= n! \sum_{k=0}^n \frac{\sum_{j=0}^k C_j^{(m)}(x) S_2(k, j)}{k!} \frac{B_{n-k}^{(r)}(x)}{(n-k)!} \\
 &= n! \sum_{k=0}^n [t^k] \left(\frac{e^t - 1}{t} \right)^m e^{xt} [t^{n-k}] \left(\frac{t}{e^t - 1} \right)^r e^{xt} \\
 &= \begin{cases} n! [t^n] \left(\frac{t}{e^t - 1} \right)^{r-m} e^{2xt}, & m < r, \\ n! [t^n] e^{2xt}, & m = r, \\ n! [t^n] \left(\frac{e^t - 1}{t} \right)^{m-r} e^{2xt}, & m > r. \end{cases}
 \end{aligned}$$

$$= \begin{cases} B_n^{(r-m)}(2x), & m < r, \\ (2x)^n, & m = r, \\ \binom{n+m-r}{n}^{-1} S_2(n+m-r, m-r; 2x), & m > r, \end{cases}$$

which completes the proof.

Theorem 3.8. For integer $n, r \geq 1$, we have

$$\sum_{k=0}^n P(r, n, k) \frac{C_k^{(m)}(x)}{k!} = \begin{cases} \binom{n+x}{m+x} P(r-m, n+x, m+x), & m < r, \\ \binom{x+n}{n-m}, & m = r, \\ \frac{(-1)^{n-r} C_{n-r}^{(m-r)}}{(n-r)!}, & m > r. \end{cases} \quad (70)$$

Proof By (29), we get

$$R\left(\binom{n}{k} P(r, n, k)\right) = \left(\frac{(-\ln(1-t))^r}{1-t}, \frac{t}{1-t}\right). \quad (71)$$

$$\begin{aligned} & \sum_{k=0}^n \binom{n}{k} P(r, n, k) \frac{C_k^{(m)}(x)}{k!} \\ &= \left[t^n\right] \frac{(-\ln(1-t))^r}{1-t} \left[\left(\frac{y}{\ln(1+y)}\right)^m (1+y)^x \mid y = \frac{t}{1-t}\right] \\ &= \left[t^n\right] \frac{(-\ln(1-t))^r}{1-t} \left(\frac{-t}{(1-t)\ln(1-t)}\right)^m (1-t)^{-x} \\ &= \begin{cases} \left[t^{n-m}\right] \frac{(-\ln(1-t))^{r-m}}{(1-t)^{m+x+1}}, & m < r, \\ (1-t)^{-(m+x+1)}, & m = r, \\ (-1)^{m-r} \left(\frac{t}{\ln(1+t)}\right)^{m-r} (1-t)^{-(m+x+1)}, & m > r. \end{cases} \\ &= \begin{cases} \binom{n+x}{m+x} P(r-m, n+x, m+x), & m < r, \\ \binom{x+n}{n-m}, & m = r, \\ \frac{(-1)^{n-r} C_{n-r}^{(m-r)}}{(n-r)!}, & m > r, \end{cases} \end{aligned}$$

which completes the proof.

Similarly, we can obtain

Theorem 3.9. For integer $n, r \geq 1$, we have

$$\sum_{k=0}^n \binom{n}{k} P(r, n, k) \frac{\hat{C}_k^{(m)}(x)}{k!} = \begin{cases} \binom{n-m+x}{x} P(r-m, n-m+x, x), & m < r, \\ \binom{x+n-m}{n-m}, & m = r, \\ (-1)^{n-r} \frac{C_{n-r}^{(m-r)}(-x-1)}{(n-r)!}, & m > r. \end{cases} \quad (72)$$

Theorem 3.10. For nonnegative integer n , we have

$$\sum_{k=0}^n \binom{n}{k} \frac{C_k^{(m)}(x)}{k!} = \frac{(-1)^n C_n^{(m)}(-m-x-1)}{n!}. \quad (73)$$

$$\sum_{k=0}^n \binom{n}{k} (-1)^{n-k} r_{n-k} C_k^{(m)}(x) = (-1)^n C_n^{(m)}(x+r). \quad (74)$$

$$\sum_{k=0}^n \binom{n}{k} (r+n-k-1)_{n-k} (-1)^k C_k^{(m)}(x) = (-1)^n C_n^{(m)}(x-r). \quad (75)$$

$$\sum_{k=0}^n \binom{n+i}{k+i} \frac{C_k^{(m)}(x)}{k!} = \frac{(-1)^n C_n^{(m)}(-m-x-i-1)}{n!}. \quad (76)$$

Proof By (29), we get

$$R\left(\binom{n}{k}\right) = \left(\frac{1}{1-t}, \frac{t}{1-t}\right), \quad R\left((-1)^{n-k} \binom{r}{n-k}\right) = \left((1-t)^r, t\right). \quad (77)$$

$$R\left(\binom{r+n-k-1}{n-k}\right) = \left(\frac{1}{(1-t)^r}, t\right), \quad R\left(\binom{n+i}{k+i}\right) = \left(\frac{1}{(1-t)^{i+1}}, \frac{t}{1-t}\right). \quad (78)$$

We only prove the first equation.

$$\begin{aligned} \sum_{k=0}^n \binom{n}{k} \frac{C_k^{(m)}(x)}{k!} &= [t^n] \frac{1}{1-t} \left[\left(\frac{y}{\ln(1+y)}\right)^m (1+y)^x \mid y = \frac{t}{1-t} \right] \\ &= [t^n] \frac{1}{1-t} \left(\frac{-t}{(1-t)\ln(1-t)}\right)^m (1-t)^{-x} \\ &= [t^n] \left(\frac{-t}{(1-t)\ln(1-t)}\right)^m (1-t)^{-(m+x+1)} \\ &= \frac{(-1)^n C_n^{(m)}(-m-x-1)}{n!}, \end{aligned}$$

which completes the proof.

Theorem 3.11. For nonnegative integer n , we have

$$\sum_{k=0}^n 2^{n-k} \binom{n}{k} \frac{Ch_{n-k}^{(r)} C_k^{(m)}(x)}{(n-k)! k!} = C_n^{(m)}(x-r). \quad (79)$$

Proof By (29), we get

$$R\left((-2)^{n-k} \frac{Ch_{n-k}^{(r)}}{(n-k)!}\right) = \left(\frac{1}{(1-t)^r}, t\right). \quad (80)$$

$$\begin{aligned}
& \sum_{k=0}^n (-2)^{n-k} \frac{Ch_{n-k}^{(r)}}{(n-k)!} (-1)^k \frac{C_k^{(m)}(x)}{k!} \\
&= [t^n] \frac{1}{(1-t)^r} \left[\left(\frac{-y}{\ln(1-y)} \right)^m (1-y)^x \mid y=t \right] \\
&= [t^n] \left(\frac{-t}{\ln(1-t)} \right)^m (1-t)^{x-r} \\
&= \frac{(-1)^n C_n^{(m)}(x-r)}{n!},
\end{aligned}$$

which completes the proof.

Theorem 3.12. For nonnegative integer $n \geq 1$, we have

$$\sum_{l=0}^n \sum_{p=0}^l \binom{n}{l} \alpha^p \beta^{l-p} S_1(n-l, p) Ch_{l-p}^{(k)} C_p^{(m)}(x) = \sum_{j=1}^n \frac{CG_n^{k,j}(\alpha, \beta) C_j^{(m)}(x)}{j!}. \quad (81)$$

Proof By (29), we get

$$R \left(\frac{CG_n^{k,j}(\alpha, \beta)}{n!} \right) = (2^k (2 + \beta t)^{-k}, \ln(1 + \alpha t)). \quad (82)$$

$$\begin{aligned}
& \sum_{j=1}^n \frac{CG_n^{k,j}(\alpha, \beta) C_j^{(m)}(x)}{n! j!} \\
&= [t^n] 2^k (2 + \beta t)^{-k} \left[\left(\frac{y}{\ln(1+y)} \right)^m (1+y)^x \mid y = \ln(1 + \alpha t) \right] \\
&= [t^n] \left(\frac{2}{2 + \beta t} \right)^k \left(\frac{\ln(1 + \alpha t)}{\ln(1 + \ln(1 + \alpha t))} \right)^m (1 + \ln(1 + \alpha t))^x \\
&= [t^n] \sum_{l=0}^{\infty} Ch_l^{(k)} \beta^l \frac{t^l}{l!} \sum_{p=0}^{\infty} C_p^{(m)}(x) \frac{(\ln(1 + \alpha t))^p}{p!} \\
&= [t^n] \sum_{l=0}^{\infty} Ch_l^{(k)} \beta^l \frac{t^l}{l!} \sum_{p=0}^{\infty} C_p^{(m)}(x) \sum_{n=p}^{\infty} S_1(n, p) \alpha^p \frac{t^n}{n!} \\
&= [t^n] \sum_{n=0}^{\infty} \sum_{l=0}^n \sum_{p=0}^l \binom{n}{l} S_1(n-l, p) \alpha^p Ch_{l-p}^{(k)} \beta^{l-p} C_p^{(m)}(x) \frac{t^n}{n!} \\
&= \sum_{l=0}^n \sum_{p=0}^l \frac{1}{(n-l)! l!} \alpha^p \beta^{l-p} S_1(n-l, p) Ch_{l-p}^{(k)} C_p^{(m)}(x),
\end{aligned}$$

which completes the proof.

Corollary 3.2. For $\alpha = \beta$ in (81), we have

$$\sum_{l=0}^n \sum_{p=0}^l \binom{n}{l} \alpha^l S_1(n-l, p) Ch_{l-p}^{(k)} C_p^{(m)}(x) = \sum_{j=1}^n \frac{CG_n^{k,j}(\alpha, \alpha) C_j^{(m)}(x)}{j!}. \quad (83)$$

Funding

Supported by the National Natural Science Foundation of China under Grant 11461050 and Natural Science Foundation of Inner Mongolia 2020MS01020.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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