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A Modified Newton Method Based on the Metric Projector for Solving SOCCVI Problem

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Abstract

In this paper, based on the second-order sufficient condition, the Clarke's generalized Jacobian of the Karush-Kuhn-Tucker system of the second-order cone constrained variational inequality (SOCCVI) problem that is nonsingular is proved by us. A modified Newton method with Armijo line search is presented. Three illustrative examples are given to show how the modified Newton method works.

Keywords

Second-Order Cone, Variational Inequality, Metric Projector, Second-Order Sufficient Condition

1. Introduction

There have been many publications about computational approaches to solve the optimization problems such as linear programming, nonlinear programming, variational inequalities, and complementarity problems, see [1]-[6] and references therein. Some complementarity functions, such as nature function and Fischer-Burmeister (FB) function, have been widely and deeply studied for dealing with nonlinear complementarity problems and variational inequality problems with polyhedral cone constraints, see the famous book by Facchiniei and Pang [7]. A lot of methods for complementarity problems, variational inequality problems and nonsmooth equations have been studied by some researchers, see [7]-[16]. Based on the above research, we used the Fischer-Burmeister operator over the second order cone to deal with second-order cone constrained variational inequality (SOCCVI) problems, see [17]. However, in [17], we have only studied the first-order necessary conditions for SOCCVI problem, and no results about the second-order sufficient conditions of SOCCVI have appeared.

In this paper, we define the second-order sufficient condition of SOCCVI based on the metric projector. Based on the second-order sufficient condition and the constraint nondegeneracy, we prove the nonsingularity of the Clarke's generalized Jacobian of the Karush-Kuhn-Tucker system, constructed by the metric projector.

The second-order cone constrained variational inequality (SOCCVI) problem is to find $a \in Q$ satisfying

$$\langle f(a), b-a \rangle \geq 0, \ \forall b \in Q, \tag{1}$$

where the set Q is finitely representable and expressed as

$$Q = \{a \in \mathbb{R}^n \mid g(a) = 0, h(a) \in \mathcal{K}\}.$$

Here $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product, $f : \mathbb{R}^n \to \mathbb{R}^n$, $g : \mathbb{R}^n \to \mathbb{R}^m$ and $h : \mathbb{R}^n \to \mathbb{R}^l$ are continuously differentiable functions, and $\mathcal{K}$ is a Cartesian product of second-order cones (or Lorentz cones), expressed as

$$\mathcal{K} = \mathcal{K}^{l_1} \times \mathcal{K}^{l_2} \times \cdots \times \mathcal{K}^{l_p}, \tag{2}$$

with $l \geq 0$, $l_1, l_2, \cdots, l_p \geq 1$, $l_1 + l_2 + \cdots + l_p = l$. We denote $h(a) = \left(h^1(a), \cdots, h^p(a)\right)^{\mathrm{T}}$ and $h^i = \left(h_0^i, \bar{h}^i\right) : \mathbb{R}^n \to \mathbb{R}^{l_i}$ for $i \in \{1, \cdots, p\}$. So we have the following equivalence relations

$$h(a) \in \mathcal{K} \Leftrightarrow h^i(a) \in \mathcal{K}^{l_i}, \forall i \in \{1, \cdots, p\} \Leftrightarrow h_0^i(a) \geq \left\|\bar{h}^i(a)\right\|, \forall i \in \{1, \cdots, p\}.$$

The convex second-order cone program (CSOCP):

$$\begin{aligned} \min \ & F(a) \\ \text{s.t.} \ & g(a) = 0 \\ & h(a) \in \mathcal{K} \end{aligned} \tag{3}$$

where $h : \mathbb{R}^n \to \mathbb{R}^m$, $g : \mathbb{R}^n \to \mathbb{R}^l$, and $F : \mathbb{R}^n \to \mathbb{R}$, is the special case of the SOCCVI problem. The SOCCVI can be solved by analyzing its KKT conditions:

$$\begin{cases} L_F(x, \mu, \lambda) = 0, \\ \langle h(a), \lambda \rangle = 0, h(a) \in \mathcal{K}, -\lambda \in \mathcal{K}, \\ g(a) = 0, \end{cases} \tag{4}$$

where $L_F(a, \mu, \lambda) = f(a) + \nabla g(a)\mu - \nabla h(a)\lambda$ is the variational inequality Lagrangian function, $\mu \in \mathbb{R}^m$ and $\lambda \in \mathbb{R}^l$. In [18], we also point out that the neural network approach for SOCCVI was already studied.

As mentioned earlier, this paper investigates the characterizations of the strong regularity of Karush-Kuhn-Tucker (KKT) for SOCCVI via the metric projector. In this paper, we use the sufficient condition of the nonsingularity the Clarke's generalized Jacobian of the KKT system of (1), which deduces the nonsingularity of the Clarke's generalized Jacobian of this system and the strong regularity of the KKT point. We employ modified Newton method for solving the SOCCVI problem and obverse the numerical performance.

2. Preliminaries

In this section, we organize some basic knowledge points. Most of these basic knowledge points can be found in [19].

We briefly recall some concepts associated with SOC, which are helpful for understanding the target problems and our analysis techniques. For any two vectors $a=(a_0,\bar{a})$ and $b=(b_0,\bar{b})$ in $\mathbb{R}\times\mathbb{R}^{n-1}$, we use the Euclidean inner product $\langle a,b\rangle := a^{\mathrm{T}}b$, and the norm $\|\cdot\|$ is induced by this inner product, *i.e.*, $\|a\|=\sqrt{a^{\mathrm{T}}a}$. And we define their Jordan product as $a\circ b := \left(ab^{\mathrm{T}}, b_0\bar{a}+a_0\bar{b}\right)$. Then, $\left(\mathbb{R}\times\mathbb{R}^{n-1},\circ\right)$ together with the element $e=(1,0,\cdots,0)^{\mathrm{T}}\in\mathbb{R}\times\mathbb{R}^{n-1}$ give rise to a Jordan algebra. Note that a^2 means $x\circ x$ and $x+y$ means the usual componentwise addition of vectors. It is known that $a^2\in\mathcal{K}^n$ for all $a\in R^n$. Moreover, if $a\in\mathcal{K}^n$, then there exists a unique vector in $\mathcal{K}^n$, denoted by $a^{1/2}$, such that $\left(a^{1/2}\right)^2=a^{1/2}\circ a^{1/2}=a$. We also denote $|a| := \left(a^2\right)^{1/2}$. Any $a=(a_0,\bar{a})\in\mathbb{R}\times\mathbb{R}^{n-1}$ has the following spectral decomposition:

$$a=\lambda_1(a)v_1(a)+\lambda_2(a)v_2(a), \tag{5}$$

where $\lambda_i(a), v_i(a)$ are the spectral values and the associated spectral vectors of *a*, given by

$$\lambda_i(a)=a_0+(-1)^i\|\bar{a}\|, v_i(a)=\begin{cases}\frac{1}{2}\left(1,(-1)^i\frac{\bar{a}}{\|\bar{a}\|}\right), & \text{if } \bar{a}\neq 0;\\ \frac{1}{2}\left(1,(-1)^i w\right), & \text{if } \bar{a}=0,\end{cases} \tag{6}$$

for $i=1,2$, where *c* is any vector in $\mathbb{R}^{n-1}$ satisfying $\|w\|=1$.

Suppose $a=(a_0,\bar{a})\in\mathbb{R}\times\mathbb{R}^{n-1}$ having the spectral decomposition as (5), then the merit projector of u onto $\mathcal{K}^n$, denoted by $\mathcal{K}^n(x)$, is

$$\Pi_{\mathcal{K}^n}(a)=\left[\lambda_1(a)\right]_+ v_1(a)+\left[\lambda_2(a)\right]_+ v_2(a), \tag{7}$$

here $\left[\lambda_i(a)\right]_+=\max\left\{0,\lambda_i(a)\right\}, i=1,2$. we have

$$\Pi_{\mathcal{K}^n}(x)=\begin{cases}\frac{1}{2}\left(1+\frac{a_0}{\|\bar{a}\|}\right)\left(\|\bar{a}\|,\bar{a}\right), & \text{if } |a_0|<\|\bar{a}\|,\\ (a_0,\bar{a}), & \text{if } \|\bar{a}\|<a_0,\\ 0, & \text{if } \|\bar{a}\|\leq -a_0.\end{cases} \tag{8}$$

Lemma 2.1. *Let the metric projection operator* $\Pi_{\mathcal{K}^n}(\cdot)$ *is directionally differentiable at x for any* $t\in\mathbb{R}^n$,

$$\Pi'_{\mathcal{K}^n}(a,t)=\begin{cases} J\Pi_{\mathcal{K}^n}(a)t, & \text{if } a\in\mathbb{R}^n\setminus\left\{\mathcal{K}^n\cup\mathcal{K}^n\right\}\\ t, & \text{if } a\in\operatorname{int}\mathcal{K}^n\\ t-2\left[v_1(a)^{\mathrm{T}}t\right]_- v_1(a), & \text{if } a\in bd\mathcal{K}^n\setminus\{0\}\\ 0, & \text{if } a\in-\operatorname{int}\mathcal{K}^n\\ 2\left[v_2(a)^{\mathrm{T}}v\right]_+ v_2(a), & \text{if } a\in -bd\mathcal{K}^n\setminus\{0\}\\ \Pi_{\mathcal{K}^n}(t), & \text{if } a=0\end{cases}$$

where

$$J\Pi_{\mathcal{K}^n}(a)=\frac{1}{2}\begin{pmatrix}1 & \frac{\bar{a}^{\mathrm{T}}}{\|\bar{a}\|}\\ \frac{\bar{a}}{\|\bar{a}\|} & I+\frac{a_0}{\|\bar{a}\|}I-\frac{a_0}{\|\bar{a}\|}\frac{\bar{a}\bar{a}^{\mathrm{T}}}{\|\bar{a}\|^2}\end{pmatrix}$$

$$\left[v_1(x)t^{\mathrm{T}}\right]_- := \min\left\{0, v_1(a)^{\mathrm{T}}t\right\}, \left[v_2(a)t^{\mathrm{T}}\right]_+ := \max\left\{0, v_2(a)^{\mathrm{T}}t\right\}.$$

We recall from Lemma 2.5 in [20] the characterization of the tangent cone of a second-order cone at a point in it.

Lemma 2.2. *Consider the second-order cone* $\mathcal{K}^n$ *and let* $a\in\mathcal{K}^n$*. Then, the tangent cone and the second-order tangent cone of* $\mathcal{K}^n$ *at a are*

$$T_{\mathcal{K}^n}(a)=\begin{cases}\mathbb{R}^n, & \text{if } a\in \operatorname{int}\mathcal{K}^n,\\ \mathcal{K}^n, & \text{if } a=0,\\ \left\{t=(t_0,\bar{t})\in\mathbb{R}\times\mathbb{R}^{n-1} \mid \langle\bar{t},\bar{a}\rangle-a_0t_0\le 0\right\}, & \text{if } a\in bd\mathcal{K}^n\setminus\{0\}.\end{cases}$$

and

$$T^2_{\mathcal{K}^n}(a,t)=\begin{cases}\mathbb{R}^n, & \text{if } a\in \operatorname{int}T_{\mathcal{K}^n},\\ T_{\mathcal{K}^n}(t), & \text{if } a=0,\\ \left\{c=(c_0,\bar{c})\in\mathbb{R}\times\mathbb{R}^{n-1} \mid \langle\bar{c},\bar{s}\rangle-c_0a_0\le t_0^2-\|\bar{t}\|^2\right\}, & \text{otherwise.}\end{cases}$$

For the convenience of discussions, we need the definition of the tangent cone, regular and normal cone of a closed set at a point, all the concepts are taken from [21].

Definition 2.1. *For a closed set* $K\subset\mathbb{R}^n$ *and a point* $\bar{a}\in K$, *define the following sets: the tangent* (*Bouligand*) *cone*

$$T_K(a):=\limsup_{d\downarrow 0}\frac{K-\bar{a}}{d},$$

the regular (*Frchet*) *normal cone*

$$\hat{N}_K(\bar{a}):=\left\{w\in\mathbb{R}^n \mid \langle w,a-\bar{a}\rangle\le o(\|a-\bar{a}\|),\forall a\in K\right\},$$

the limiting (*in the sense of Mordukhovich*) *normal cone*

$$N_K(\bar{a}):=\limsup_{a\xrightarrow{K}\bar{a}}\hat{N}_K(\bar{a}).$$

If K is a closed convex set, then
$T_K(\bar{a})=v(K+\mathbb{R}\bar{a})\cdot\hat{N}_K(\bar{a})=N_K(\bar{a})=T_K(\bar{a})^{\circ}=\left\{w\in K^{\circ} \mid \langle w,a\rangle\le 0\right\}$.

Let A,B,Z and N be finite-dimensional real Hilbert spaces and f is a mapping from $A\times B\times Z$ to N. If f is Fréchet differentiable at $(a,b,z)\in A\times B\times Z$, then we use $Jf(a,b,z)$ (respectively, $J_af(a,b,z)$) to denote the Fréchet derivative of f at (a,b,z) (respectively, the partial Fréchet derivative of f at (a,b,z) with respect to a). And let $\nabla f(a,b,z):=Jf(a,b,z)^{\mathrm{T}}$ be the adjoint of $Jf(a,b,z)$ (respectively, $\nabla_a f(a,b,z):=J_af(a,b,z)^{\mathrm{T}}$), where the adjoint operator $(\cdot)^{\mathrm{T}}$ satisfies the following formula: if the operation is consistent, then $\langle a, Mb\rangle = \langle M^{\mathrm{T}}a, b\rangle$. If *f* is twice Fréchet differentiable at $(a,b,z)\in A\times B\times Z$, we define

$$J^2 f(a,b,z) := J(Jf)(a,b,z), J_{aa}^2 F(a,b,z) := J_a(J_a f)(a,b,z),$$

$$\nabla^2 f(a,b,z) := J(\nabla f)(a,b,z), \nabla_{aa}^2 F(a,b,z) := J_a(\nabla_a F)(a,b,z).$$

3. The Optimality Condition and Nonsingularity Theorem

Let $L_f(a,\mu,\lambda) = f(a) + Jg(a)^{\mathrm{T}}\mu - Jh(a)^{\mathrm{T}}\lambda$ be the Lagrangian of (1), where $(\mu,\lambda) = (\mu,\lambda_1,\cdots,\lambda_p)\in\mathbb{R}^m\times\mathbb{R}^{l_1}\times\cdots\times\mathbb{R}^{l_p} = \mathbb{R}^m\times\mathbb{R}^l$. Let a^* be a locally optimal solution to (1). Then, a^* satisfies the following Karush-Kuhn-Tucker condition. Using the NR function and the definition of the normal cone, the KKT condition can be expressed as

$$H(a,\mu,\lambda) = \begin{pmatrix} L_f(a^*,\mu,\lambda) \\ g(a^*) \\ h(a) - \Pi_{\mathcal{K}}(h(a)-\lambda) \end{pmatrix} = 0$$

Mimicking the arguments described as in [17], we can verify that the KKT system (4) is equivalent to the following unconstrained smooth minimization problem:

$$\min \Phi(z) := \frac{1}{2}\|H(z)\|^2, \tag{9}$$

where $z = (a,\mu,\lambda)\in R^{n+m+l}$ and $S(z)$ is given by

$$H(z) = \begin{bmatrix} L_f(a,\mu,\lambda) \\ -g(x) \\ \phi_{\mathrm{NR}}(-h_{l_1}(a),\lambda_{l_1}) \\ \vdots \\ \phi_{\mathrm{NR}}(-h_{l_q}(a),\lambda_{l_q}) \end{bmatrix},$$

with $h_{l_i}(a),\lambda_{l_i}\in\mathbb{R}^{l_i}$. In other words, $\Phi(z)$ is a smooth merit function for the KKT system (4).

By [20], we give the following definition and theorem.

Definition 3.1. *Let* a^* *be a feasible point of* (1) *such that* $\Lambda(a^*) = (\mu,\lambda)\neq\phi$. *We say that the second-order sufficient condition holds at* a^* *if*

$$\sup_{(\mu,\lambda)\in\Lambda(a^*)}\left\{\left\langle J_a L_f(a^*,\mu,\lambda)t, t\right\rangle + t^{\mathrm{T}}\mathcal{H}(a^*,\lambda)t\right\} > 0, \forall t\in\mathcal{Q}(a^*)\setminus\{0\} \tag{10}$$

Theorem 3.1. *Let* (a^*,μ^*,λ^*) *is the KKT point of* (1). *Assume that*

(i) $\Lambda(a^*)\neq\phi$;

(ii) *the second-order sufficient condition* (10) *holds*;

(iii) $\lambda^*\in\operatorname{int} N_{\mathcal{K}}(h(a^*))$ *holds*;

(iv) *the following constraint nondegeneracy holds*,

$$\begin{pmatrix} Jg\left(a^*\right) \\ Jh\left(a^*\right) \end{pmatrix}\mathbb{R}^n + linT_{\{0_m\}\times\mathcal{K}}\left(g\left(a^*\right),h\left(a^*\right)\right) = \mathbb{R}^m \times \mathbb{R}^l, \tag{11}$$

then $JH\left(a^*,\mu^*,\lambda^*\right)$ *is nonsingular.*

Proof. It follows from $\lambda^* \in \operatorname{int} N_{\mathcal{K}}\left(h\left(a^*\right)\right)$ that $\Pi_{\mathcal{K}}$ is differentiable at $\left(h\left(a^*\right)-\lambda^*\right)$. Let $(ta,t\mu,t\lambda)\in\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^l$. we have

$$JH\left(a^*,\mu^*,\lambda^*\right)\begin{pmatrix} ta \\ t\mu \\ t\lambda \end{pmatrix} = \begin{pmatrix} J_a L_f\left(a^*,\mu^*,\lambda^*\right)ta + Jg\left(a^*\right)^{\mathrm{T}} t\mu - Jh\left(a^*\right)^{\mathrm{T}} t\lambda \\ Jg\left(a^*\right)ta \\ Jh\left(a^*\right)ta - \Pi'_{\mathcal{K}}\left(h\left(a^*\right)-\lambda^*; Jh\left(a^*\right)ta - t\lambda\right) \end{pmatrix} \tag{12}$$

Let $JH\left(a^*,\mu^*,\lambda^*\right)\begin{pmatrix} ta \\ t\mu \\ t\lambda \end{pmatrix} = 0$ (We need to show $ta=0$, $t\mu=0$, $t\lambda=0$).

We get

$$\begin{cases} Jg\left(a^*\right)ta = 0 \\ Jh\left(a^*\right)ta = \Pi'_{\mathcal{K}}\left(h\left(a^*\right)-\lambda^*; Jh\left(a^*\right)ta - t\lambda\right) \end{cases} \tag{13}$$

which implies that $ta \in \mathcal{Q}\left(a^*\right)$. From the first line of (12), we get that

$$\left\langle J_a L_f\left(a^*,\mu^*,\lambda^*\right)ta, ta\right\rangle + \left\langle Jh\left(a^*\right)ta, t\lambda\right\rangle = 0 \tag{14}$$

Define the index sets:

$$I^* = \left\{i : h^i(a) \in \operatorname{int}\mathcal{K}^{l_i}, i = 1,\cdots,p\right\};$$
$$B^* = \left\{i : h^i(a) \in bdry\mathcal{K}^{l_i}, h_i(a) \neq 0\right\};$$
$$Z^* = \left\{i : h^i(a) = 0\right\}.$$

If $\varphi(b_0,b) = b_0 - \|b\|$, $(b_0,y)\in\mathbb{R}^n$, we have

$$\nabla\varphi(b_0,b) = \begin{pmatrix} 1 \\ -\dfrac{b}{\|b\|} \end{pmatrix}$$

Note that

$$\mathcal{Q}_{\mathcal{K}}\left(h\left(a^*\right)\right) = \left\{t\in\mathbb{R}^n : Jh\left(a^*\right)t \in T_{\mathcal{K}}\left(h\left(a^*\right)\right)\right\}$$

and

$$T_{\mathcal{K}}\left(h\left(a^*\right)\right) = \left\{ t \,\middle|\, \begin{matrix} \nabla h_0^i\left(a^*\right)^{\mathrm{T}} t - \dfrac{J\bar{h}^i\left(a^*\right)t}{h_0^i\left(a^*\right)} \geq 0, i\in B^* \\ \nabla h_0^i\left(a^*\right)t - Jh^i\left(a^*\right)t \geq 0, i \in Z^* \end{matrix} \right\}$$

From $-\lambda \perp h(a)$, we can deduce that

$$\lambda = \left\{ \lambda \middle| \begin{array}{l} \lambda^i = 0, i \in I^* \\ \lambda^i = \sigma\left(h_0^i\left(a^*\right), -\bar{h}^i\left(a^*\right)\right), \sigma > 0, i \in B^* \\ \lambda^i \in \operatorname{int} \mathcal{K}_i, i \in Z^* \end{array} \right\}$$

Hence

$$\left[h\left(a^*\right) - \lambda^*\right]_i = \begin{cases} h_i\left(a^*\right) \in \operatorname{int} \mathcal{K}_i, i \in I^* \\ \left(\left(1-\sigma\right)h_0^i\left(a^*\right), \left(1+\sigma\right)\bar{h}^i\left(a^*\right)\right), i \in B^* \\ \lambda^i \in \operatorname{int} \mathcal{K}_i, i \in Z^* \end{cases}$$

Condition (iii) of Theorem 3.1 means

$$\mathcal{Q}\left(a^*\right) = \left\{ t \middle| \begin{array}{l} Jg\left(a^*\right) = 0, Jh^i\left(a^*\right)t = 0, i \in Z^* \\ Jh^i\left(a^*\right)t \in T_{\mathcal{K}}\left(h^i\left(a^*\right)\right), \left\langle \lambda^i, Jh^i\left(a^*\right)t \right\rangle = 0, i \in B^* \end{array} \right\}$$

and $Q\left(a^*\right)$ is a linear space. Therefore,

$$-\delta^*\left(\lambda \middle| T_{\mathcal{K}}^2\left(h\left(a^*\right), Jh\left(a^*\right)t\right)\right) = \sum_{i \in \mathbb{B}^*} \frac{\lambda_0^i}{h_0^i\left(a^*\right)} \left[\left\|\nabla h_0^i\left(a^*\right)^{\mathrm{T}} ta\right\|^2 - \left\|J\bar{h}^i\left(a^*\right)ta\right\|^2\right]$$

In addition, by Lemma 2.3 and (13), we can deduce that

$$\begin{aligned} &\Pi'_{\mathcal{K}}\left(h^i\left(a^*\right) - \lambda^{*i}; Jh^i\left(a^*\right)ta - t\lambda^i\right) \\ &= \frac{1}{2}\begin{pmatrix} 1 & \frac{\left(\bar{b}^i\right)^{\mathrm{T}}}{\left\|\bar{b}\right\|} \\ \frac{\bar{b}^i}{\left\|\bar{b}\right\|} & I_{l_i} + \frac{b_0^i}{\left\|\bar{b}\right\|} I_{l_i} - \frac{b_0^i \bar{b}^i \left(\bar{b}^i\right)^{\mathrm{T}}}{\left\|\bar{b}\right\| \cdot \left\|\bar{b}\right\|^2} \end{pmatrix} \begin{pmatrix} \nabla h_0^i\left(a^*\right)^{\mathrm{T}} ta - t\lambda_0^i \\ Jh^i\left(a^*\right)ta - t\bar{\lambda}^i \end{pmatrix} \\ &= M\left(Jh^i\left(a^*\right)ta - t\lambda^i\right) = Jh^i\left(a^*\right)ta, \end{aligned} \tag{15}$$

where $b = h^i\left(a^*\right) - \lambda^{*i}$. From (15), we have

$$\left[I - M\right] Jh^i\left(a^*\right)t = -Mt\lambda^i,$$

that is

$$\begin{aligned} &\frac{1}{2}\nabla h_0^i\left(a^*\right)^i ta - \frac{1}{2}\frac{\bar{b}_i^{\mathrm{T}}}{\left\|\bar{b}^i\right\|} J\bar{h}^i\left(a^*\right)t \\ &= -\frac{1}{2}t\lambda_0^i - \frac{\bar{b}_i^{\mathrm{T}}}{\left\|\bar{b}^i\right\|^2} t\bar{\lambda}^i - \frac{1}{2}\frac{\bar{b}^i}{\left\|\bar{b}^i\right\|} \nabla h_0^i\left(a^*\right)^{\mathrm{T}} ta - \frac{1}{2}c_i J\bar{h}_i\left(a^*\right)ta - \frac{1}{2}J\bar{h}^i\left(a^*\right)ta \\ &= -\frac{1}{2}\frac{\bar{b}^i}{\left\|\bar{b}\right\|} t\lambda_0^{*i} - \frac{1}{2}t\lambda^{*i} - \frac{1}{2}c_i d\lambda^i, \end{aligned}$$

where $c_i = \dfrac{\bar{h}^i\left(a^*\right)}{\left\|\bar{h}^i\left(a^*\right)\right\|}$.

Case (I). If $i \in B^*$, we have

$$
\begin{aligned}
&\Pi'_{\mathcal{K}_i}\left(h^i\left(a^*\right)-\bar{\lambda}^i; Jh^i\left(a^*\right)ta-t\lambda^i\right) \\
&=\frac{1}{2}\begin{pmatrix} 1 & c_i^{\mathrm{T}} \\ c_i & \frac{2}{1+\sigma}I-\frac{1-\sigma}{1+\sigma}c_ic_i^{\mathrm{T}} \end{pmatrix}\left(Jh^i\left(a^*\right)ta-t\lambda^i\right) \\
&=M_i\left(Jh^i\left(a^*\right)ta-t\lambda^i\right)=Jh^i\left(a^*\right)ta
\end{aligned} \tag{16}
$$

When $i\in B^*$, $\lambda^{*i}=\left(\sigma h_0^i\left(a^*\right),-\sigma\bar{h}^i\left(a^*\right)\right)$. Now we need to prove that $ta\in T_Q\left(a^*\right)$ and

$$
Jh_0^i\left(a^*\right)ta\geq\frac{\bar{h}^i\left(a^*\right)^{\mathrm{T}}J\bar{h}^i\left(a^*\right)ta}{\left\|\bar{h}^i\left(a^*\right)\right\|} \tag{17}
$$

From $h_0^i\left(a^*\right)=\left\|h^i\left(a^*\right)\right\|$, we have

$$
\lambda^{*i}=\begin{pmatrix}\sigma h_0^i\left(a^*\right) \\ -\sigma\bar{h}^i\left(a^*\right)\end{pmatrix}=\sigma h_0^i\left(a^*\right)\begin{pmatrix}1 \\ -c_i\end{pmatrix}
$$

where $\left\|c_i\right\|=1$ and $c_i=\frac{\bar{h}^i\left(a^*\right)}{h_0^i\left(a^*\right)}$. Hence

$$
\lambda^{*i}M_i=\left(1-\left\|c_i\right\|^2, c_i^{\mathrm{T}}-\frac{2}{1+\sigma}c_i^{\mathrm{T}}+\frac{1-\sigma}{1+\sigma}c_i^{\mathrm{T}}\left\|c_i\right\|^2\right)=\left(0,0\right). \tag{18}
$$

(16) and (18) imply

$$
\left\langle\lambda^{*i}, Jh^i\left(a^*\right)ta\right\rangle=0
$$

which means $ta\in Q\left(a^*\right)$.

It follows from (16) that

$$
\begin{aligned}
&M_i\left(Jh^i\left(a^*\right)ta-t\lambda^i\right)=Jh^i\left(a^*\right)ta \\
&\Leftrightarrow\left(M_i-I\right)Jh^i\left(a^*\right)ta=M_it\lambda^i \\
&\Leftrightarrow\left(1,c_i^{\mathrm{T}}\right)\begin{pmatrix}-\frac{1}{2} & \frac{1}{2}c_i^{\mathrm{T}} \\ \frac{1}{2}c_i & \frac{-\sigma}{1+\sigma}I-\frac{1}{2}\frac{1-\sigma}{1+\sigma}c_ic_i^{\mathrm{T}}\end{pmatrix}\begin{pmatrix}\nabla h_0^i\left(a^*\right)^{\mathrm{T}}ta \\ Jh^i\left(a^*\right)^{\mathrm{T}}ta\end{pmatrix} \\
&=\left(1,c_i^{\mathrm{T}}\right)\begin{pmatrix}\frac{1}{2} & \frac{1}{2}c_i^{\mathrm{T}} \\ \frac{1}{2}c_i & \frac{1}{1+\sigma}I-\frac{1}{2}\frac{1-\sigma}{1+\sigma}c_ic_i^{\mathrm{T}}\end{pmatrix}\begin{pmatrix}t\lambda_0^i \\ t\lambda^i\end{pmatrix}
\end{aligned} \tag{19}
$$

Therefore, we can deduce that

$$
\left(1,\frac{1}{2}c_i^{\mathrm{T}}+\frac{\sigma}{1+\sigma}c_i^{\mathrm{T}}+\frac{1}{2}\frac{1-\sigma}{1+\sigma}c_i^{\mathrm{T}}\right)\begin{pmatrix}\nabla h_0^i\left(a^*\right)^{\mathrm{T}}ta \\ J\bar{h}^i\left(a^*\right)ta\end{pmatrix}=0
$$

which is

$$
\left(-1,c_i^{\mathrm{T}}\right)\begin{pmatrix}\nabla h_0^i\left(a^*\right)^{\mathrm{T}}ta \\ J\bar{h}^i\left(a^*\right)ta\end{pmatrix}=0 \tag{20}
$$

From (20), we get

$$\nabla h_0^i\left(a^*\right)^{\mathrm{T}} ta = \frac{\bar{h}^i\left(a^*\right)^{\mathrm{T}} \bar{h}^i\left(a^*\right)ta}{\left\|\bar{h}^i\left(a^*\right)\right\|}.$$

Hence, (17) holds.

Case II. Let $i \in Z^*$. From the second equation of (13), we can get that

$$\Pi'_{\mathcal{K}_i}\left(0-\lambda^i; Jh^i\left(a^*\right)ta - t\lambda^i\right) = Jh^i\left(a^*\right)ta$$

It follows from the projection of the second-order cone that

$$Jh^i\left(a^*\right)ta = 0.$$

Case III. Let $i \in I^*$. From the second equation of (13), we can get that

$$\Pi'_{\mathcal{K}_i}\left(h^i\left(a^*\right), Jh^i\left(a^*\right)ta - t\lambda^i\right) = Jh^i\left(a^*\right)ta - t\lambda^i = Jh^i\left(a^*\right)ta$$

Then $t\lambda^i = 0$.

To sum up the above three situations, $ta \in \mathcal{Q}\left(a^*\right)$ implies

$$\begin{cases} Jh^i\left(a^*\right)ta = 0, i \in Z^* \\ h_0^i\left(a^*\right)\nabla h_0^i\left(a^*\right)^{\mathrm{T}} ta = \bar{h}^i\left(a^*\right)J\bar{h}^i\left(a^*\right)ta, i \in B^*. \end{cases}$$

From (12) and (13), we assume that

$$J_a L_f\left(a^*, \mu^*, \lambda^*\right)ta + Jg\left(a^*\right)^{\mathrm{T}} t\mu - Jh\left(a^*\right)^{\mathrm{T}} a\lambda = 0 \tag{21}$$

$$Jg\left(a^*\right)ta = 0 \tag{22}$$

$$Jh\left(a^*\right)ta - \Pi'_{\mathcal{K}}\left(h\left(a^*\right) - \lambda^*; Jh\left(a^*\right)ta - t\lambda\right) = 0 \tag{23}$$

By (21) and (22), we have

$$\begin{aligned} 0 &= \left\langle ta, J_a L_f\left(a^*, \mu^*, \lambda^*\right)ta + Jg\left(a^*\right)^{\mathrm{T}} t\mu - Jh\left(a^*\right)^{\mathrm{T}} t\lambda \right\rangle \\ &= \left\langle ta, J_a L - f\left(a^*, \mu^*, \lambda^*\right)ta \right\rangle - \sum_{i \in B^*} \left\langle Jh^i\left(a^*\right)^{\mathrm{T}} ta, t\lambda^i \right\rangle. \end{aligned}$$

For $i \in B^*$,

$$\begin{aligned} &\left\langle Jh^i\left(a^*\right)^{\mathrm{T}} ta, t\lambda^i \right\rangle = \nabla h_0^i\left(a^*\right)tat\lambda_0^i + \left\langle J\bar{h}^i\left(a^*\right)^{\mathrm{T}} ta, t\bar{\lambda}^i \right\rangle \\ &= -\nabla h_0^i\left(a^*\right)^{\mathrm{T}} ta \cdot \frac{\bar{h}^i\left(a^*\right)}{\left\|\bar{h}^i\left(a^*\right)\right\|} t\bar{\lambda}^i + \left\langle J\bar{h}^i\left(a^*\right)^{\mathrm{T}} ta, t\bar{\lambda}^i \right\rangle \\ &= -\frac{\bar{h}^i\left(a^*\right)^{\mathrm{T}} J\bar{h}^i\left(a^*\right)ta}{\left\|\bar{h}^i\left(a^*\right)\right\|^2} \bar{h}^i\left(a^*\right)^{\mathrm{T}} t\bar{\lambda}^i + \left\langle J\bar{h}^i\left(a^*\right)^{\mathrm{T}} ta, t\bar{\lambda}^i \right\rangle \\ &= \left\langle \left(J\bar{h}^i\left(a^*\right)ta\right)^{\mathrm{T}} \left[I - \frac{\bar{h}^i\left(a^*\right)\bar{h}^i\left(a^*\right)^{\mathrm{T}}}{\left\|\bar{h}^i\left(a^*\right)\right\|^2} \right], t\bar{\lambda}^i \right\rangle \end{aligned} \tag{24}$$

By (19), we get

$$\begin{pmatrix} -\frac{1}{2}\nabla h_0^i\left(a^*\right)^{\mathrm{T}} ta+\frac{1}{2}\frac{\bar{h}^i\left(a^*\right)^{\mathrm{T}}}{\left\|\bar{h}^i\left(a^*\right)\right\|} J\bar{h}^i\left(a^*\right)ta \\ \frac{1}{2}c_i\left(\nabla h_0^i\left(a^*\right)^{\mathrm{T}} ta-c_i^{\mathrm{T}} J\bar{h}^i\left(a^*\right)ta\cdot\frac{1-\sigma}{1+\sigma}\right)-\frac{\sigma}{1+\sigma}J\bar{h}^i\left(a^*\right)ta \end{pmatrix} = \begin{pmatrix} \frac{1}{2}t\lambda_0^i+\frac{1}{2}c_i^{\mathrm{T}} t\bar{\lambda}^i \\ \frac{1}{2}c_i\left(t\lambda_0^i-\frac{1-\sigma}{1+\sigma}c_i^{\mathrm{T}} d\lambda^i\right)+\frac{1}{1+\sigma}t\bar{\lambda}^i \end{pmatrix} \tag{25}$$

Since

$$\begin{aligned} &\frac{1}{2}c_i\left(\nabla h_0^i\left(a^*\right)^{\mathrm{T}} ta-c_i^{\mathrm{T}} J\bar{h}^i\left(a^*\right)ta\cdot\frac{1-\sigma}{1+\sigma}\right)-\frac{\sigma}{1+\sigma}J\bar{h}^i\left(a^*\right)ta \\ &=\frac{1}{2}c_i\nabla h_0^i\left(\bar{a}\right)^{\mathrm{T}} ta-\frac{1}{2}\frac{1-\sigma}{1+\sigma}c_ic_i^{\mathrm{T}} J\bar{h}^i\left(a^*\right)^{\mathrm{T}} ta-\frac{\sigma}{1+\sigma}J\bar{h}^i\left(a^*\right)ta \\ &=\frac{1}{2}c_i\left(\nabla h_0^i\left(a^*\right)^{\mathrm{T}} ta-\frac{1-\sigma}{1+\sigma}v_i^{\mathrm{T}} J\bar{h}^i\left(a^*\right)ta\right)-\frac{\sigma}{1+\sigma}J\bar{h}^i\left(a^*\right)ta \\ &=\frac{1}{2}c_i\left(\nabla h_0^i\left(a^*\right)^{\mathrm{T}} ta-\frac{1-\sigma}{1+\sigma}\nabla h^i\left(a^*\right)^{\mathrm{T}} ta\right)-\frac{\sigma}{1+\sigma}J\bar{h}^i\left(a^*\right)ta \\ &=\frac{\sigma}{1+\sigma}\left(c_i\nabla h_0^i\left(a^*\right)^{\mathrm{T}} ta-J\bar{h}^i\left(a^*\right)ta\right) \end{aligned} \tag{26}$$

and

$$\begin{aligned} &\frac{1}{2}c_i\left(t\lambda_0^i-\frac{1-\sigma}{1+\sigma}c_i^{\mathrm{T}} t\lambda^i\right)+\frac{1}{1+\sigma}t\bar{\lambda}^i \\ &=\frac{1}{2}c_i\left(t\lambda_0^i+\frac{1-\sigma}{1+\sigma}t\lambda_0^i\right)+\frac{1}{1+\sigma}t\bar{\lambda}^i \\ &=\frac{1}{1+\sigma}c_it\lambda_0^i+\frac{1}{1+\sigma}t\bar{\lambda}^i \\ &=\frac{1}{1+\sigma}\left(c_it\lambda_0^i+t\bar{\lambda}^i\right) \end{aligned} \tag{27}$$

From (25), (26) and (27), we get that

$$\frac{1}{1+\sigma}\left(c_it\lambda_0^i+t\bar{\lambda}^i\right)=\frac{\sigma}{1+\sigma}\left(c_i\nabla h_0^i\left(a^*\right)^{\mathrm{T}} ta-J\bar{h}^i\left(a^*\right)ta\right),$$

that is

$$c_it\lambda_0^i+t\bar{\lambda}^i=\sigma\left(c_i\nabla h_0^i\left(a^*\right)^{\mathrm{T}} ta-J\bar{h}^i\left(a^*\right)ta\right). \tag{28}$$

Note that

$$c_it\lambda_0^i+t\bar{\lambda}^i=\left(I-c_ic_i^{\mathrm{T}}\right)t\bar{\lambda}^i=\left(I-\frac{\bar{h}^i\left(a^*\right)\bar{h}^i\left(a^*\right)^{\mathrm{T}}}{\left\|\bar{h}^i\left(a^*\right)\right\|^2}\right)t\bar{\lambda}^i \tag{29}$$

It follows from (24), (28) and (29) that

$$\begin{aligned}
\left\langle Jh^i\left(a^*\right)^{\mathrm{T}} ta, t\lambda^i \right\rangle &= \left\langle J\bar{h}^i\left(a^*\right)ta, \left(I - \frac{\bar{h}^i\left(a^*\right)\bar{h}^i\left(a^*\right)^{\mathrm{T}}}{\left\|\bar{h}^i\left(a^*\right)\right\|^2}\right)t\bar{\lambda}^i \right\rangle \\
&= \sigma\left(\left\langle Jh^i\left(a^*\right)ta, c_i \nabla h_0^i\left(a^*\right)^{\mathrm{T}} ta\right\rangle - \left\|J\bar{h}^i\left(a^*\right)ta\right\|^2\right) \\
&= \sum_{i\in B^*} \frac{\lambda_0^i}{h_0^i\left(a^*\right)}\left(\left\|\nabla h_0^i\left(a^*\right)^{\mathrm{T}} ta\right\|^2 - \left\|J\bar{h}_0^i\left(a^*\right)ta\right\|^2\right) \qquad (30) \\
&= -(ta)^{\mathrm{T}}\left(\sum_{j=1}^{p} \frac{\lambda_0^j}{h_0^j(a)} \nabla h^j(a) \begin{pmatrix} 1 & 0^{\mathrm{T}} \\ 0 & -I \end{pmatrix} \left(\nabla h^j(a)\right)^{\mathrm{T}}\right) ta. \\
&= (ta)^{\mathrm{T}} \mathcal{H}\left(a^*, \lambda\right) ta
\end{aligned}$$

From (30) and (14), we have

$$\left\langle J_a L_f\left(a^*, \mu^*, \lambda^*\right)ta, ta\right\rangle + t^{\mathrm{T}}\mathcal{H}\left(a^*, \lambda\right)t = 0.$$

From the second-order sufficient condition, we have $ta = 0$. Hence, from (21), we deduce

$$Jg\left(a^*\right)^{\mathrm{T}} t\mu + Jh\left(a^*\right)^{\mathrm{T}} t\lambda = 0.$$

(23) and condition (iv) of Theorem 3.1 imply $d\mu = 0$, $d\lambda = 0$. Therefore, $JH\left(a^*, \mu^*, \lambda^*\right)$ is nonsingular. □

4. A Modified Newton Method and Numerical Experiments

In this section, we use a modified Newton algorithm to solve the unconstrained smooth minimization problem (9).

The presented algorithm is actually a counterpart in the case of second order cone constrained VI problems of ([7], Algorithm 9.1.10), which is used to solve polyhedral cone constrained VI problems. Note that although $S(z)$ is non-smooth, the merit function $\Phi(z)$ is continuously differentiable if f is, see ([7], Proposition 1.5.3). In the following proposition we give the relationship between the merit function and the KKT condition.

Proposition 4.1. *Suppose that* f, h *and* g *are continuously differentiable. If* $JH(a, \mu, \lambda)$ *is nonsingular, then every stationary point of the merit function* $\Phi(a, \mu, \lambda)$ *is a KKT triple of the SOCCVI.*

The proof of Proposition 4.1 is similar to that of Proposition 4.1 in [17], so we omit it here.

Algorithm 4.1

Data Given $z^0 = \left(a^0, \mu^0, \lambda^0\right) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^l$, $\sigma > 0$, $p > 1$ and $\gamma \in (0,1)$.

Step 1 Set $k = 0$.

Step 2 If $z^k = \left(a^k, \mu^k, \lambda^k\right)$ is a stationary point of Φ stop.

Step 3 Find a solution t^k of the system

$$H\left(z^k\right) + JH\left(z^k\right)t = 0. \qquad (31)$$

If the system (31) is not solvable or if the condition

$$\nabla\Phi\left(z^k\right)^{\mathrm{T}} t^k \le -\sigma\left\|t^k\right\|^p \tag{32}$$

is not satisfied, (re)set $t^k \equiv -\nabla\Phi\left(z^k\right)$.

Step 4 Find the smallest nonnegative integer i_k such that, with $i = i_k$,

$$\Phi\left(z^k + 2^{-i}t^k\right) \le \Phi\left(z^k\right) + \gamma 2^{-i}\nabla\Phi\left(z^k\right)^{\mathrm{T}} t^k; \tag{33}$$

set $\tau_k \equiv 2^{-i_k}$.

Step 5 Set $z^{k+1} \equiv z^k + \tau_k t^k$ and $k \leftarrow k+1$; go to step 2.

The superlinear convergence of Algorithm 4.1 can be obtained by reference to Algorithm 4.1 in [17]. To demonstrate effectiveness of the Newton method, some illustrative SOCCVI problems are tested. Observe that all the trajectories were successfully able to converge to the SOCCVI solution.

5. Numerical Experiments

Example 5.1. *Consider the SOCCVI problem* (1) *where*

$$f(x) = \begin{pmatrix} x_1 - 5 \\ 2x_2 - 1 \\ x_3 - 2 \\ 2^{x_4} - 2 \\ x_5 + 1 \end{pmatrix}$$

and

$$Q = \left\{x \in \mathbb{R}^5 : h(x) = x \in K^5\right\}.$$

This problem has a solution $x^* = (5, 0.5, 2, 1, -1)^{\mathrm{T}}$. It can be verified that the Lagrangian function for this example is

$$L(x, \mu, \lambda) = f(x) - \lambda.$$

The gradient of the Lagrangian function is

$$\nabla L(x, \mu, \lambda) = \begin{bmatrix} \nabla f(x) \\ I_{5\times 5} \end{bmatrix},$$

where I is the identity mapping and $\nabla f(x)$ is the gradient of $f(x)$. For example 5.1, **Table 1** is a comparison chart of the results of different complementarity functions.

In our simulations, the initial points are $x^0 = (1,0,0,0,0)^{\mathrm{T}}$, $\lambda^0 = (0,0,0,0,0)^{\mathrm{T}}$, and the stopping criterion is set at $\left\|\nabla\Phi(z)\right\| \le 1\times 10^{-6}$. The trajectories of Algorithm 4.1 for the above problems are shown in **Figure 1** and **Figure 2**.

Example 5.2. *Consider the SOCCVI problem* (1) *where*

Table 1. The results of Example 5.1.

n	time	*k*	*e*
FB	9.69147	63	3.7417e−4
NR	7.21559	39	1.4142e−4

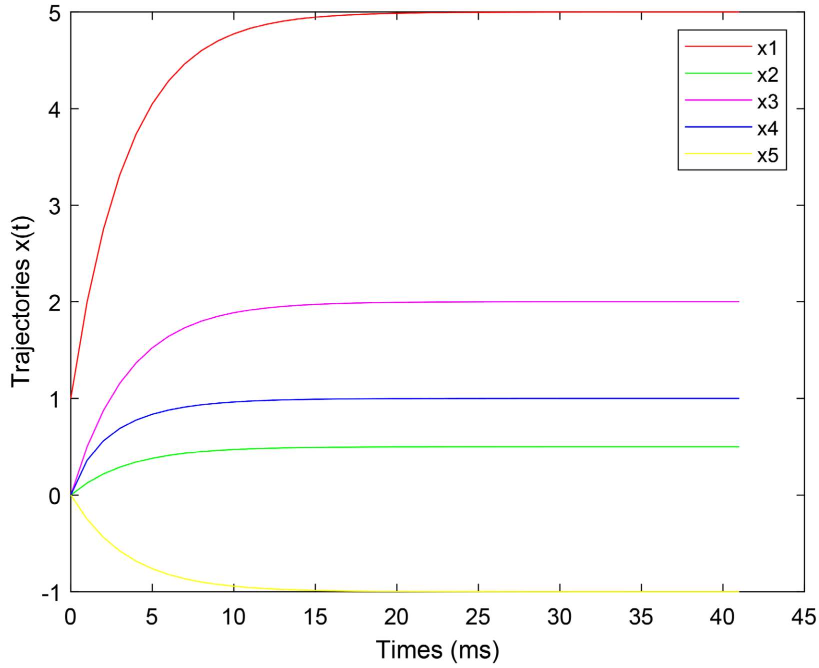

Figure 1. Transient behavior of x in Example 5.1 with $\sigma = 0.01$ and $\gamma = 0.45$.

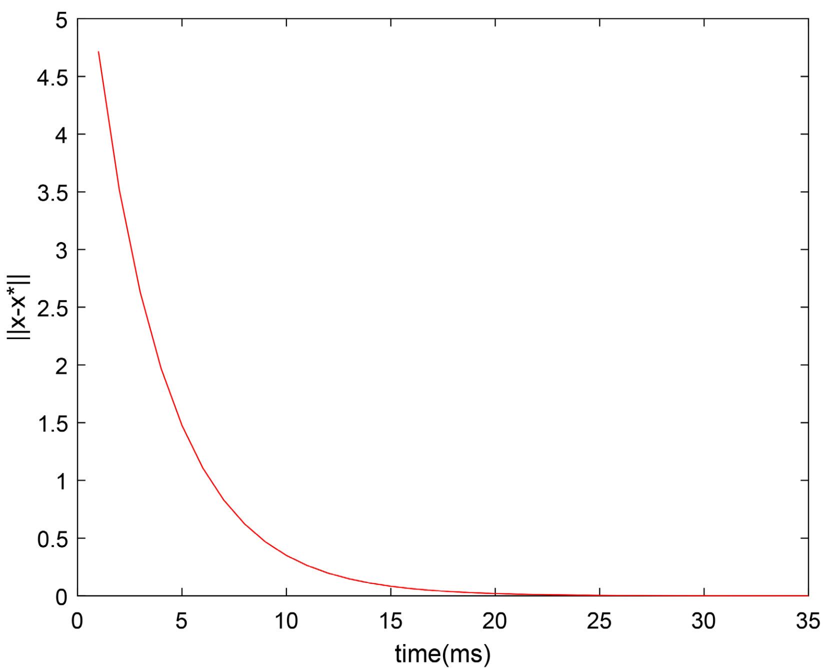

Figure 2. Convergence behavior of the error $\|x - x^*\|$ for Example 5.1.

$$f(x) = \begin{pmatrix} x_1 - 4 \\ e^{x_2} - 2.178 \\ 3x_3 + 4 \\ \tan x_4 + 1 \\ x_5 + 2 \\ 2x_6 - 1 \end{pmatrix}$$

and

$$Q=\left\{x\in\mathbb{R}^6:h(x)=x\in K^6\right\}.$$

This problem has an approximate solution $x^*=(4,0.9999,-1.3333,-0.7854,-2,0.5)^{\mathrm{T}}$. It can be verified that the Lagrangian function for this example is

$$L(x,\mu,\lambda)=f(x)-\lambda.$$

The gradient of the Lagrangian function is

$$\nabla L(x,\mu,\lambda)=\begin{bmatrix}\nabla f(x)\\ I_{5\times5}\end{bmatrix},$$

where I is the identity mapping and $\nabla f(x)$ is the gradient of $f(x)$. For example 5.2, **Table 2** is a comparison chart of the results of different complementarity functions.

In our simulations, the initial points are $x^0=(1,0,0,0,0,0)^{\mathrm{T}}$, $\lambda^0=(0,0,0,0,0,0)^{\mathrm{T}}$, and the stopping criterion is set at $\|\nabla\Phi(z)\|\le 1\times10^{-6}$. The trajectories of Algorithm 4.1 for the above problems are shown in **Figure 3** and **Figure 4**.

Table 2. The results of Example 5.2.

n	time	*k*	*e*
FB	8.06143	57	2.4872e−4
NR	6.191146	31	1.0423e−4

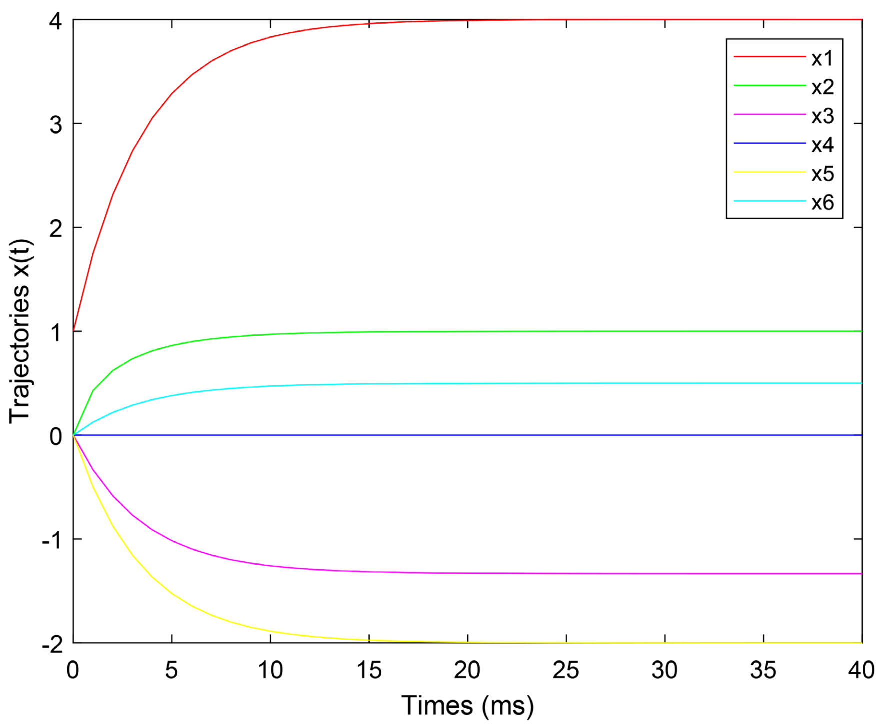

Figure 3. Transient behavior of x in Example 5.2 with $\sigma=0.01$ and $\gamma=0.45$.

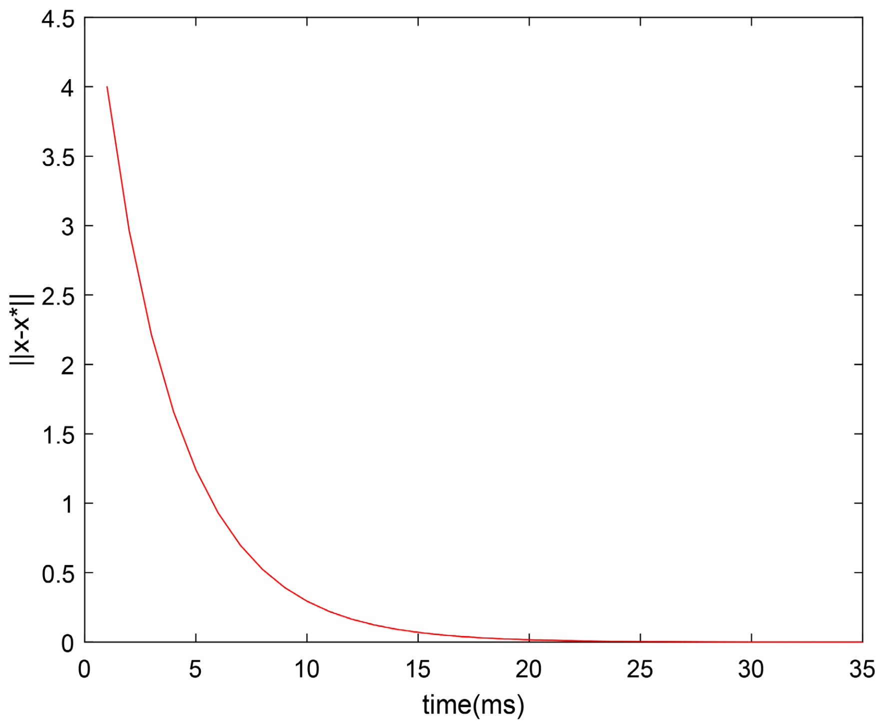

Figure 4. Convergence behavior of the error $\|x - x^*\|$ for Example 5.2.

Table 3. The results of Example 5.3.

n	time	*k*	*e*
FB	9.86342	65	2.9713e−4
NR	7.291559	40	1.1275e−4

Example 5.3. *Consider the SOCCVI problem* (1) *where*

$$f(x) = \begin{pmatrix} x_1 - 6 \\ 4x_2 - 3 \\ e^{x_3} - 2.718 \\ \tan x_4 - 1 \\ x_5 - 2 \\ 3x_6 - 1 \end{pmatrix}$$

and

$$Q = \left\{ x \in \mathbb{R}^5 : h(x) = x \in K^5 \right\}.$$

For example 5.3, **Table 3** is a comparison chart of the results of different complementarity functions.

This problem has a solution $x^* = (6, 0.75, 0.9999, 0.7854, 2, -0.3333)^{\mathrm{T}}$. In our simulations, the initial points are $x^0 = (1, 0, 0, 0, 0, 0)^{\mathrm{T}}$, $\lambda^0 = (0, 0, 0, 0, 0, 0)^{\mathrm{T}}$, and the stopping criterion is set at $\|\nabla \Phi(z)\| \le 1 \times 10^{-6}$. The trajectories of Algorithm 4.1 for the above problems are shown in **Figure 5** and **Figure 6**.

Example 5.4. *Consider the SOCCVI problem* (1) *where*

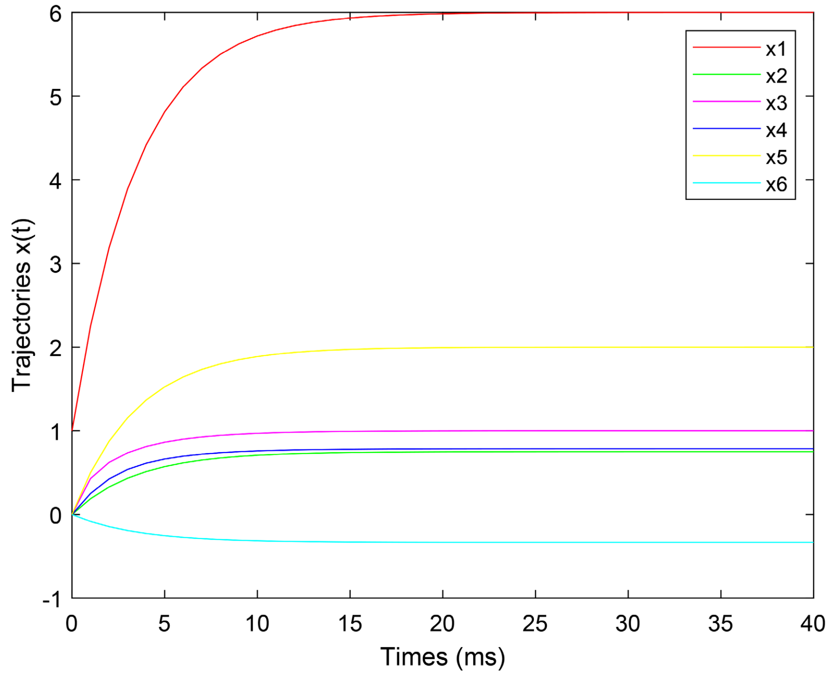

Figure 5. Transient behavior of x in Example 5.3 with $\sigma = 0.01$ and $\gamma = 0.45$.

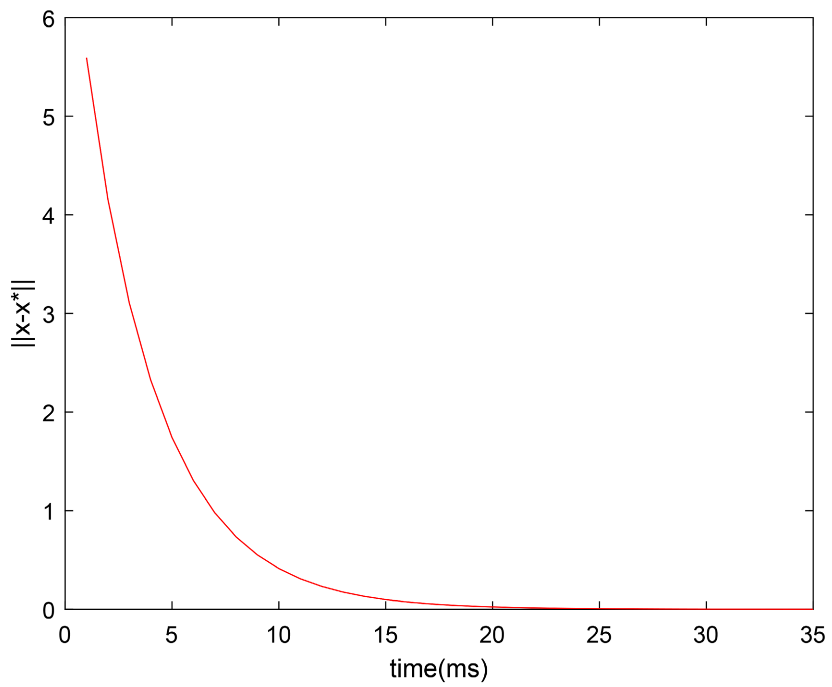

Figure 6. Convergence behavior of the error $\|x - x^*\|$ for Example 5.3.

$$f(x) = \begin{pmatrix} 2x_1 - 4 \\ e^{x_2} - 1 \\ 3x_3 - 4 \\ -\sin x_4 \\ x_5 \end{pmatrix}$$

and

$$Q = \left\{ x \in \mathbb{R}^5 : h(x) = x \in K^5 \right\}.$$

This problem has a solution $x^* = (2, 0, 1.3333, 0, 0)^{\mathrm{T}}$. In our simulations, the initial points are $x^0 = (1, 0, 0, 0, 0)^{\mathrm{T}}$, $\lambda^0 = (0, 0, 0, 0, 0)^{\mathrm{T}}$, and the stopping criterion is set at $\|\nabla\Phi(z)\| \le 1 \times 10^{-6}$. Comparison of decay rates of $\|x(t) - x^*\|$ for the two complementary function (FB function and NR function) in **Figure 7** The error plots shown in **Figure 7** reveal that the NR function and FB function have almost the same convergence rates.

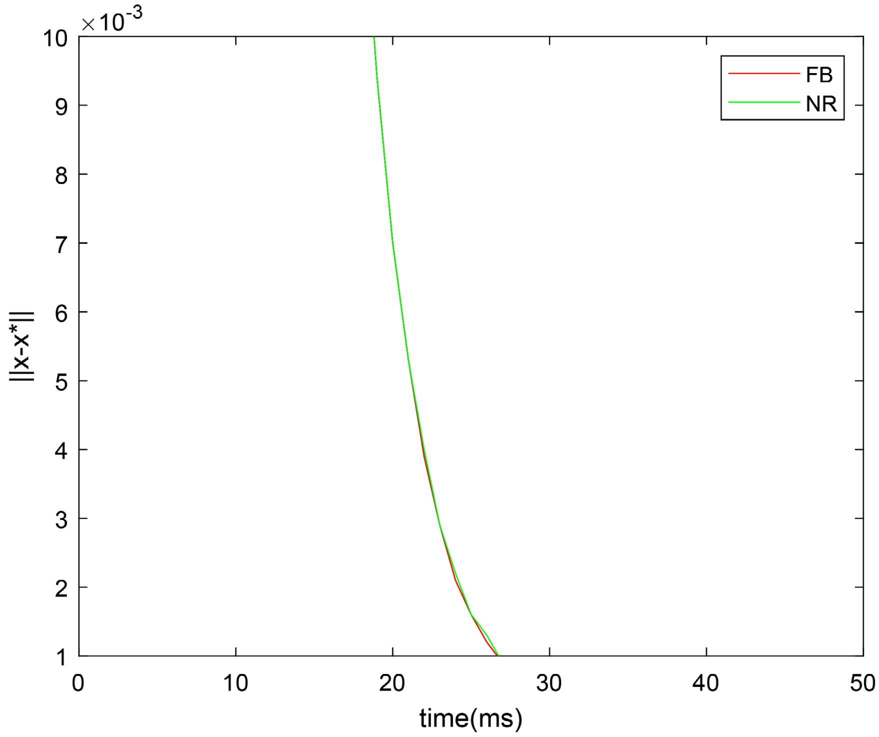

Figure 7. Convergence behavior of the error $\|x - x^*\|$ for Example 4.

6. Conclusion

In this paper, we target the second-order cone constrained variational inequality (SOCCVI) problem by using the modified Newton algorithm which is applied in [17], based on the metric projector. We have established that for a locally optimal solution to a SOCCVI problem, the nonsingularity of the Clarke's generalized Jacobian of the KKT system, constructed by the metric projector, is equivalent to the second-order sufficient condition and constraint nondegeneracy. Three numerical simulations are conducted which show the efficiency of the Newton algorithm. This paper improves our previous work [17].

Founding

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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