Journal of Applied Mathematics and Physics, 2022, 10, 930-951
https://www.scirp.org/journal/jamp
ISSN Online: 2327-4379
ISSN Print: 2327-4352

A Family of Global Attractors for a Class of Generalized Kirchhoff-Beam Equations

Yuhuai Liao[1], Guoguang Lin[2], Jie Liu[2]

[1]Wenshan Universiry, Wenshan, China
[2]School of Mathematics and Statistics, Yunnan University, Kunming, China
Email: 15925159599@163.com

How to cite this paper: Liao, Y.H., Lin, G.G. and Liu, J. (2022) A Family of Global Attractors for a Class of Generalized Kirchhoff-Beam Equations. *Journal of Applied Mathematics and Physics*, **10**, 930-951.
https://doi.org/10.4236/jamp.2022.103064

Received: February 24, 2022
Accepted: March 27, 2022
Published: March 30, 2022

Abstract

The initial boundary value problem for a class of high-order Beam equations with quasilinear and strongly damped terms is studied. Firstly, the existence and uniqueness of the global solution of the equation are proved by prior estimation and Galerkin finite element method. Then the bounded absorption set is obtained by prior estimation, and the family of global attractors for the high-order Kirchhoff-Beam equation is obtained. The Frechet differentiability of the solution semigroup is proved after the linearization of the equation, and the decay of the volume element of the linearization problem is further proved. Finally, the Hausdorff dimension and Fractal dimension of the family of global attractors are proved to be finite.

Keywords

High-Order Kirchhoff-Beam Equation, Galerkin's Method, Family of Global Attractors, The Hausdorff Dimension

1. Introduction

In order to study the global stability of a wide norm model for vertical beam vibration, the initial boundary value problems of the following Kirchhoff-Beam equations are studied:

$$u_{tt}+\beta(-\Delta)^{2m}u_t+M\left(\|D^m u\|_p^p\right)u_t+\alpha\Delta^{2m}u+N\left(\|D^m u\|_p^p\right)(-\Delta)^m u=f(x), \quad (1)$$

$$u(x,t)=0,\frac{\partial^i u}{\partial v^i}=0,i=1,2,\cdots,m-1,x\in\partial\Omega,t>0, \quad (2)$$

$$u(x,0)=u_0(x),u_t(x,0)=u_1(x),x\in\Omega\subset R^n. \quad (3)$$

where $m>1$ is a positive integer, Ω is the bounded region in R^n with smooth boundary $\partial\Omega$. $f(x)$ is the external force term. $\beta(-\Delta)^{2m}u_t$ is the strongly damped term, α,β are positive constants, $\alpha\geq\frac{3}{2}$, $M\left(\|D^m u\|_p^p\right), N\left(\|D^m u\|_p^p\right)$ are the general non-negative real-valued functions, $\|D^m u\|_p^p=\int_\Omega |D^m u|^p \mathrm{d}x$, and the relevant assumptions will be given later. In this paper, we study the existence and uniqueness of global solutions for problems (1) - (3), prove the existence of a family global attractor and estimate its Hausdorff dimension and Fractal dimension.

In 1883, Kirchhoff [1] proposed the following model when studying the free vibration of elastic strings:

$$u_{tt}-\sigma\left(\|Du\|^2\right)\Delta u_t-\varphi\left(\|Du\|^2\right)\Delta u+g(u)=h(x).$$

This model more accurately describes the motion of the elastic rod and reveals the physical significance of elastic vibration, from which the Kirchhoff equation model becomes a kind of classical problems in infinite-dimensional dynamic systems. Many scholars have also achieved some good results in their research on the global attractor of Kirchhoff equation and the estimation of Hausdorff dimension. For details, please refer to references [1] [2] [3].

Igor Chueshov [4]: the long time behavior of the following Kirchhoff wave equations with strong nonlinear damping is studied

$$\partial_{tt}u-\sigma\left(\|\nabla u\|^2\right)\Delta\partial_t u-\phi\left(\|\nabla u\|^2\right)\Delta u+f(u)=h(x).$$

Tokio Matsuyama and Ryo Ikehata [5]: the attenuation of global solution of Kirchhoff type wave equation with nonlinear damping is proved:

$$u_{tt}+(-\Delta)^m u_t+\left(\|D^m u\|^2\right)(-\Delta)^m u+g(u)=f(x).$$

$$u(x,t)\big|_{\partial\Omega}=0, t\geq 0.$$

where $M(s)\in C^1[0,\infty), M(s)\geq m_0>0; \delta>0, \mu\in R$ are constants.

Guoguang Lin, Yunlong Gao [6]: the initial boundary value problem of Kirchhoff type equation with strongly damped term is studied

$$u_{tt}+M\left(\|\nabla u(t)\|_2^2\right)\Delta u+\delta|u_t|^{p-1}u_t=\mu|u|^{q-1}u.$$

The existence and uniqueness of knowledge are proved by prior estimation and Galerkin's method, and then the existence of global attractor is obtained. The Hausdorff dimension and Fractal dimension of global attractor are estimated.

Lin Chen, Guoguang Lin [7]: study the well-posedness and long-time behavior of solutions for a class of nonlinear higher-order Kirchhoff equations:

$$u_{tt}+(-\Delta)^m u_t+\phi\left(\|\nabla^m u\|^2\right)(-\Delta)^m u+g(u)=f(x).$$

The existence and uniqueness of global solutions are proved by prior estimation and Galerkin's method, and the Hausdorff dimension and Fractal dimension of global attractors are estimated. More studies of wave equations can be found in reference [8]-[19].

For the convenience of statement, the following Spaces and notations are defined:

$H = L^2(\Omega)$, $D = \nabla$, $H_0^m(\Omega) = H^m(\Omega) \cap H_0^1(\Omega)$, $H_0^{2m+k}(\Omega) = H^{2m+k}(\Omega) \cap H_0^1(\Omega)$. $C_i\,(i = 1,2,\cdots)$ are the different positive constants. $E_k = H_0^{2m+k}(\Omega) \times H_0^k(\Omega), (k = 1,2,\cdots,2m)$, $k = 0$, $E_0 = H_0^{2m} \times L^2(\Omega)$.

Defines $(\cdot,\cdot)$ and $\|\cdot\|$ represents H the inner product and norm respectively: $(u,v) = \int_\Omega u(x)v(x)\,\mathrm{d}x$, $\|\cdot\| = \|\cdot\|_{L^2}$, $(u,u) = \|u\|^2$.

A_k is (E_0, E_k)-global attractors, B_{0k} is the bounded absorption set in E_k, $k = 1,2,\cdots,2m$.

Kirchhoff stress $M(s), N(s)$ meet the conditions:

1) $M(s) \in C^2([0,+\infty), R)$, $2\varepsilon \le \sigma_0 \le M(s) \le \sigma_1$, σ_0, σ_1 are the positive constants;

2) $N(s) \in C^2([0,+\infty), R)$, $\mu_0 \le N(s) \le \mu_1$, μ_0, μ_1 are the positive constants;

3) $\dfrac{2n}{n+2m} \le p < \begin{cases} \dfrac{2n}{n-2m}, n > 2m \\ \infty, n \le 2m \end{cases}$, $0 < \varepsilon < \min\left\{ \dfrac{4\lambda_1^m - 1}{4\beta\lambda_1^m}, \dfrac{\alpha}{3\beta}, 2\lambda_1^m \mu_0 \right\}$,

$$\beta \ge \max\left\{ \frac{4\mu_1^2\lambda_1^m}{(4\alpha-1)\varepsilon\lambda_1^{2m} + 2\varepsilon(\varepsilon - \sigma_1)}, \frac{\varepsilon\sigma_1 - 2\sigma_0 + 2\varepsilon + 2\varepsilon^2}{2\lambda_1^{2m}}, \frac{2\varepsilon\sigma_1}{\lambda_1^m}, \frac{2C_4}{\lambda_1^m}, \right.$$
$$\frac{1}{2\lambda_1^m - 1}, \frac{3\mu_0}{2\lambda_1^m} + \frac{\sigma_1}{\lambda_1^{2m}}, \frac{\alpha - 1}{\varepsilon + 1}, \frac{\varepsilon^2 + \phi + \kappa - \varepsilon\sigma_0}{\varepsilon\lambda_1^{2m}} + \frac{\mu_0 + \alpha - 1}{\varepsilon} - \varepsilon,$$
$$\left. \frac{\mu_0 + \varepsilon^2 + \phi + \kappa - (\varepsilon + 2)\sigma_0}{\lambda_1^{2m}} \right\}.$$

2. The Existence of the Family of Globals

For the initial boundary value problem (1) - (3), the existence of global solutions is proved by prior estimation and Galerkin's finite element method, and then the uniqueness of global solutions is proved. Finally, the operator semigroup theory is used to prove that the solution semigroup of this problem has a family of global attractors.

Lemma 1. Assumes that (a), (b), (c) are held, $f(x) \in H$, $(u_0, u_1) \in E_0$, then the initial boundary value problem (1) - (3) has global smooth solutions $(u,v) \in E_0$,

$$\|(u,v)\|_{E_0}^2 = \|D^{2m}u\|^2 + \|v\|^2 \le \left(\|D^{2m}u_0\|^2 + \|v_0\|^2\right)\mathrm{e}^{-\alpha_1 t} + \frac{C_1}{\alpha_1}\left(1 - \mathrm{e}^{-\alpha_1 t}\right), \tag{4}$$

$$\|(u,v)\|_{E_0}^2 = \|D^{2m}u\|^2 + \|v\|^2 \le R_0^2, (t > t_1) \tag{5}$$

where $v = u_t + \varepsilon u$, $C_1 = \dfrac{1}{\varepsilon}\|f(x)\|^2$, $\alpha_1 = \min\left\{\theta_1, \dfrac{\theta_2}{\alpha}\right\}$, a non-negative real number R_0 and $t_1 = t_1(\Omega) > 0$.

Proof. Take the inner product of $v = u_t + \varepsilon u$ and Equation (1) both sides,

$$\begin{aligned}&\left(u_{tt} + \beta(-\Delta)^{2m} u_t + M\left(\left\|D^m u\right\|_p^p\right)u_t + \alpha\Delta^{2m}u + N\left(\left\|D^m u\right\|_p^p\right)(-\Delta)^m u, v\right)\\&=\left(f(x), v\right).\end{aligned} \tag{6}$$

By using Holder's Inequality, Young's Inequality, and Poincare's Inequality, The items in (6) can be obtained by successive processing:

$$\left(u_{tt}, v\right) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v\|^2 - \varepsilon\|v\|^2 + \varepsilon^2(u, v). \tag{7}$$

$$\begin{aligned}\left(\beta(-\Delta)^{2m} u_t, v\right) &= \beta\left\|D^{2m}v\right\|^2 - \varepsilon\beta\left(D^{2m}u, D^{2m}v\right)\\&\geq \beta\left\|D^{2m}v\right\|^2 - \frac{\varepsilon}{4}\left\|D^{2m}u\right\|^2 - \varepsilon\beta^2\left\|D^{2m}v\right\|^2,\end{aligned} \tag{8}$$

$$\begin{aligned}\left(M\left(\left\|D^m u\right\|_p^p\right)u_t, v\right) &= M\left(\left\|D^m u\right\|_p^p\right)\|v\|^2 - \varepsilon M\left(\left\|D^m u\right\|_p^p\right)(u, v).\\&\geq \sigma_0\|v\|^2 - \varepsilon M\left(\left\|D^m u\right\|_p^p\right)(u, v).\end{aligned} \tag{9}$$

$$\begin{aligned}\left(N\left(\left\|D^m u\right\|_p^p\right)(-\Delta)^m u, v\right) &\geq -N\left(\left\|D^m u\right\|_p^p\right)\left\|D^m u\right\|\left\|D^m v\right\|\\&\geq -\frac{\beta}{4\lambda_1^{m}}\left\|D^{2m}v\right\|^2 - \frac{\mu_1^2}{\beta}\left\|D^m u\right\|^2,\end{aligned} \tag{10}$$

$$\left(\alpha\Delta^{2m}u, v\right) = \frac{\alpha}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|D^{2m}u\right\|^2 + \alpha\varepsilon\left\|D^{2m}u\right\|^2. \tag{11}$$

$$\left(h(x), v\right) \leq \frac{1}{2\varepsilon}\left\|f(x)\right\|^2 + \frac{\varepsilon}{2}\|v\|^2. \tag{12}$$

$$\left(\varepsilon^2 - \varepsilon M\left(\left\|D^m u\right\|_p^p\right)\right)(u, v) \geq \frac{\varepsilon^2 - \varepsilon\sigma_1}{2}\|u\|^2 + \frac{\varepsilon^2 - \varepsilon\sigma_1}{2}\|v\|^2. \tag{13}$$

where λ_1 be the first eigenvalue of $-\Delta$ with a homogeneous Dirichlet boundary.

Substitute Formulas (7) - (13) into Equation (6) to obtain

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|v\|^2 + \alpha\left\|D^{2m}u\right\|^2\right) + \left(\sigma_0 - \frac{3\varepsilon}{2} + \frac{\varepsilon^2 - \varepsilon\sigma_1}{2}\right)\|v\|^2\\&+\left(\alpha\varepsilon - \frac{\varepsilon}{4} + \frac{\varepsilon^2 - \varepsilon\sigma_1}{2\lambda_1^{2m}} - \frac{\mu_1^2}{\beta\lambda_1^m}\right)\left\|D^{2m}u\right\|^2 + \left(\beta - \frac{\beta}{4\lambda_1^m} - \varepsilon\beta^2\right)\left\|D^{2m}v\right\|^2\\&\leq \frac{\left\|f(x)\right\|^2}{2\varepsilon^2},\end{aligned} \tag{14}$$

According to the hypothesis (a), (c),

$$\sigma_0 + \frac{\varepsilon^2 - \varepsilon\sigma_1}{2} - \frac{3\varepsilon}{2} \geq 0,\ \beta - \frac{\beta}{4\lambda_1^m} - \varepsilon\beta^2 \geq 0,\ \alpha\varepsilon + \frac{\varepsilon^2 - \varepsilon\sigma_1}{2\lambda_1^{2m}} \geq \frac{\varepsilon}{4} + \frac{\mu_1^2}{\beta\lambda_1^m}. \tag{15}$$

$$\theta_1 = 2\sigma_0 - 3\varepsilon + \varepsilon^2 - \varepsilon\sigma_1,\ \theta_2 = 2\left(\alpha\varepsilon + \frac{\varepsilon^2 - \varepsilon\sigma_1}{2\lambda_1^{2m}} - \frac{\varepsilon}{4} - \frac{\mu_1^2}{\beta\lambda_1^m}\right). \tag{16}$$

Thus

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|v\|^2+\alpha\left\|(-\Delta)^m u\right\|^2\right)+\alpha_1\left(\|v\|^2+\alpha\left\|(-\Delta)^m u\right\|^2\right)\le C_1, \tag{17}$$

Using Gronwall's inequation,

$$\|v\|^2+\alpha\left\|(-\Delta)^m u\right\|^2\le\left(\|v\|^2+\alpha\left\|(-\Delta)^m u\right\|^2\right)\mathrm{e}^{-\alpha_1 t}+\frac{C_1}{\alpha_1}\left(1-\mathrm{e}^{-\alpha_1 t}\right), \tag{18}$$

$$\overline{\lim_{t\to\infty}}\|(u,v)\|_{E_0}^2\le\frac{C_1}{\alpha_1}. \tag{19}$$

So there's a positive constant R_0, $t_1=t_1(\Omega)>0$, such that

$$\|(u,v)\|_{E_0}^2=\left\|D^{2m}u\right\|^2+\|v\|^2\le R_0^2,(t>t_1). \tag{20}$$

Lemma 1 is proved.

Lemma 2. Assumes that (a), (b), (c) are held, $h(x)\in H_1^0$, $(u_0,u_1)\in E_k$, then the initial boundary value problem (1) - (3) has global smooth solutions $(u,v)\in E_k$,

$$\begin{aligned}\|(u,v)\|_{E_k}^2&=\left\|D^{2m+k}u\right\|^2+\left\|D^k v\right\|^2\\&\le\left(\left\|D^{2m+k}u_0\right\|^2+\left\|D^k v_0\right\|^2\right)\mathrm{e}^{-\alpha_2 t}+\frac{C_2}{\alpha_2}\left(1-\mathrm{e}^{-\alpha_2 t}\right),\end{aligned} \tag{21}$$

$$\|(u,v)\|_{E_k}^2=\left\|D^{2m+k}u\right\|^2+\left\|D^k v\right\|^2\le R_1^2,(t>t_2). \tag{22}$$

where $v=u_t+\varepsilon u$, $C_2=\frac{1}{\varepsilon^2}\left\|D^k f(x)\right\|^2$, $\alpha_2=\min\left\{\theta_3,\frac{\theta_4}{\sigma}\right\}$, $t_2=t_2(\Omega)>0$.

Proof. Take the inner product of $(-\Delta)^k v=(-\Delta)^k u_t+(-\Delta)^k\varepsilon u$ and Equation (1) both sides,

$$\begin{aligned}&\left(u_{tt}+\beta(-\Delta)^{m2m}u_t+M\left(\left\|D^m u\right\|_p^p\right)u_t+\alpha\Delta^{2m}u+N\left(\left\|D^m u\right\|_p^p\right)(-\Delta)^m u,(-\Delta)^{mk}v\right)\\&=\left(f(x),(-\Delta)^{mk}v\right),\end{aligned} \tag{23}$$

By using Holder's Inequality, Young's Inequality, and Poincare's Inequality, The items in (23) can be obtained by successive processing:

$$\begin{aligned}\left(u_{tt},(-\Delta)^k v\right)&=\left(v_t-\varepsilon u_t,(-\Delta)^k v\right)=\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|D^k v\right\|^2-\varepsilon\left\|D^k v\right\|^2+\varepsilon^2\left(u,(-\Delta)^k v\right)\\&\ge\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|D^k v\right\|^2-\left(\varepsilon+\frac{\varepsilon^2}{2}\right)\left\|D^k v\right\|^2-\frac{\varepsilon^2}{2\lambda_1^m}\left\|D^{m+k}u\right\|^2,\end{aligned} \tag{24}$$

$$\begin{aligned}&\left(M\left(\left\|D^m u\right\|_p^p\right)u_t,(-\Delta)^k v\right)\\&=M\left(\left\|D^m u\right\|_p^p\right)\left\|D^k v\right\|^2-\varepsilon M\left(\left\|D^m u\right\|_p^p\right)\left\|D^k u\right\|\left\|D^k v\right\|\\&\ge M\left(\left\|D^m u\right\|_p^p\right)\left\|D^k v\right\|^2-\frac{\varepsilon M\left(\left\|D^m u\right\|_p^p\right)}{2}\left\|D^k v\right\|^2-\frac{\varepsilon M\left(\left\|D^m u\right\|_p^p\right)}{2\lambda_1^{2m}}\left\|D^{2m+k}u\right\|^2\\&\ge M\left(\left\|D^m u\right\|_p^p\right)\left\|D^k v\right\|^2-\frac{\varepsilon M\left(\left\|D^m u\right\|_p^p\right)}{2}\left\|D^k v\right\|^2-\frac{\alpha\varepsilon}{2}\left\|D^{2m+k}u\right\|^2\\&\ge\sigma_0\left\|D^k v\right\|^2-\frac{\varepsilon\sigma_1}{2}\left\|D^k v\right\|^2-\frac{\alpha\varepsilon}{2}\left\|D^{2m+k}u\right\|^2,\end{aligned} \tag{25}$$

with $\alpha \ge \frac{\sigma_1}{\lambda_1^{2m}}$.

$$\begin{aligned}\left(\beta(-\Delta)^{2m} u_t,(-\Delta)^k v\right) &\ge \beta\lambda_1^{2m}\left\|D^k v\right\|^2-\frac{\beta\varepsilon}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|D^{2m+k}u\right\|^2-\beta\varepsilon^2\left\|D^{2m+k}u\right\|^2 \\ &\ge \beta\lambda_1^{2m}\left\|D^k v\right\|^2-\frac{\beta\varepsilon}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|D^{2m+k}u\right\|^2-\frac{\alpha\varepsilon}{3}\left\|D^{2m+k}u\right\|^2,\end{aligned} \tag{26}$$

$$\begin{aligned}&\left(N\left(\left\|D^m u\right\|_p^p\right)(-\Delta)^m u,(-\Delta)^k v\right) \\ &=\frac{N\left(\left\|D^m u\right\|_p^p\right)}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|D^{m+k}u\right\|^2+\varepsilon N\left(\left\|D^m u\right\|_p^p\right)\left\|D^{m+k}u\right\|^2.\end{aligned} \tag{27}$$

The following two cases are used for prior estimation of Equation (25):

1) As $\frac{\mathrm{d}}{\mathrm{d}t}\left\|D^{m+k}u\right\|^2 \ge 0$, By assuming (b),

$$\begin{aligned}&\frac{N\left(\left\|D^{m+k}u\right\|_p^p\right)}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|D^m u\right\|^2+\varepsilon N\left(\left\|D^{m+k}u\right\|_p^p\right)\left\|D^m u\right\|^2 \\ &\ge \frac{\mu_0}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|D^m u\right\|^2+\varepsilon\mu_0\left\|D^m u\right\|^2,\end{aligned} \tag{28}$$

let $\mu=\mu_0$,

$$\left(N\left(\left\|D^m u\right\|_p^p\right)(-\Delta)^m u,v\right) \ge \frac{\mu}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|D^{m+k}u\right\|^2+\varepsilon\mu_0\left\|D^{m+k}u\right\|^2. \tag{29}$$

2) As $\frac{\mathrm{d}}{\mathrm{d}t}\left\|D^{m+k}u\right\|^2 < 0$, By assuming (b),

$$\begin{aligned}&\frac{N\left(\left\|D^{m+k}u\right\|_p^p\right)}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|D^m u\right\|^2+\varepsilon N\left(\left\|D^{m+k}u\right\|_p^p\right)\left\|D^m u\right\|^2 \\ &\ge \frac{\mu_1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|D^m u\right\|^2+\varepsilon\mu_0\left\|D^m u\right\|^2,\end{aligned} \tag{30}$$

let $\mu=\mu_1$,

$$\left(N\left(\left\|D^m u\right\|_p^p\right)(-\Delta)^m u,v\right) \ge \frac{\mu}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|D^{m+k}u\right\|^2+\varepsilon\mu_0\left\|D^{m+k}u\right\|^2. \tag{31}$$

Similarly discussion (27) can be obtained

$$\left(N\left(\left\|D^m u\right\|_p^p\right)(-\Delta)^m u,(-\Delta)^k v\right) \ge \frac{\mu}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|D^{m+k}u\right\|^2+\varepsilon\mu_0\left\|D^{m+k}u\right\|^2. \tag{32}$$

$$\begin{aligned}\left(\alpha\Delta^{2m}u,(-\Delta)^k v\right) &= \left(\alpha\Delta^{2m}u,(-\Delta)^k u_t+(-\Delta)^k \varepsilon u\right) \\ &= \frac{\alpha}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|D^{2m+k}u\right\|^2+\alpha\varepsilon\left\|D^{2m+k}u\right\|^2,\end{aligned} \tag{33}$$

$$\left(f(x),(-\Delta)^k v\right) \le \frac{\varepsilon^2}{2}\left\|D^k v\right\|^2+\frac{1}{2\varepsilon^2}\left\|D^k f(x)\right\|^2. \tag{34}$$

The comprehensive Equations (24) - (34) and (23) can be written as

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\left\|D^k v\right\|^2+(\alpha-\varepsilon\beta)\left\|D^{2m+k}u\right\|^2+\mu\left\|D^{m+k}u\right\|^2\right)$$
$$+\left(\beta\lambda_1^{2m}+\sigma_0-\frac{\varepsilon\sigma_1}{2}-\varepsilon-\varepsilon^2\right)\left\|D^k v\right\|^2+\left(\varepsilon\mu_0-\frac{\varepsilon^2}{2\lambda_1^m}\right)\left\|D^{m+k}u\right\|^2 \tag{35}$$
$$+\frac{\alpha\varepsilon}{6}\left\|D^{2m+k}u\right\|^2\le\frac{\left\|D^k f(x)\right\|^2}{2\varepsilon^2}.$$

According to the hypothesis (c),

$$\alpha-\beta\varepsilon>0,\ \varepsilon\mu_0-\frac{\varepsilon^2}{2\lambda_1^m}\ge 0,\ \beta\lambda_1^{2m}+\sigma_0-\frac{\varepsilon\sigma_1}{2}-\varepsilon-\varepsilon^2\ge 0. \tag{36}$$

$$\theta_3=2\left(\beta\lambda_1^{2m}+\sigma_0-\frac{\varepsilon\sigma_1}{2}-\varepsilon-\varepsilon^2\right),\theta_4=2\varepsilon\mu_0-\frac{\varepsilon^2}{\lambda_1^m},\theta_5=\frac{1}{3}\alpha\varepsilon>0.$$

Let $\alpha_2=\min\left\{\theta_3,\frac{\theta_4}{\mu},\frac{\theta_5}{\alpha-\beta\varepsilon}\right\}$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\left\|D^k v\right\|^2+(\alpha-\beta\varepsilon)\left\|D^{2m+k}u\right\|^2+\mu\left\|D^{m+k}u\right\|^2\right)$$
$$+\alpha_2\left(\left\|D^k v\right\|^2+(\alpha-\beta\varepsilon)\left\|D^{2m+k}u\right\|^2+\mu\left\|D^{m+k}u\right\|^2\right)\le C_2. \tag{37}$$

By Gronwall's inequality, we can get

$$Y_1\le Y_1\mathrm{e}^{-\alpha_2 t}+\frac{C_2}{\alpha_2}\left(1-\mathrm{e}^{-\alpha_2 t}\right). \tag{38}$$

where $Y_1=\left\|D^k v\right\|^2+(\alpha-\beta\varepsilon)\left\|D^{2m+k}u\right\|^2+\mu\left\|D^{m+k}u\right\|^2$.

$$\left\|(u,v)\right\|_{E_k}^2=\left\|D^{2m+k}u\right\|^2+\left\|D^k v\right\|^2$$
$$\le\left(\left\|D^{2m+k}u_0\right\|^2+\left\|D^k v_0\right\|^2\right)\mathrm{e}^{-\alpha_2 t}+\frac{C_2}{\alpha_2}\left(1-\mathrm{e}^{-\alpha_2 t}\right), \tag{39}$$

$$\overline{\lim_{t\to\infty}}\left\|(u,v)\right\|_{E_k}^2\le\frac{C_2}{\alpha_2}. \tag{40}$$

So there's a positive constant R_1, $t_2=t_2(\Omega)>0$, such that

$$\left\|(u,v)\right\|_{E_k}^2=\left\|D^{2m+k}u\right\|^2+\left\|D^k v\right\|^2\le R_1^2,(t>t_2). \tag{41}$$

Lemma 2 is proved.

Theorem 1. (existence and uniqueness of solutions) Under the hypothesis of Lemma 1 and Lemma 2, $f\in H$, $(u_0,u_1)\in E_k$. Then there exists a unique global solution to the initial boundary value problem (1) - (3), $(u,v)\in L^\infty([0,+\infty);E_k)$.

Proof. Existence: The existence of global solution is proved by Galerkin's finite element method.

The first step: To construct approximate solutions

Let $(-\Delta)^{2m+k}w_j=\lambda_j^{2m+k}w_j,k=1,2,\cdots,2m$, where λ_j is the eigenvalues of $-\Delta$ with homogeneous Dirichlet boundary on Ω, w_j is the eigenfunctions determined by the corresponding eigenvalues and $w_1,w_2,\cdots,w_l$ are the orthonormal basis formed by the eigenvalue theory.

Let the approximate solution of problems (1) - (3) be $u_l(t)=\sum_{j=1}^{l} g_{jl}(t)w_j$, and $g_{jl}(t)$ is determined by the following nonlinear ordinary differential equations

$$\begin{aligned}&\left(u_{tt}+\beta(-\Delta)^{2m}u_t+M\left(\left\|D^m u\right\|_p^p\right)u_t+\alpha\Delta^{2m}u+N\left(\left\|D^m u\right\|_p^p\right)(-\Delta)^m u,w_j\right)\\&=\left(h(x),w_j\right),j=1,2,\cdots,l.\end{aligned} \tag{42}$$

It satisfies the initial conditions $u_{l0}(0)=u_{l0},u_{lt}(0)=u_{l1}$. As $l\to+\infty$, In E_k, $(u_{l0},u_{l1})\to(u_0,u_1)$ in E_k. It is known from the basic theory of ordinary differential that approximate solutions exist on $(0,t_l)$.

The second step: A prior estimation

Because we want to prove the existence of weak solutions in space $E_k(k=1,2,\cdots,2m-1)$.

So we multiply both sides of Equation (42) with $\lambda_j^k\left(g'_{jl}(t)+\varepsilon g_{jl}(t)\right)$, and the sum of *j*, let $v_l(t)=u_{lt}(t)+\varepsilon u_l(t)$.

As $k=0$, Get a priori estimate of the solution in space E_0:

$$\left\|(u_l,v_l)\right\|_{E_0}^2=\left\|D^{2m}u_l\right\|^2+\left\|v_l\right\|^2\le R_0^2, \tag{43}$$

As $k=1,2,\cdots,2m-1$, Get a priori estimate of the solution in space E_k:

$$\left\|(u_l,v_l)\right\|_{E_k}^2=\left\|D^{2m+k}u_l\right\|^2+\left\|v_l\right\|^2\le R_1^2. \tag{44}$$

It can be known that the prior estimates of lemma 1 and lemma 2 of Equations (43) and (44) are valid respectively. According to Equations (43) and (44), (u_l,v_l) be bounded in $L^\infty([0,+\infty];E_0)$, (u_l,v_l) be bounded in $L^\infty([0,+\infty];E_k)$.

The third step: Limit process

In the space $E_k(k=1,2,\cdots,2m-1)$, Selecting subcolumns u_μ from the sequence u_l,

$$(u_\mu,v_\mu)\to(u,v)\ \text{Weak*-convergence, in}\ L^\infty([0,+\infty];E_k) \tag{45}$$

By Rellich-Kohdrachov Compact embedding theorem, E_k compact embedded E_0, $(u_\mu,v_\mu)\to(u,v)$

$$\text{Strong convergence almost everywhere in } E_0. \tag{46}$$

Let $l=\mu\to+\infty$, From (45),

$$\left(u_{\mu t},(-\Delta)^k w_j\right)\to\left(v,\lambda_j^k w_j\right)-\left(\varepsilon u,\lambda_j^k w_j\right)$$

Weak*-convergence in $L^\infty[0,+\infty)$.

$$\left(u_{\mu tt},(-\Delta)^k w_j\right)=\frac{\mathrm{d}}{\mathrm{d}t}\left(u_{\mu t},(-\Delta)^k w_j\right)$$

Thus, $\left(u_{\mu tt},(-\Delta)^k w_j\right)\to\left(u_t,(-\Delta)^k w_j\right)$ convergence in $D'[0,+\infty)$, here $D'[0,+\infty)$ is $D[0,+\infty)$ Conjugate space of infinitely differentiable space $D[0,+\infty)$.

$$\left(\beta(-\Delta)^{2m} u_{\mu t},(-\Delta)^k w_j\right) \to \beta\left((-\Delta)^{\frac{2m+k}{2}} v-\varepsilon(-\Delta)^{\frac{2m+k}{2}} u, \lambda_j^{\frac{2m+k}{2}} w_j\right)$$

Weak*-convergence in $L^\infty[0,+\infty)$.

$$\left(M\left(\left\|D^m u\right\|_p^p\right) u_{\mu t},(-\Delta)^k w_j\right) \to M\left(\left\|D^m u\right\|_p^p\right)\left((-\Delta)^{\frac{k}{2}} v-\varepsilon(-\Delta)^{\frac{k}{2}} u_t, \lambda_j^{\frac{k}{2}} w_j\right)$$

Weak*-convergence in $L^\infty[0,+\infty)$.

$$\left(N\left(\left\|D^m u\right\|_p^p\right)(-\Delta)^m u_\mu,(-\Delta)^k w_j\right) \to\left(N\left(\left\|D^m u\right\|_p^p\right)(-\Delta)^{\frac{m+k}{2}} u, \lambda_j^{\frac{m+k}{2}} w_j\right)$$

Weak*-convergence in $L^\infty[0,+\infty)$.

$$\left(\alpha \Delta^{2m} u_\mu,(-\Delta)^k w_j\right) \to\left(\alpha \Delta^{\frac{2m+k}{2}} u, \lambda_j^{\frac{2m+k}{2}} w_j\right).$$

Weak*-convergence in $L^\infty[0,+\infty)$.

In particular $u_{\mu 0} \to u_0$ weak convergence in E_k, $u_{\mu t} \to u_t=u_1$ weak convergence in E_k. For all *j* and $\mu \to+\infty$, It follows that

$$\begin{aligned} & \left(u_{tt}+\beta(-\Delta)^{2m} u_t+M\left(\left\|D^m u\right\|_p^p\right) u_t+\alpha \Delta^{2m} u+N\left(\left\|D^m u\right\|_p^p\right)(-\Delta)^m u, w_j\right) \\ & =\left(f(x), w_j\right), j=1,2, \cdots, l. \end{aligned} \tag{47}$$

Therefore, the existence of the weak solution of the problem (1) - (3) is obtained, and the existence is proved.

The uniqueness of the solution of the following proof.

Let u, v be two solutions of Equation (1), set $w=u-v$ and take the inner product of $w_t+\varepsilon w$ in *H*,

$$\begin{aligned} & \left(w_{tt}+\beta(-\Delta)^{2m} w_t+M\left(\left\|D^m u\right\|_p^p\right) u_t-M\left(\left\|D^m v\right\|_p^p\right) v_t+\alpha \Delta^{2m} w\right. \\ & \left.+N\left(\left\|D^m u\right\|_p^p\right)(-\Delta)^m u-N\left(\left\|D^m v\right\|_p^p\right)(-\Delta)^m v, w_t+\varepsilon w\right)=0 \end{aligned} \tag{48}$$

Similar to Lemma 1, can be obtained

$$\left(w_{tt}, w_t+\varepsilon w\right)=\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d} t}\left\|w_t\right\|^2+\varepsilon \frac{\mathrm{d}}{\mathrm{d} t}\left(w_t, w\right)-\varepsilon\left\|w_t\right\|^2, \tag{49}$$

$$\left(\beta(-\Delta)^{2m} w_t, w_t+\varepsilon w\right)=\beta\left\|D^{2m} w_t\right\|^2+\frac{\beta \varepsilon}{2} \frac{\mathrm{d}}{\mathrm{d} t}\left\|D^{2m} w\right\|^2, \tag{50}$$

$$\begin{aligned} & \left(M\left(\left\|D^m u\right\|_p^p\right) u_t-M\left(\left\|D^m v\right\|_p^p\right) v_t, w_t+\varepsilon w\right) \\ & =M\left(\left\|D^m u\right\|_p^p\right)\left(w_t, w_t\right)+\varepsilon M\left(\left\|D^m u\right\|_p^p\right)\left(w_t, w\right) \\ & \quad+M'\left(\left\|D^m \xi\right\|_p^p\right)\left(\left\|D^m \xi\right\|_p^p\right)'\left(D^m w v_t, w_t+\varepsilon w\right) \\ & \geq M\left(\left\|D^m u\right\|_p^p\right)\left\|w_t\right\|^2-\varepsilon M\left(\left\|D^m u\right\|_p^p\right)\left(w_t, w\right)-C_3 v_t\left(D^m w, \varepsilon w+w_t\right) \end{aligned}$$

$$\geq M\left(\left\|D^m u\right\|_p^p\right)\left\|w_t\right\|^2-\frac{\varepsilon M\left(\left\|D^m u\right\|_p^p\right)}{2\lambda_1^m}\left\|D^m w_t\right\|^2-\frac{\varepsilon M\left(\left\|D^m u\right\|_p^p\right)}{2}\|w\|^2$$

$$-C_4\left\|D^m w\right\|\cdot\left\|w_t\right\|-\varepsilon C_4\left\|D^m w\right\|\cdot\|w\|$$

$$\geq M\left(\left\|D^m u\right\|_p^p\right)\left\|w_t\right\|^2-\frac{\varepsilon M\left(\left\|D^m u\right\|_p^p\right)}{2\lambda_1^m}\left\|D^m w_t\right\|^2-\frac{\varepsilon M\left(\left\|D^m u\right\|_p^p\right)}{2}\|w\|^2 \tag{51}$$

$$-\frac{C_4}{2}\left\|D^m w\right\|^2-\frac{C_4}{2\lambda_1^m}\left\|D^m w_t\right\|^2-\frac{\varepsilon C_4}{2}\left\|D^m w\right\|^2-\frac{\varepsilon C_4}{2}\|w\|^2$$

$$\geq \sigma_0\left\|w_t\right\|^2-\frac{\beta}{2}\left\|D^m w_t\right\|^2-\left(\frac{\varepsilon\sigma_1}{2}+\frac{\varepsilon C_4}{2}\right)\|w\|^2-\frac{C_4(1+\varepsilon)}{2\lambda_1^m}\left\|D^{2m} w\right\|^2,$$

$$C_3=\left\|M'\left(\left\|D^m\xi\right\|_p^p\right)\left(\left\|D^m\xi\right\|_p^p\right)'\right\|_\infty, C_4=\left\|M'\left(\left\|D^m\xi\right\|_p^p\right)\left(\left\|D^m\xi\right\|_p^p\right)'\right\|_\infty\left\|v_t\right\|.$$

$$\xi=sD^m u+(1-s)D^m v, s\in(0,1).$$

$$\left(\alpha\Delta^{2m}w, w_t+\varepsilon w\right)\geq\frac{\alpha}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|D^{2m}w\right\|^2+\frac{\alpha\varepsilon}{2}\left\|D^{2m}w\right\|^2+\frac{\alpha\varepsilon\lambda_1^{2m}}{2}\|w\|^2, \tag{52}$$

$$\left(N\left(\left\|D^m u\right\|_p^p\right)(-\Delta)^m u-N\left(\left\|D^m v\right\|_p^p\right)(-\Delta)^m v, w_t+\varepsilon w\right)$$

$$=\left(N\left(\left\|D^m u\right\|_p^p\right)(-\Delta)^m w, w_t\right)+\left(N\left(\left\|D^m u\right\|_p^p\right)(-\Delta)^m v-N\left(\left\|D^m v\right\|_p^p\right)(-\Delta)^m v, w_t\right)$$

$$+\varepsilon\left(N\left(\left\|D^m u\right\|_p^p\right)(-\Delta)^m w, w\right)$$

$$+\varepsilon\left(N\left(\left\|D^m u\right\|_p^p\right)(-\Delta)^m v-N\left(\left\|D^m v\right\|_p^p\right)(-\Delta)^m v, w\right)$$

$$\geq\frac{N\left(\left\|D^m u\right\|_p^p\right)}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|D^m w\right\|^2-C_5\left\|D^m w\right\|\left\|D^m v\right\|\left\|D^m w_t\right\|$$

$$+\varepsilon N\left(\left\|D^m u\right\|_p^p\right)\left\|D^m w\right\|^2-\varepsilon C_5\left\|D^m w\right\|\left\|D^m v\right\|\left\|D^m w\right\|$$

$$\geq\frac{N\left(\left\|D^m u\right\|_p^p\right)}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|D^m w\right\|^2-\frac{C_6^2}{2}\left\|D^m w\right\|^2-\frac{1}{2}\left\|D^m w_t\right\|^2$$

$$+\varepsilon N\left(\left\|D^m u\right\|_p^p\right)\left\|D^m w\right\|^2-\varepsilon C_6\left\|D^m w\right\|^2 \tag{53}$$

$$\geq\frac{N\left(\left\|D^m u\right\|_p^p\right)}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|D^m w\right\|^2+\left(\varepsilon N\left(\left\|D^m u\right\|_p^p\right)-\frac{C_6^2}{2}-\varepsilon C_6\right)\left\|D^m w\right\|^2-\frac{1}{2}\left\|D^m w_t\right\|^2$$

$$\geq\frac{\mu}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|D^m w\right\|^2+\left(\varepsilon\mu_0-\frac{C_6^2}{2}-\varepsilon C_6\right)\left\|D^m w\right\|^2-\frac{1}{2}\left\|D^m w_t\right\|^2,$$

where $C_5=\left\|N'\left(\left\|D^m\xi\right\|_p^p\right)\left(\left\|D^m\xi\right\|_p^p\right)'\right\|_\infty$.

Substitute (49) - (53) into Equation (48)

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\|w_t\|^2+2\varepsilon(w_t,w)+\mu\|D^m w\|^2+(\varepsilon\beta+\alpha)\|(-\Delta)^m w\|^2\right]$$
$$+2(\sigma_0-\varepsilon)\|w_t\|^2+\left(\alpha\varepsilon-\frac{C_4+\varepsilon C_4}{\lambda_1^m}\right)\|(-\Delta)^m w\|^2+\left(2\beta\lambda_1^m-\beta-1\right)\|D^m w_t\|^2 \quad (54)$$
$$+2\left(\varepsilon\mu_0-\varepsilon C_6-\frac{C_6^2}{2}\right)\|D^m w\|^2+\left(\alpha\varepsilon\lambda_1^{2m}-\varepsilon\sigma_1-\varepsilon C_4\right)\|w\|^2\le 0,$$

$$\varepsilon(w_t,w)\ge-\frac{\varepsilon}{2}\|w_t\|^2-\frac{\varepsilon}{2}\|w\|^2\ge-\frac{\varepsilon}{2}\|w_t\|^2-\frac{\varepsilon}{2\lambda_1^m}\|D^m w\|^2,$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[(1-\varepsilon)\|w_t\|^2+\left(\mu-\frac{\varepsilon}{\lambda_1^m}\right)\|D^m w\|^2+(\beta\varepsilon+\alpha)\|(-\Delta)^m w\|^2\right]$$
$$+2(\sigma_0-\varepsilon)\|w_t\|^2+\left(\alpha\varepsilon-\frac{C_4+\varepsilon C_4}{\lambda_1^m}\right)\|(-\Delta)^m w\|^2 \quad (55)$$
$$+2\left(\varepsilon\mu_0-\varepsilon C_3-\frac{{C_3}^2}{2}\right)\|D^m w\|^2\le 0,$$

$$\alpha\ge\max\left\{\frac{\sigma_1+C_4}{\lambda_1^{2m}},\frac{C_4+\varepsilon C_4}{\lambda_1^m}\right\},\quad \mu_0\ge\max\left\{\frac{\varepsilon}{\lambda_1^m},\frac{C_6^2}{2\varepsilon}+C_6\right\}$$

and (c),

$$\mu-\frac{\varepsilon}{\lambda_1^m}\ge 0, 2\beta\lambda_1^m-1-\beta\ge 0, \alpha\lambda_1^{2m}-\sigma_1-\varepsilon C_4\ge 0, \theta_5=\sigma_0-\varepsilon,$$

$$\theta_6=\alpha\varepsilon-\frac{C_4+\varepsilon C_4}{\lambda_1^m}, \theta_7=\varepsilon\mu_0-\varepsilon C_6-\frac{C_6^2}{2}\ge 0.$$

Take $\alpha_3=\max\left\{\frac{2\theta_5}{1-\varepsilon},\frac{\theta_6}{\alpha+\beta\varepsilon},\frac{2\theta_7}{\mu-\frac{\varepsilon}{\lambda_1^m}}\right\}$

Then, according to Gronwall's inequality, get

$$(1-\varepsilon)\|w_t\|^2+(\beta\varepsilon+\alpha)\|D^{2m}w\|^2$$
$$\le\alpha_3\left(\|w_t(0)\|^2+(\beta\varepsilon+\alpha)\|D^{2m}w(0)\|^2\right)\mathrm{e}^{-\alpha_3 t}=0, \quad (56)$$

Then $\|w_t\|^2=\|D^{2m}w\|^2=0$, that is $w(t)=0, u=v$, So the uniqueness is proved. Theorem 1 is proved.

Theorem 2. According to lemma 1 and theorem 1, then the initial boundary value problem (1) - (3) has a family of global attractors

$$A_k=\omega(B_{0k})=\bigcap_{\tau\ge 0}\overline{\bigcup_{t\ge\tau}S(t_k)B_{0k}},(k=1,2,3,\cdots,2m),$$

where $B_{0k}=\left\{(u,v)\in E_k:\|(u,v)\|_{E_k}^2=\|D^{2m+k}u\|^2+\|D^k v\|^2\le R_k^2+R_0^2\right\}$ is a bounded absorbing set in E_k and satisfies the following conditions:

1) $S(t)A_k=A_k, t>0$;
2) $\lim_{t\to\infty} dist(S(t)B_k,A_k)=0\ (\forall B_k\subset E_k)$ B_k is a bounded set;

Where $dist\left(S(t)B_k, A_k\right)=\sup_{x\in B_k}\inf_{y\in A_k}\left\|S(t)x-y\right\|_{E_k}$, $S(t)$ is the solution semigroup generated by the initial boundary value problem (1) - (3).

Proof. It is necessary to verify the conditions (I), (II) and (III) for the existence of attractors in reference [8]. Under the condition of Theorem 1, there exists a solution semigroup $S(t): E_k \to E_k$ of the initial boundary value problem (1) - (3).

From lemma 1, we can obtain that $\forall B_k \subset E_k$ is a bounded set that includes in the ball $\left\{\left\|(u,v)\right\|_{E_k} \le R_k\right\}$.

$$\begin{aligned}\left\|S(t)(u_0,v_0)\right\|_{E_k}^2 &= \|u\|_{H_0^{2m+k}(\Omega)}^2+\|v\|_{H_0^k(\Omega)}^2\\ &\le \|u_0\|_{H_0^{2m+k}(\Omega)}^2+\|v_0\|_{H_0^k(\Omega)}^2+C\\ &\le R_k^2+C,\end{aligned} \tag{57}$$

where $t\ge 0$, $(u_0,v_0)\in B_k$, this shows that $\{S(t)\}(t\ge 0)$ is uniformly bounded in E_k.

Furthermore, for any $(u_0,v_0)\in E_k$, when $t\ge \max\{t_1,t_k\}$, we have

$$\left\|S(t)(u_0,v_0)\right\|_{E_k}^2 = \|u\|_{H_0^{2m+k}(\Omega)}^2+\|v\|_{H_0^k(\Omega)}^2 \le R_k^2+R_0^2, \tag{58}$$

Therefore, $B_{0k}=\left\{(u,v)\in E_k : \left\|(u,v)\right\|_{E_k}^2=\left\|D^{2m+k}u\right\|^2+\left\|D^k v\right\|^2\le R_k^2+R_0^2\right\}$ is a bounded absorbing set in semigroup $S(t)$.

According to the rellich kondrachov's compact embedding theorem, if E_k is compactly embedded in E_0, then the bounded set in E_k is the compact set in E_0. Therefore, the solution semigroup $S(t)$ is a completely continuous operator, thus the family of global attractors A_k of solution semigroup $S(t)$ is obtained. Where $A_k=\omega(B_{0k})=\bigcap_{\tau\ge 0}\overline{\bigcup_{t\ge\tau}S(t)B_{0k}}$.

The proof is completed.

3. The Dimension Estimation for the Family of Global Attractors

In this part, we first linearize the equation into a first-order variational equation and prove that the solution semigroup $s(t)$ is Fréchet differentiable on E_k. furthermore, we prove the decay of the volume element of the linearization problem. Finally, we estimate the upper bound of the Hausdorff dimension and fractal dimension of A_k

Linearize problems (1) - (3),

$$\begin{aligned}&U_{tt}+M\left(\left\|D^m u\right\|_p^p\right)U_t+M'\left(\left\|D^m u\right\|_p^p\right)\left(\left\|D^m u\right\|_p^p\right)' u_t+\beta(-\Delta)^{2m}U_t+\alpha\Delta^{2m}U\\ &+N\left(\left\|D^m u\right\|_p^p\right)(-\Delta)^m U+N'\left(\left\|D^m u\right\|_p^p\right)\left(\left\|D^m u\right\|_p^p\right)'(-\Delta)^m u=0,\end{aligned} \tag{59}$$

$$U(x,t)=0, x\in\partial\Omega, t>0, \tag{60}$$

$$U(x,0)=\xi, U_t(x,0)=\eta. \tag{61}$$

where $(\xi,\eta)\in E_k$, $(u,u_t)=S(t)(u_0,u_1)$ is the solution of problems (1) - (3) with $(u_0,u_1)\in A_k$.

Given $(u_0,u_1)\in A_k$, $S(t):E_k\to E_k$, Verifiable to any $(\xi,\eta)\in E_k$, Linearization of initial boundary value problems (59) - (61) has a unique solution:

$$(U(t),U_t(t))\in L^\infty([0,+\infty);E_k). \tag{62}$$

Lemma 3. If $S(t):E_k\to E_k$, the Frechet differential on $\phi_0=(u_0,u_1)^{\mathrm{T}}$ is a linear operator $F:(\xi,\eta)^{\mathrm{T}}\to(U(t),U_t(t))^{\mathrm{T}}$, then $\forall t>0$, $R>0$, the mapping $S(t):E_k\to E_k$ is Fréchet differentiable on E_k, where $U(t),U_t(t)$ is the solution of the linearized initial boundary value problem (59) - (61).

Proof. Set $\phi_0=(u_0,u_1)^{\mathrm{T}}\in E_k,\bar{\phi}_0=(u_0+\xi,u_1+\eta)^{\mathrm{T}}\in E_k$,
$\phi_0=(u_0,u_1)^{\mathrm{T}}\in E_k,\bar{\phi}_0=(u_0+\xi,u_1+\eta)^{\mathrm{T}}\in E_k$ and $\|\phi_0\|_{E_k}\le R,\|\bar{\phi}_0\|_{E_k}\le R$,
the semigroup $S(t)$ is Lipschitz continuous on the bounded set of E_k, *i.e.*

$$\left\|S(t)\phi_0-S(t)\bar{\phi}_0\right\|_{E_k}^2\le \mathrm{e}^{ct}\left\|(\xi,\eta)^{\mathrm{T}}\right\|_{E_k}^2. \tag{63}$$

Let $\theta=U_t+\varepsilon U$, then

$$\begin{aligned}&\theta_t-\varepsilon\theta+\varepsilon^2u+M\left(\|D^mu\|_p^p\right)\theta-\varepsilon M\left(\|D^mu\|_p^p\right)U\\&+M'\left(\|D^mu\|_p^p\right)\left(\|D^mu\|_p^p\right)'u_t+\beta(-\Delta)^{2m}(\theta-\varepsilon u)\\&+N\left(\|D^mu\|_p^p\right)(-\Delta)^mU+N'\left(\|D^mu\|_p^p\right)\left(\|D^mu\|_p^p\right)'(-\Delta)^mu+\alpha\Delta^{2m}U=0,\end{aligned} \tag{64}$$

$$\theta(0)=0,\theta_t(0)=0. \tag{65}$$

Set $(\psi,\phi)=\bar{\eta}-\eta-\omega=(\bar{u}-u-U,\bar{v}-v-\theta)$,

$$\begin{cases}\bar{u}_{tt}+\beta(-\Delta)^{2m}\bar{u}_t+M\left(\|D^m\bar{u}\|_p^p\right)\bar{u}_t+\alpha\Delta^{2m}\bar{u}+N\left(\|D^m\bar{u}\|_p^p\right)(-\Delta)^m\bar{u}=f(x),\\ u_{tt}+\beta(-\Delta)^{2m}u_t+M\left(\|D^mu\|_p^p\right)u_t+\alpha\Delta^{2m}u+N\left(\|D^mu\|_p^p\right)(-\Delta)^mu=f(x),\\ U_{tt}+\beta(-\Delta)^{2m}\bar{U}_t+M\left(\|D^mu\|_p^p\right)U_t+M'\left(\|D^mu\|_p^p\right)\left(\|D^mu\|_p^p\right)'D^mU\cdot u_t\\ +\alpha\Delta^{2m}U+N\left(\|D^mu\|_p^p\right)(-\Delta)^mU+N'\left(\|D^mu\|_p^p\right)\left(\|D^mu\|_p^p\right)'D^mU\cdot(-\Delta)^mu=0.\end{cases}$$

Then the three equations can be subtracted

$$\begin{aligned}&\psi_{tt}+\beta(-\Delta)^{2m}\psi_t+\alpha\Delta^{2m}\psi+M\left(\|D^m\bar{u}\|_p^p\right)\bar{u}_t-M\left(\|D^mu\|_p^p\right)u_t\\&-M\left(\|D^mu\|_p^p\right)U_t-M'\left(\|D^mu\|_p^p\right)\left(\|D^mu\|_p^p\right)'D^mU\cdot u_t\\&+N\left(\|D^m\bar{u}\|_p^p\right)(-\Delta)^m\bar{u}-N\left(\|D^mu\|_p^p\right)(-\Delta)^mu-N\left(\|D^mu\|_p^p\right)(-\Delta)^mU\\&-N'\left(\|D^mu\|_p^p\right)\left(\|D^mu\|_p^p\right)'D^mU\cdot(-\Delta)^mu=0,\end{aligned} \tag{66}$$

$$
\begin{aligned}
H = {} & M\left(\left\|D^m\bar{u}\right\|_p^p\right)\bar{u}_t - M\left(\left\|D^m u\right\|_p^p\right)u_t - M\left(\left\|D^m u\right\|_p^p\right)U_t \\
& - M'\left(\left\|D^m u\right\|_p^p\right)\left(\left\|D^m u\right\|_p^p\right)' D^m U u_t + N\left(\left\|D^m\bar{u}\right\|_p^p\right)(-\Delta)^m \bar{u} \\
& - N\left(\left\|D^m u\right\|_p^p\right)(-\Delta)^m u - N\left(\left\|D^m u\right\|_p^p\right)(-\Delta)^m U \\
& - N'\left(\left\|D^m u\right\|_p^p\right)\left(\left\|D^m u\right\|_p^p\right)' D^m U(-\Delta)^m u.
\end{aligned}
$$

then $H = h_1 + h_2$,

$$
\begin{aligned}
h_1 = {} & M\left(\left\|D^m\bar{u}\right\|_p^p\right)\bar{u}_t - M\left(\left\|D^m u\right\|_p^p\right)u_t - M\left(\left\|D^m u\right\|_p^p\right)U_t \\
& - M'\left(\left\|D^m u\right\|_p^p\right)\left(\left\|D^m u\right\|_p^p\right)' D^m U u_t,
\end{aligned}
\tag{67}
$$

$$
\begin{aligned}
h_2 = {} & N\left(\left\|D^m\bar{u}\right\|_p^p\right)(-\Delta)^m \bar{u} - N\left(\left\|D^m u\right\|_p^p\right)(-\Delta)^m u - N\left(\left\|D^m u\right\|_p^p\right)(-\Delta)^m U \\
& - N'\left(\left\|D^m u\right\|_p^p\right)\left(\left\|D^m u\right\|_p^p\right)' D^m U(-\Delta)^m u,
\end{aligned}
\tag{68}
$$

$$
\begin{aligned}
h_1 = {} & M\left(\left\|D^m\bar{u}\right\|_p^p\right)\bar{u}_t - M\left(\left\|D^m u\right\|_p^p\right)\bar{u}_t + M\left(\left\|D^m u\right\|_p^p\right)\psi_t \\
& - M'\left(\left\|D^m u\right\|_p^p\right)\left(\left\|D^m u\right\|_p^p\right)' D^m(\bar{u} - u - \psi)u_t \\
= {} & M'\left(\left\|D^m \bar{\xi}\right\|_p^p\right)\left(\left\|D^m \bar{\xi}\right\|_p^p\right)' D^m(\bar{u} - u)\bar{u}_t \\
& - M'\left(\left\|D^m u\right\|_p^p\right)\left(\left\|D^m u\right\|_p^p\right)' D^m(\bar{u} - u)u_t + M'\left(\left\|D^m u\right\|_p^p\right)\left(\left\|D^m u\right\|_p^p\right)' \\
& D^m\psi u_t + M\left(\left\|D^m u\right\|_p^p\right)\psi_t \\
= {} & M''\left(\left\|D^m \varsigma\right\|_p^p\right)\left(\left\|D^m \varsigma\right\|_p^p\right)'(1-s)\left(D^m(\bar{u} - u)\right)^2 \bar{u}_t \\
& + M'\left(\left\|D^m u\right\|_p^p\right)\left(\left\|D^m u\right\|_p^p\right)' D^m(\bar{u} - u)(\bar{u}_t - u_t) + M' \\
& \left(\left\|D^m u\right\|_p^p\right)\left(\left\|D^m u\right\|_p^p\right)' D^m\psi u_t + M\left(\left\|D^m u\right\|_p^p\right)\psi_t,
\end{aligned}
\tag{69}
$$

$$
\bar{\xi} = sD^m\bar{u} + (1-s)D^m u, s \in (0,1),
$$

$$
\varsigma = sD^m\ \bar{\xi} + (1-s)D^m u, s \in (0,1).
$$

$$
\begin{aligned}
& \left|\left(M''\left(\left\|D^m \varsigma\right\|_p^p\right)\left(\left\|D^m \varsigma\right\|_p^p\right)'(1-s)\left(D^m(\bar{u} - u)\right)^2 \bar{u}_t, (-\Delta)^k \phi\right)\right| \\
& \le C_7 \left|\int_\Omega \left(D^m(\bar{u} - u)\right)^2 D^k\bar{u}_t D^k\phi \mathrm{d}x\right| \\
& \le C_8 \left\|\bar{u} - u\right\|_{H^m}^2 \left\|D^k\phi\right\| \\
& \le \frac{C_8^2\lambda_1^{-3m}}{2\mu_0}\left\|\bar{u} - u\right\|_{E_k}^4 + \frac{\mu_0}{2}\left\|D^{m+k}\phi\right\|^2,
\end{aligned}
\tag{70}
$$

$$C_7 = \left\| M''\left(\left\|D^m \varsigma\right\|_p^p\right)\left(\left\|D^m \varsigma\right\|_p^p\right)'(1-s)\right\|_\infty,$$

$$C_8 = \left\| M''\left(\left\|D^m \varsigma\right\|_p^p\right)\left(\left\|D^m \varsigma\right\|_p^p\right)'(1-s)\right\|_\infty \left\|D^k \overline{u}_t\right\|.$$

$$\begin{aligned}
&\left|\left(M'\left(\left\|D^m u\right\|_p^p\right)\left(\left\|D^m u\right\|_p^p\right)' D^m(\overline{u}-u)(\overline{u}_t-u_t),(-\Delta)^k\phi\right)\right| \\
&\le C_3\left|\int_\Omega D^m(\overline{u}-u)D^k(\overline{u}_t-u_t)D^k\phi \mathrm{d}x\right| \\
&\le \frac{C_3}{\lambda_1^{\frac{m}{2}}}\left\|D^m(\overline{u}-u)\right\|\cdot\left\|D^k(\overline{u}_t-u_t)\right\|\cdot\left\|D^{m+k}\phi\right\| \\
&\le \frac{C_3}{2\lambda_1^{\frac{2m+k}{2}}}\left(\left\|D^{2m+k}(\overline{u}-u)\right\|^2+\left\|D^k(\overline{u}_t-u_t)\right\|^2\right)\left\|D^{m+k}\phi\right\| \\
&\le \frac{C_3}{2\lambda_1^{\frac{2m+k}{2}}}\left(\left\|D^{2m+k}(\overline{u}-u)\right\|^2+\left\|D^k(\overline{v}-v)\right\|^2\right. \\
&\quad\left.+\varepsilon\lambda_1^{-2m}\left\|D^{2m+k}(\overline{u}-u)\right\|^2\right)\left\|D^{m+k}\phi\right\| \\
&\le C_9\left\|\overline{u}-u\right\|_{E_k}^2\cdot\left\|D^{m+k}\phi\right\| \\
&\le \frac{\mu_0}{2}\left\|D^{m+k}\phi\right\|^2+\frac{C_9^2}{2\mu_0}\left\|\overline{u}-u\right\|_{E_k}^4,
\end{aligned} \tag{71}$$

$$\begin{aligned}
&\left|\left(M'\left(\left\|D^m u\right\|_p^p\right)\left(\left\|D^m u\right\|_p^p\right)' D^m\psi u_t,(-\Delta)^k\phi\right)\right| \\
&\le C_3\left|\int_\Omega D^m\psi D^k u_t D^k\phi \mathrm{d}x\right| \\
&\le C_3\left\|D^m\psi\right\|_\infty\cdot\left\|D^k u_t\right\|\cdot\left\|D^k\phi\right\| \\
&\le C_3\left\|D^k u_t\right\|\cdot\lambda_1^{-m-\frac{k}{2}}\left\|D^{2m+k}\psi\right\|\cdot\left\|D^{m+k}\phi\right\| \\
&\le \frac{C_{10}^2}{2\mu_0}\left\|D^{2m+k}\psi\right\|^2+\frac{\mu_0}{2}\left\|D^{m+k}\phi\right\|^2,
\end{aligned} \tag{72}$$

$$\begin{aligned}
&\left|\left(M\left(\left\|D^m u\right\|_p^p\right)\psi_t,(-\Delta)^k\phi\right)\right| \\
&=\left|\left(M\left(\left\|D^m u\right\|_p^p\right)(\phi-\varepsilon\psi),(-\Delta)^k\phi\right)\right| \\
&\le \lambda_1^{-m}\sigma_1\left\|D^{m+k}\phi\right\|^2+\frac{\sigma_1}{4}\left\|D^k\phi\right\|^2+\varepsilon^2\sigma_1\left\|D^k\psi\right\|^2,
\end{aligned} \tag{73}$$

we obtain from Equations (69) - (73),

$$\begin{aligned}
\left(h_1,(-\Delta)^k\phi\right) \le &\left(\frac{3\mu_0}{2}+\frac{\sigma_1}{\lambda_1^m}\right)\left\|D^{m+k}\phi\right\|^2+\frac{C_8^2\lambda_1^{-3m}+C_9^2}{2\mu_0}\left\|\overline{u}-u\right\|_{E_k}^4 \\
&+C_{11}\left(\left\|D^k\psi\right\|^2+\left\|D^{2m+k}\psi\right\|^2+\left\|D^k\phi\right\|^2\right),
\end{aligned}$$

with $C_{11}=\max\left\{\frac{C_{10}^2}{2\mu_0},\varepsilon^2\sigma_1,\frac{\sigma_1}{4}\right\}$.

Do the same thing,

$$\begin{aligned}
h_2 &= N\left(\left\|D^m\bar{u}\right\|_p^p\right)(-\Delta)^m\bar{u}-N\left(\left\|D^m u\right\|_p^p\right)(-\Delta)^m u-N\left(\left\|D^m u\right\|_p^p\right)(-\Delta)^m U\\
&\quad -N'\left(\left\|D^m u\right\|_p^p\right)\left(\left\|D^m u\right\|_p^p\right)' D^m U(-\Delta)^m u\\
&= N\left(\left\|D^m\bar{u}\right\|_p^p\right)(-\Delta)^m\bar{u}-N\left(\left\|D^m u\right\|_p^p\right)(-\Delta)^m\bar{u}\\
&\quad -N'\left(\left\|D^m u\right\|_p^p\right)\left(\left\|D^m u\right\|_p^p\right)' D^m(\bar{u}-u)(-\Delta)^m u\\
&\quad +N'\left(\left\|D^m u\right\|_p^p\right)\left(\left\|D^m u\right\|_p^p\right)' D^m\psi(-\Delta)^m u+N\left(\left\|D^m u\right\|_p^p\right)(-\Delta)^m\psi\\
&= N''\left(\left\|\varsigma\right\|_p^p\right)\left(\left\|\varsigma\right\|_p^p\right)'(1-s)\left(D^m(\bar{u}-u)\right)^2(-\Delta)^m\bar{u}\\
&\quad +N'\left(\left\|D^m u\right\|_p^p\right)\left(\left\|D^m u\right\|_p^p\right)' D^m(\bar{u}-u)(-\Delta)^m(\bar{u}-u)\\
&\quad +N'\left(\left\|D^m u\right\|_p^p\right)\left(\left\|D^m u\right\|_p^p\right)' D^m\psi(-\Delta)^m u+N\left(\left\|D^m u\right\|_p^p\right)(-\Delta)^m\psi
\end{aligned}\tag{74}$$

Take inner product h_2 with $(-\Delta)^k\phi$, by usingYoung's inequality, Poincare's inequality,

$$\begin{aligned}
&\left|\left(N''\left(\left\|\varsigma\right\|_p^p\right)\left(\left\|\varsigma\right\|_p^p\right)'(1-s)\left(D^m(\bar{u}-u)\right)^2(-\Delta)^m\bar{u},(-\Delta)^k\phi\right)\right|\\
&\le C_{12}\left\|D^m(\bar{u}-u)\right\|^2\cdot\left\|D^{2m+k}\bar{u}\right\|\cdot\left\|D^k\phi\right\|\\
&\le C_{12}\left\|D^{2m+k}\bar{u}\right\|\lambda_1^{-m-k}\cdot\left\|D^{2m+k}(\bar{u}-u)\right\|^2\cdot\left\|D^k\phi\right\|\\
&\le\frac{C_{13}^2}{2\mu_0}\left\|D^{2m+k}(\bar{u}-u)\right\|^4+\frac{\mu_0}{2}\left\|D^k\phi\right\|^2,
\end{aligned}\tag{75}$$

where

$$C_{12}=\left\|N''\left(\left\|D^m\varsigma\right\|_p^p\right)\left(\left\|D^m\varsigma\right\|_p^p\right)'(1-s)\right\|_\infty,$$

$$C_{13}=\left\|N''\left(\left\|D^m\varsigma\right\|_p^p\right)\left(\left\|D^m\varsigma\right\|_p^p\right)'(1-s)\right\|_\infty\left\|D^{2m+k}\bar{u}\right\|\lambda_1^{-m-k}.$$

$$\begin{aligned}
&\left(N'\left(\left\|D^m u\right\|_p^p\right)\left(\left\|D^m u\right\|_p^p\right)' D^m(\bar{u}-u)(-\Delta)^m(\bar{u}-u),(-\Delta)^k\phi\right)\\
&\le C_5\lambda_1^{\frac{m+k}{-2}}\left\|D^{2m+k}(\bar{u}-u)\right\|^2\left\|D^k\phi\right\|\\
&\le\frac{\mu_0}{2}\left\|D^k\phi\right\|^2+\frac{C_5^2\cdot\lambda_1^{-m-k}}{2\mu_0}\left\|D^{2m+k}(\bar{u}-u)\right\|^4.
\end{aligned}\tag{76}$$

$$\begin{aligned}&\left(N'\left(\left\|D^m u\right\|_p^p\right)\left(\left\|D^m u\right\|_p^p\right)' D^m\psi(-\Delta)^m u,(-\Delta)^k\phi\right)\\&\le C_5\lambda_1^{\frac{-k}{2}}\left\|D^{m+k}\psi\right\|\cdot\left\|D^{2m+k}u\right\|\cdot\left\|D^k\phi\right\|\\&\le\frac{\mu_0}{2}\left\|D^k\phi\right\|^2+\frac{C_{14}^2}{2\mu_0}\left\|D^{m+k}\psi\right\|^2,\end{aligned}\tag{77}$$

$$\left(N\left(\left\|D^m u\right\|_p^p\right)(-\Delta)^m\psi,(-\Delta)^k\phi\right)\le\frac{\mu_0}{2}\left\|D^k\phi\right\|^2+\frac{\mu_1^2}{2\mu_0}\left\|D^{2m+k}\psi\right\|^2,\tag{78}$$

From Equation (75) - (78), it follows

$$\left|\left(h_2,(-\Delta)^k\phi\right)\right|\le C_{15}\left(\left\|D^{2m+k}\psi\right\|^2+\left\|D^k\phi\right\|^2\right)+\frac{C_{13}^2+C_5\lambda_1^{-m-k}}{2\mu_0}\left\|D^{2m+k}(\bar u-u)\right\|^4,\tag{79}$$

where $C_{15}=\max\left\{2\mu_0,\frac{C_{14}^2\lambda_1^{-m}+\mu_1^2}{2\mu_0}\right\}$.

Equation (73) and Equation (79) deduce that,

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\left\|D^k\phi\right\|^2+\varepsilon^2\left\|D^k\psi\right\|^2+(\alpha-\beta\varepsilon)\left\|D^{2m+k}\psi\right\|^2\right)+\varepsilon^3\left\|D^k\psi\right\|^2\\&+\left(\alpha\varepsilon-\beta\varepsilon^2\right)\left\|D^{2m+k}\psi\right\|^2+\left(\beta\lambda_1^m-\frac{3}{2}\mu_0-\frac{\sigma_1}{\lambda_1^m}\right)\left\|D^{m+k}\phi\right\|^2\\&\le\left(C_{11}+C_{15}\right)\left(\left\|D^k\phi\right\|^2+\left\|D^{2m+k}\psi\right\|^2\right)+C_{11}\left\|D^k\psi\right\|^2\\&\quad+\frac{C_8^2\lambda_1^{-3m}+C_9^2+C_{13}^2+C_5\lambda_1^{-m-k}}{2\mu_0}\left\|\bar u-u\right\|_{E_k}^4.\end{aligned}\tag{80}$$

According to the hypothesis (c), $\beta\lambda_1^m-\frac{3}{2}\mu_0-\frac{\sigma_1}{\lambda_1^m}\ge 0$,

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\left\|D^k\phi\right\|^2+\varepsilon^2\left\|D^k\psi\right\|^2+(\alpha-\beta\varepsilon)\left\|D^{2m+k}\psi\right\|^2\right)\\&\le C_{16}\left(\left\|D^k\phi\right\|^2+\varepsilon^2\left\|D^k\psi\right\|^2+(\alpha-\beta\varepsilon)\left\|D^{2m+k}\psi\right\|^2\right)\\&\quad+\frac{C_8^2\lambda_1^{-3m}+C_9^2+C_{13}^2+C_5\lambda_1^{-m-k}}{2\mu_0}\left\|\bar u-u\right\|_{E_k}^4,\end{aligned}\tag{81}$$

with $C_{16}=2\max\left\{C_{11}+C_{15},C_{11}\right\}=2\left(C_{11}+C_{15}\right)$,

Obtained by Gronwall inequality

$$\left\|D^k\phi\right\|^2+\varepsilon^2\left\|D^k\psi\right\|^2+(\alpha-\beta\varepsilon)\left\|D^{2m+k}\psi\right\|^2\le C_{17}\mathrm{e}^{C_{18}t}\left\|(\xi,\zeta)^{\mathrm{T}}\right\|_{E_k}^4.$$

$$\left\|D^k\phi\right\|^2+(\alpha-\beta\varepsilon)\left\|D^{2m+k}\psi\right\|^2\le C_{17}\mathrm{e}^{C_{18}t}\left\|(\xi,\zeta)^{\mathrm{T}}\right\|_{E_k}^4\tag{82}$$

From (82), as $\left\|(\xi,\eta)^{\mathrm{T}}\right\|_{E_k}^2\to 0$,

$$\frac{\left\|S(t)\bar\eta-S(t)\eta-DS(t)(\xi,\zeta)\right\|_{E_k}}{\left\|(\xi,\zeta)\right\|_{E_k}^2}\le C_{19}\mathrm{e}^{C_{20}t}\left\|(\xi,\zeta)\right\|_{E_k}^2\to 0.\tag{83}$$

Lemma 3 is proved.

Theorem 3. Under the assumptions and conditions of theorem 2, then a family of global attractors A_k of initial boundary value problem (1) - (3) has Hausdorff dimension and fractal dimension, and

$$d_H(A_k) < \frac{2}{7}n,\ d_F(A_k) < \frac{4}{7}n.$$

Proof. Let $\psi = R_\varepsilon\varphi = (u,v)^{\mathrm{T}}, \varphi = (u,u_t)^{\mathrm{T}}, v = u_t + \varepsilon u$, then $R_\varepsilon : \{u,u_t\} \to \{u,u_t+\varepsilon u\}$ is an isomorphic mapping. If $A_i(i=1,2,\cdots,2m)$ is the global attractor of $\{S(t)\}$, then $A_{\varepsilon i}$ is the global attractor of $\{S_\varepsilon(t)\}$, and they have the same dimension.

From Lemma 3, we can get that $S(t): E_k \to E_k$ is Fréchet differentiable, then the linearized first order variational Equation (59) can be rewritten as

$$P_t + \Lambda(\psi)P = 0, \tag{84}$$

$$\psi_t + \Lambda_\varepsilon\psi = 0. \tag{85}$$

$$\Lambda_\varepsilon = \begin{pmatrix} \varepsilon I & -I \\ \left(N\left(\left\|A^{\frac{m}{2}}u\right\|_p^P\right)A^m + (\alpha-\beta\varepsilon)A^{2m} + \varepsilon^2 - \varepsilon M + \phi + \kappa\right)I & \left(\beta A^{2m} + M - \varepsilon\right)I \end{pmatrix} \tag{86}$$

where

$$\phi = M'\int_\Omega p\left|A^{\frac{m}{2}}u\right|_p^{p-2} A^{\frac{m}{2}}uA^{\frac{m}{2}}U\mathrm{d}xu_t = M'\int_\Omega p\left|A^{\frac{m}{2}}u\right|_p^{p-2} A^{\frac{m}{2}}uA^{\frac{m}{2}}\cdot\mathrm{d}xu_tU,$$

$$\kappa = N'\int_\Omega p\left|A^{\frac{m}{2}}u\right|_p^{p-2} A^{\frac{m}{2}}uA^{\frac{m}{2}}U\mathrm{d}xA^mu = N'\int_\Omega p\left|A^{\frac{m}{2}}u\right|_p^{p-2} A^{\frac{m}{2}}uA^{\frac{m}{2}}\cdot\mathrm{d}xA^muU,$$

$$-\Delta = A.$$

For a fixed $(u_0,v_0)\in E_k$, let $\gamma_1,\gamma_2,\cdots,\gamma_n$ be n elements of E_k, and $U_1(t),U_2(t),\cdots,U_n(t)$ be n solutions of linear Equation (84), whose initial value is $U_1(0)=\gamma_1, U_2(0)=\gamma_2,\cdots,U_n(0)=\gamma_n$.

Therefore,

$$\begin{aligned}&\left\|U_1(t)\Lambda U_2(t)\Lambda\cdots\Lambda U_n(t)\right\|_{\Lambda E_k}^2\\&=\left\|\gamma_1\Lambda\gamma_1\Lambda\cdots\Lambda\gamma_n\right\|_{\Lambda E_k}\exp\left(\int_0^t trF'(\psi(\tau))\cdot Q_n(\tau)\mathrm{d}\tau\right).\end{aligned} \tag{87}$$

where Λ is the outer product, tr is the trace, $Q_n(\tau)$ is an orthogonal projection from space E_k to $span\{U_1(t),U_2(t),\cdots,U_n(t)\}$.

For a given time τ, let $\omega_j(\tau) = (\xi_j(\tau),\eta_j(\tau))^{\mathrm{T}}$ $(j=1,2,\cdots,n)$ be the standard orthogonal basis of spance $span\{w_1(t),w_2(t),\cdots,w_n(t)\}$.

We define the inner product of E_k as

$$\left((\xi,\eta),(\bar\xi,\bar\eta)\right) = \left(\left(D^{2m+k}\xi, D^{2m+k}\bar\xi\right) + \left(D^k\eta, D^k\bar\eta\right)\right). \tag{88}$$

To sum up, it can be concluded that

$$\begin{aligned} trF'(\psi(\tau))\cdot Q_n(\tau) &= \sum_{j=1}^{n}\left(F'(\psi(\tau))\cdot Q_n(\tau)\omega_j(\tau),\omega_j(\tau)\right)_{E_k} \\ &= \sum_{j=1}^{n}\left(F'(\psi(\tau))\omega_j(\tau),\omega_j(\tau)\right)_{E_k}, \end{aligned} \tag{89}$$

where

$$\left(F'(\psi(\tau))\omega_j(\tau),\omega_j(\tau)\right)_{E_k} = -\left(\Lambda_\varepsilon \omega_j,\omega_j\right).$$

$$\begin{aligned} -\left(\Lambda_\varepsilon\omega_j,\omega_j\right) &= -\Bigg(\Bigg(\varepsilon\xi_j-\eta_j, N\left(\left\|A^{\frac{m}{2}}u\right\|_p^p\right)A^m\xi_j+(\alpha-\beta\varepsilon)A^{2m}\xi_j \\ &+\left(\varepsilon^2-\varepsilon M\left(\left\|A^{\frac{m}{2}}u\right\|_p^p\right)+\phi+\kappa\right)\xi_j+\beta A^{2m}\eta_j+M\left(\left\|A^{\frac{m}{2}}u\right\|_p^p\right)\eta_j-\varepsilon\eta_j\Bigg),\left(\xi_j,\eta_j\right)\Bigg) \\ &+\frac{\varepsilon^2}{2}\left\|D^k\eta_j\right\|^2-\lambda_1^m\mu_0\left\|D^k\eta_j\right\|^2+\varepsilon\left\|D^k\eta_j\right\|^2+\frac{\varepsilon}{2}\left\|D^{2m+k}\xi_j\right\|^2+\frac{C_{14}^2\lambda_1^{-m-k}}{2\varepsilon}\left\|D^k\eta_j\right\|^2 \\ &= -\varepsilon\left\|D^{2m+k}\xi_j\right\|^2-(\alpha-1-\beta\varepsilon)\left(D^{2m+k}\xi_j,D^{2m+k}\eta_j\right) \\ &-N\left(\left\|A^{\frac{m}{2}}u\right\|_p^p\right)\left(D^{2m+k}\xi_j,D^k\eta_j\right)-\left(\varepsilon^2+\phi+\kappa\right)\left(D^k\xi_j,D^k\eta_j\right) \\ &-\beta\left\|D^{2m+k}\eta_j\right\|^2-M\left(\left\|A^{\frac{m}{2}}u\right\|_p^p-\varepsilon\right)\left\|D^k\eta_j\right\|^2 \\ &\le -\varepsilon\left\|D^{2m+k}\xi_j\right\|^2+\frac{\alpha-1-\beta\varepsilon}{2}\left\|D^{2m+k}\xi_j\right\|^2+\frac{\alpha-1-\beta\varepsilon}{2}\left\|D^{2m+k}\eta_j\right\|^2 \\ &+\frac{\mu_0}{2}\left\|D^{2m+k}\xi_j\right\|^2+\frac{\mu_0}{2}\left\|D^k\eta_j\right\|^2+\frac{\varepsilon^2+\phi+\kappa-\varepsilon\sigma_0}{2\lambda_1^{2m}}\left\|D^{2m+k}\xi_j\right\|^2 \\ &+\frac{\varepsilon^2+\phi+\kappa-\varepsilon\sigma_0}{2}\left\|D^k\eta_j\right\|^2-\beta\left\|D^{2m+k}\eta_j\right\|^2-(\sigma_0-\varepsilon)\left\|D^k\eta_j\right\|^2 \\ &= \left(\frac{\alpha-1-\beta\varepsilon}{2}-\varepsilon+\frac{\mu_0}{2}+\frac{\varepsilon^2+\phi+\kappa-\varepsilon\sigma_0}{2\lambda_1^{2m}}\right)\left\|D^{2m+k}\xi_j\right\|^2 \\ &+\left(\frac{\alpha-1-\beta\varepsilon-\beta}{2}\right)\left\|D^{2m+k}\eta_j\right\|^2 \\ &+\frac{1}{2}\left(\mu_1+\varepsilon^2+\phi+\kappa-\varepsilon\sigma_0-\beta\lambda_1^{2m}-2\sigma_0\right)\left\|D^k\eta_j\right\|^2+\varepsilon\left\|D^k\eta_j\right\|^2 \\ &\le \left(\frac{\alpha-1-\beta\varepsilon}{2}-\varepsilon+\frac{\mu_1}{2}+\frac{\varepsilon^2+\phi+\kappa-\varepsilon\sigma_0}{2\lambda_1^{2m}}\right)\left\|D^{2m+k}\xi_j\right\|^2 \\ &+\frac{1}{2}\left((\alpha-1-\beta\varepsilon-2\beta)\lambda_1^{2m}+\mu_0+\varepsilon^2+\phi+\kappa-\varepsilon\sigma_0-2\sigma_0\right)\left\|D^k\eta_j\right\|^2+\varepsilon\left\|D^k\eta_j\right\|^2 \\ &\le -C_{21}\left(\left\|D^{2m+k}\xi_j\right\|^2+\left\|D^k\eta_j\right\|^2\right)+\varepsilon\left\|D^k\eta_j\right\|^2, \end{aligned} \tag{90}$$

$$\begin{aligned} C_{21} = \min\Bigg\{&\frac{\beta\varepsilon+1+\alpha}{2}+\varepsilon-\frac{\mu_0}{2}-\frac{\varepsilon^2+\phi+\kappa-\varepsilon\sigma_0}{2\lambda_1^{2m}},\frac{-1}{2}\Big((\alpha-1-\beta\varepsilon-2\beta)\lambda_1^{2m} \\ &+\mu_0+\varepsilon^2+\phi+\kappa-\varepsilon\sigma_0-\beta\lambda_1^{2m}-2\sigma_0\Big)\left\|D^k\eta_j\right\|^2+\varepsilon\left\|D^k\eta_j\right\|^2\Bigg\}. \end{aligned} \tag{91}$$

According to the hypothesis (c),

$$\alpha-\beta\varepsilon-1\ge 0,\alpha-\beta\varepsilon-1-\beta\le 0,$$

$$\frac{\alpha-1-\beta\varepsilon}{2}-\varepsilon+\frac{\mu_1}{2}+\frac{\varepsilon^2+\phi+\kappa-\varepsilon\sigma_0}{2\lambda_1^{2m}}\le 0,$$

$$\mu_0+\varepsilon^2+\phi+\kappa-\varepsilon\sigma_0-2\sigma_0\le 0.$$

From (84) and (85),

$$\begin{aligned}\left(F'(\psi(\tau))\omega_j(\tau),\omega_j(\tau)\right)_{E_k}&\le -C_{21}\left(\left\|D^{2m+k}\xi_j\right\|^2+\left\|D^k\eta_j\right\|^2\right)+\varepsilon\left\|D^k\eta_j\right\|^2\\&\le -C_{22}\left(\left\|\xi_j\right\|^2+\left\|\eta_j\right\|^2\right)+r\left\|D^k\eta_j\right\|^2.\end{aligned}\tag{92}$$

Owing to the $\omega_j(\tau)=\left(\xi_j(\tau),\eta_j(\tau)\right)^{\mathrm{T}}$, $j=1,2,\cdots,n$ is the standard orthonormal basis of $span\{U_1(t),U_2(t),\cdots,U_n(t)\}$, so

$$\left\|\xi_j\right\|^2+\left\|\eta_j\right\|^2=1,\tag{93}$$

$$\sum_{j=1}^{n}\left(F'(\psi(\tau))\omega_j(\tau),\omega_j(\tau)\right)_{E_k}\le -nC_{22}+r\sum_{j=1}^{n}\left\|D^k\eta_j\right\|^2,\tag{94}$$

$$\sum_{j=1}^{n}\left\|D^k\eta_j\right\|^2\le\sum_{j=1}^{n}\lambda_j^{s-1},\tag{95}$$

Therefore

$$TrF'(\psi(\tau))\cdot Q_n(\tau)\le -nC_{22}+r\sum_{j=1}^{n}\lambda_j^{s-1}.\tag{96}$$

$$q_n(t)=\sup_{\psi_0\in B_{0k}}\sup_{\substack{\eta_j\in E_k\\ \|\eta_j\|\le 1}}\left(\frac{1}{t}\int_0^t trF'(S(\tau)\psi_0)\cdot Q_n(\tau)\,\mathrm{d}\tau\right),\quad q_n=\lim_{t\to\infty}q_n(t),\tag{97}$$

And

$$q_n\le -nC_{22}+r\sum_{j=1}^{n}\lambda_j^{s-1}.\tag{98}$$

Thus, the Lyapunov exponent $\kappa_1,\kappa_2,\cdots,\kappa_n\ (n>1)$ of B_{0k} is uniformly bounded

$$\kappa_1+\kappa_2+\cdots+\kappa_n\le -nC_{22}+r\sum_{j=1}^{n}\lambda_j^{s-1}.\tag{99}$$

then, there is a $s\in[0,1]$, such that

$$\left(q_j\right)_+\le -nC_{22}+r\sum_{j=1}^{n}\lambda_j^{s-1}\le r\sum_{j=1}^{n}\lambda_j^{s-1}\le\frac{nC_{22}}{9},\tag{100}$$

where λ_j is the eigenvalue of A_k^m, $\lambda_1<\lambda_2<\cdots<\lambda_n$.

$$q_n\le -\frac{nC_{22}}{2}\left(1-\frac{2r}{nC_{22}}\sum_{j=1}^{n}\lambda_j^{s-1}\right)\le -\frac{7}{18}nC_{22}.\tag{101}$$

$$\max_{1\le j\le n}\frac{\left(q_j\right)_+}{\left|q_n\right|}\le\frac{2}{7}.\tag{102}$$

Then

$$d_H(A_k) < \frac{2}{7}n, d_F(A_k) < \frac{4}{7}n,$$

the Hausdorff dimension and Fractal dimension of the family of global attractors are finite. Theorem 3 is proved.

4. Conclusion

We have shown the existence and uniqueness of global solutions, the existence of the family of global attractors and the upper bound estimates of Hausdoff and Fractal dimensions for the wide norm model of vertical beam vibration. The global stability of the problem is obtained.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

References

[1] Kirchhoff, G. (1883) Vorlesungen Uber Mechanik, Teubner, Leipzig. Stutgaety, 56-70.

[2] Teman, R. (1988) Infinite-Dimensional Dynamical Systems in Mechanics and Physics. Springer-Verlag, New York. https://doi.org/10.1007/978-1-4684-0313-8

[3] Guo, B. and Jing, Z. (1996) Infinite-Dimensional Dynamical Systems. *China Academic Journal*, **2**, 290-296.

[4] Chueshov, I. (2012) Long-Time Dynamics of Kirchhoff Wave Models with Strong Nonlinear Damping. *Journal of Differential Equations*, **252**, 1229-1262. https://doi.org/10.1016/j.jde.2011.08.022

[5] Matsuyama, T. and Ikehata, R. (1996) On Global Solutions and Energy Decay for the Wave Equations of Kirchhoff Type with Nonlinear Damping Terms. *Journal of Mathematical Analysis and Applications*, **204**, 729-753. https://doi.org/10.1006/jmaa.1996.0464

[6] Gao, Y., Sun, Y. and Lin, G. (2016) The Global Attractors and Their Hausdorff and Fractal Dimensions Estimation for the Higer-Order Nonlinear Kirchhoff-Type Equation with Strong Linear Damping. *International Journal of Modern Nonlinear Theory and Application*, **5**, 185-202. https://doi.org/10.4236/ijmnta.2016.54018

[7] Chen, L., Wang, W. and Lin, G. (2016) The Global Attractors and the Hausdorff and Fractal Demensions Estimation for the Higher-Order Nonlinear Kirchhoff-Type Equation. *Journal of Advances in Mathematics*, **12**, 6608-6621. https://doi.org/10.24297/jam.v12i9.133

[8] Lin, G. (2011) Nonlinear Evolution Equation. Yunnan University Press, Kunming, 123-184.

[9] Zhou, S. (1999) Global Attractor for Strongly Damped Nonlinear Wave Equations. *Functional Differential Equations*, **6**, 451-470.

[10] Xu, G. and Lin, G. (2011) Global Attractors and Their Dimensions Estimation for the Generalized Boussinesq Equation. *China Science and Technology Information*, **10**, 54-61.

[11] Lyu, P. (2016) Long-Time Behavior of a Class of Generalized Nonlinear Kirchhoff-Boussinesq Equations. Yunnan University, Kunming.

[12] Lin, G. and Yang, S. (2017) Hausdorff Dimension and Fractal Dimension of the Global Attractor for the Higher-Order Coupled Kirchhoff-Type Equations. *Journal of Applied Mathematics and Physics*, **5**, 2411-2424.

[13] Lyu, P., Lu, J. and Lin, G. (2016) Global Attractor for a Class of Nonlinear Generalized Kirchhoff Models. *Journal of Advances in Mathematics*, **12**, 6452-6462. https://doi.org/10.24297/jam.v12i8.135

[14] Lyu, P., Lou, R. and Lin, G. (2016) Global Attractor for a Class of Nonlinear Generalized Kirchhoff-Boussinesq Model. *International Journal of Modern Nonlinear Theory and Application*, **5**, 82-92. https://doi.org/10.4236/ijmnta.2016.51009

[15] Sun, Y., Gao, Y. and Lin, G. (2016) The Global Attractors for the Higher-Order Kirchhoff-Type Equation with Nonlinear Strongly Damped Term. International Journal of Modern Nonlinear *Theory and Application*, **5**, 203-217. https://doi.org/10.4236/ijmnta.2016.54019

[16] Li, Q.X. and Zhong, T. (2002) Existence of Global Solutions for Kirchhoff Type Equations with Dissipation and Damping Terms. *Journal of Xiamen University*: *Natural Science Edition*, **41**, 419-422.

[17] Silva, M.A.J. and Ma, T.F. (2013) Long-Time Dynamics for a Class of Kirchhoff Models with Memory. *Journal of Mathematical Physics*, **54**, Article ID: 021505. https://doi.org/10.1063/1.4792606

[18] Alves, C.O. and Figueiredo, G.M. (2016) Nonlinear Perturbations of a Periodic Kirchhoff Equation in R^N. *Journal of Mathematical Analysis and Applications*, **435**, 1826-1851.

[19] Lin, G. and Li, Z. (2019) A Class of Attractor Families and Their Dimensions for Higher Order Nonlinear Kirchhoff Equations. *Journal of Shandong University* (*Natural Science*), **54**, 1-11.