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A Family of Exponential Attractors and Inertial Manifolds for a Class of Higher Order Kirchhoff Equations

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Abstract

In this paper, the global dynamics of a class of higher order nonlinear Kirchhoff equations under n-dimensional conditions is studied. Firstly, the Lipschitz property and squeezing property of the nonlinear semigroup associated with the initial boundary value problem are proved, and the existence of a family of exponential attractors is obtained. Then, by constructing the corresponding graph norm, the condition of a spectral interval is established when N is sufficiently large. Finally, the existence of the family of inertial manifolds is obtained.

Keywords

Kirchhoff Equation, Lipschitz Property, Squeezing Property, a Family of the Exponential Attractors, a Family of Inertial Manifolds

1. Introduction

In the 1990s, C. Foiao proposed the concept of an exponential attractor. The exponential attractor has a compact positive invariant set with finite fractal dimension and is exponentially attractive to the solution orbit. The exponential attractor has deeper and more practical properties. Compared with the global attractor, the exponential attractor has a uniform exponential convergence rate on the invariant absorption set of its solution. The exponential attractor is more robust under numerical approximation and perturbation. The family of inertial manifolds is concerned with the long-time behavior of the solution of a dissipative evolution equation, which is a finite-dimensional invariant Lipschitz manifold and attracts all solution orbits in the phase space with an exponential rate. The family of inertial manifolds is an important link between finite-dimensional and infinite-dimensional dynamical systems. In this paper, we study the family of exponential attractors and inertial manifolds of the following nonlinear Kirchhoff equations

$$u_{tt}+M\left(\left\|D^m u\right\|_p^p\right)(-\Delta)^{2m}u+\beta(-\Delta)^{2m}u_t+g(u_t)=f(x), \tag{1.1}$$

$$u(x,t)=0,\frac{\partial^i u}{\partial v^i}=0, i=1,2,\cdots,2m-1, x\in\partial\Omega, t>0, \tag{1.2}$$

$$u(x,0)=u_0(x), u_t(x,0)=u_t(x), x\in\Omega\subset R^n. \tag{1.3}$$

where $m>1$, and $m\in N^+$, $\Omega\in R^n\ (n\geq 1)$ is a bounded domain, $\partial\Omega$ denotes the boundary of Ω, $g(u_t)$ is a nonlinear source term, $\beta(-\Delta)^{2m}u_t$ is a strongly dissipative term, $\beta>0$, $f(x)$ is an external force term.

In reference [1], Fan Xiaoming constructed the spatial discretization based on the wave equation on R^+, and studied the exponential attractor of the second-order lattice dynamical system with nonlinear damping:

$$u_{tt}-\beta\Delta u_t+h(u_t)-\Delta u+\lambda u+\bar{g}u=\bar{q}$$

In reference [2], Yang Zhijian *et al.* studied the exponential attractor of Kirchhoff equation with strong damping of nonlinear term and supercritical nonlinear term:

$$u_{tt}-\sigma\left(\|\nabla u\|^2\right)\Delta u_t-\phi\left(\|\nabla u\|^2\right)\Delta u+f(u)=h(x),$$
$$u|_{\partial\Omega}=0, u(x,0)=u_0(x), u_t(x,0)=u_1(x), x\in\Omega.$$

They prove the existence of the exponential attractor by using the weak quasi-stability estimate; in reference [3], Xu Guigui, Wang Libo and Lin Guoguang studied a class of second-order nonlinear wave equations with time delay under the assumption that the delay time is small:

$$\frac{\partial^2 u}{\partial^2 t}+\alpha\frac{\partial u}{\partial t}-\beta\Delta\frac{\partial u}{\partial t}-\Delta u+g(u)=f(x)+h(t,u_t),$$
$$t>0,\alpha>0,\beta>0.$$

The existence of inertial manifolds. Need to know more references [4]-[17].

2. Basic Assumption

For convenience, space and symbols are defined as follows:

$H=L^2(\Omega)$, $H_0^{2m}(\Omega)=H^{2m}(\Omega)\cap H_0^1(\Omega)$,
$H_0^{2m+k}(\Omega)=H^{2m+k}(\Omega)\cap H_0^1(\Omega)$, $E_0=H^{2m}(\Omega)\times L^2(\Omega)$,
$E_k=H_0^{2m+k}(\Omega)\times H_0^k(\Omega), k=1,2,\cdots,2m$. $(\cdot,\cdot)$ and $\|\cdot\|$ represent the inner product and the norm on H space, then $(u,v)=\int_\Omega u(x)v(x)\mathrm{d}x$, $(u,u)=\|u\|^2$.

Nonlinear function $g(u_t)$ meets the following conditions:

(H_1) $2<p\leq\dfrac{2n}{n-2m}, n>2m; p\geq 2, n<2m.$

(H_2) $g\in C^k, k=1,2,\cdots,2m.$

(H$_3$) $M(s)\in C^2([0,+\infty),R^+),1\le\mu_0<M(s)<\mu_1,\mu=\begin{cases}\mu_0,\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla^{2m}u\|^2\ge 0,\\ \mu_1,\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla^{2m}u\|^2<0.\end{cases}$

(H$_4$) $\|g'(s)\|_\infty\le C_2$.

3. Exponential Attractors

In order to prove the need of later, so the inner product and norm of E_k are defined

$$\forall U_i=(u_i,v_i)\in E_k, i=1,2,$$

$$(U_1,U_2)_{E_k}=(\nabla^{2m+k}u_1,\nabla^{2m+k}u_2)+(\nabla^k v_1,\nabla^k v_2) \tag{3.1}$$

$$\|U_1\|_{E_k}^2=(U_1,U_1)_{E_k}=\|\nabla^{2m+k}u_1\|^2+\|\nabla^k v_1\|^2 \tag{3.2}$$

Let $U=(u,v)\in E_k$, $v=u_t+\varepsilon u$,

$$\frac{1-\sqrt{1-\beta\lambda_1^{-2m}}}{\beta}\le\varepsilon\le\min\left\{\frac{\beta\lambda_1^{2m}}{4},\frac{1+\sqrt{1-\beta\lambda_1^{-2m}}}{\beta},\sqrt[4]{\frac{\beta}{2\lambda_1^{-2m}}}\right\}$$

then the Equation (1.1) is equivalent to

$$U_t+HU=F(U) \tag{3.3}$$

where

$$HU=\begin{pmatrix}\varepsilon u-v\\ \beta(-\Delta)^{2m}v+(1-\beta\varepsilon)(-\Delta)^{2m}u-\varepsilon v+\varepsilon^2 u\end{pmatrix}$$

$$F(U)=\begin{pmatrix}0\\ \left(1-M\left(\|D^m u\|_p^p\right)\right)(-\Delta)^{2m}u+f(x)-g(u_t)\end{pmatrix}$$

By definition, we know that E_0,E_k are two Hilbert spaces, E_k is dense and compact in E_0, let $S(t)$ is the mapping of E_i to E_i, $i=0,k$.

Definition 3.1. [4] If there is a compact set $A_k\subset E_k$, A_k attracts all bounded sets in E_k, and it is an invariant set under $S(t)$, $S(t)A_k=A_k$, $\forall t\ge 0$. Then we say that a semigroup $S(t)$ has a family of (E_k,E_0)-compact attractors A_k.

Definition 3.2. [5] If $A_k\subset M_k\subset B_k$ and

1) $S(t)M_k\subseteq M_k$, $\forall t\ge 0$;

2) M_k has finite fractal dimension, $d_f(M_k)<+\infty$;

3) There exist universal constants $\eta>0,\delta>0$, such that

$$dist(S(t),B_k,M_k)\le\eta\mathrm{e}^{-\delta t},t>0,$$

where, $dist_{E_k}(A_k,B_k)=\sup_{x\in A_k}\inf_{y\in B_k}|x-y|_{E_k}$, B_k is the positive invariant set of $S(t)$ in E_k. The compact set $M_k\in E_k$ is called a family of (E_k,E_0) exponential attractors for the system $(S(t),E_k)$.

Definition 3.3. [5] if there exists limited function $l(t)$, such that

$$\left|S(t)u-S(t)v\right|_{E_k}\le l(t)\left|u-v\right|_{E_k},\forall u,v\in E_k. \tag{3.4}$$

Then the semigroup $S(t)$ is Lipschitz continuous in E_k.

Definition 3.4. [5] If $\delta\le\left(0,\frac{1}{8}\right)$ and exists an orthogonal projection P_N of rank *N*, such that for $\forall(u,v)\in E_k$,

$$\left|S(t_*)u-S(t_*)v\right|_{E_k}\le\delta\left|u-v\right|_{E_k}, \tag{3.5}$$

or

$$\left|Q_N\left(S(t_*)u-S(t_*)v\right)\right|_{E_k}\le\left|P_N\left(S(t_*)u-S(t_*)v\right)\right|_{E_k}. \tag{3.6}$$

Then $S(t)$ is said to satisfy the discrete squeezing property in E_k, where $Q_N=I-P_N$.

Theorem 3.1. [5] Assume that

1) $S(t)$ possesses a family of (E_k,E_0)-compact attractors A_k;

2) In E_k, there exists a compact set B_k with positive invariance to the action of $S(t)$;

3) $S(t)$ is Lipschitz continuous and is squeezed in B_k. $S(t)B_k$ possesses a family of (E_k,E_0)-compact attractors M_k, $M_k=\bigcup_{0\le t\le t_*}S(t)M_*$, $M_*=A_k\cup\left(\bigcup_{j=1}^{\infty}\bigcup_{i=1}^{\infty}S(t_*)^j\left(E^{(i)}\right)\right)$. The fractal dimension of M_k satisfies $d_f(M_k)\le cN_0+1$, where, N_0 is the smallest N which makes the discrete squeezing property established.

Proposition 3.2. [6] After making reasonable assumptions about $M(s)$ and $g(u_t)$, the initial boundary value problem (1.1)-(1.3) has a unique smooth solution and the solution has the following properties:

$$\left\|(u,v)\right\|_{E_0}^2=\left\|\nabla^{2m}u\right\|^2+\left\|v\right\|^2\le C(R_0),\left\|(u,v)\right\|_{E_k}^2=\left\|\nabla^{2m+k}u\right\|^2+\left\|\nabla^k v\right\|^2\le C(R_1).$$

The solution $S(t)(u_0,v_0)=(u(t),v(t))$ of the equation is expressed by Theorem 3.1, then $S(t)$ is a semigroup of continuous operators in E_k, we have the ball

$$B_1=\left\{(u,v)\in E_0:\left\|(u,v)\right\|_{E_0}^2\le C(R_0)\right\} \tag{3.7}$$

$$B_2=\left\{(u,v)\in E_k:\left\|(u,v)\right\|_{E_k}^2\le C(R_1)\right\} \tag{3.8}$$

are absorbing sets of $S(t)$ in E_0 and E_k respectively.

Lemma 3.1. For any $U=(u,v)\in E_k$, there is $(HU,U)\ge k_1\left\|U\right\|_{E_k}^2+k_2\left\|\nabla^{2m+k}v\right\|^2$.

Proof. By

$$(U_1,U_2)_{E_k}=\left(\nabla^{2m+k}u_1,\nabla^{2m+k}u_2\right)+\left(\nabla^k v_1,\nabla^k v_2\right)$$

and

$$\left\|U_1\right\|_{E_k}^2=(U_1,U_1)_{E_k}=\left\|\nabla^{2m+k}u_1\right\|^2+\left\|\nabla^k v_1\right\|^2,$$

then

$$\begin{aligned}(HU,U)&=\left(\nabla^{2m+k}(\varepsilon u-v),\nabla^{2m+k}u\right)+\left(\nabla^{k}H,\nabla^{k}v\right)\\&=\left(\varepsilon\nabla^{2m+k}u,\nabla^{2m+k}u\right)-\left(\nabla^{2m+k}v,\nabla^{2m+k}u\right)+\left(\nabla^{k}H,\nabla^{k}v\right).\end{aligned}\tag{3.9}$$

where $H=\beta(-\Delta)^{2m}v+(1-\beta\varepsilon)(-\Delta)^{2m}u-\varepsilon v+\varepsilon^2u$.

From the Young's inequality and the Poincare's inequality, we can get

$$\begin{aligned}\left(\nabla^{k}H,\nabla^{k}v\right)&=\left(\nabla^{k}\left(\beta(-\Delta)^{2m}v+(1-\beta\varepsilon)(-\Delta)^{2m}u-\varepsilon v+\varepsilon^2u\right),\nabla^{k}v\right)\\&\ge\frac{\beta}{2}\left\|\nabla^{2m+k}v\right\|^2+\frac{\beta\lambda_1^{2m}}{2}\left\|\nabla^{k}v\right\|^2-\frac{\beta\varepsilon}{2}\left\|\nabla^{2m+k}u\right\|^2-\frac{\beta\varepsilon}{2}\left\|\nabla^{2m+k}v\right\|^2\\&\quad+\left(\nabla^{2m+k}u,\nabla^{2m+k}v\right)-\varepsilon\left\|\nabla^{k}v\right\|^2+\varepsilon^2\left(\nabla^{k}u,\nabla^{k}v\right),\end{aligned}\tag{3.10}$$

$$\begin{aligned}\varepsilon^2\left(\nabla^{k}u,\nabla^{k}v\right)&\ge-\varepsilon^2\lambda_1^{-m}\left\|\nabla^{m+k}u\right\|\left\|\nabla^{m+k}v\right\|\\&\ge-\varepsilon^2\lambda_1^{-2m}\left(\frac{1}{2\varepsilon^2}\left\|\nabla^{2m+k}u\right\|^2+\frac{\varepsilon^2}{2}\left\|\nabla^{2m+k}v\right\|^2\right)\\&=-\frac{\lambda_1^{-2m}}{2}\left\|\nabla^{2m+k}u\right\|^2-\frac{\lambda_1^{-2m}\varepsilon^4}{2}\left\|\nabla^{2m+k}v\right\|^2,\end{aligned}\tag{3.11}$$

where $\lambda_1(>0)$ is the first eigenvalue of the operator $-\Delta$.

Derived from Formula (3.9) to Formula (3.11)

$$\begin{aligned}(HU,U)\ge&\left(\frac{\beta\lambda_1^{2m}}{2}-\varepsilon\right)\left\|\nabla^{k}v\right\|^2+\left(\varepsilon-\frac{\beta\varepsilon}{2}-\frac{\lambda_1^{-2m}}{2}\right)\left\|\nabla^{2m+k}u\right\|^2.\\&+\left(\frac{\beta}{2}-\frac{\beta\varepsilon}{2}-\frac{\lambda_1^{-2m}\varepsilon^4}{2}\right)\left\|\nabla^{2m+k}v\right\|^2.\end{aligned}$$

Due to

$$\frac{\lambda_1^{-2m}}{2-\beta}\le\varepsilon\le\min\left\{\frac{\beta\lambda_1^{2m}}{2},\frac{1+\sqrt{1-\beta\lambda_1^{-2m}}}{\beta},\sqrt[4]{\frac{\beta}{2\lambda_1^{-2m}}}\right\},$$

then

$$\frac{\beta\lambda_1^{2m}}{2}-\varepsilon\ge0,\varepsilon-\frac{\beta\varepsilon}{2}-\frac{\lambda_1^{-2m}}{2}\ge0,\frac{\beta}{2}-\frac{\beta\varepsilon}{2}-\frac{\lambda_1^{-2m}\varepsilon^4}{2}\ge0.$$

thus

$$(HU,U)\ge k_1\|U\|_{E_k}^2+k_2\left\|\nabla^{2m+k}v\right\|^2.$$

where

$$k_1=\min\left\{\left(\frac{\beta\lambda_1^{2m}}{2}-\varepsilon\right),\left(\varepsilon-\frac{\beta\varepsilon}{2}-\frac{\lambda_1^{-2m}}{2}\right)\right\},k_2=\left(\frac{\beta}{2}-\frac{\beta\varepsilon}{2}-\frac{\lambda_1^{-2m}\varepsilon^4}{2}\right)\ge0.$$

Let $S(t)U_0=U(t)=(u(t),v(t))^{\mathrm{T}}$, where $v=u_t(t)+\varepsilon u(t)$; $S(t)V_0=V(t)=(\tilde{u}(t),\tilde{v}(t))^{\mathrm{T}}$, where $\tilde{v}=\tilde{u}_t(t)+\varepsilon\tilde{u}(t)$;

Let $W(t)=S(t)U_0-S(t)V_0=U(t)-V(t)=(w(t),z(t))^{\mathrm{T}}$; where $z(t)=w_t(t)+\varepsilon w(t)$, then $W(t)$ satisfies

$$W_t(t)+HU-HV+F(V)-F(U)=0,\tag{3.12}$$

$$W(0)=U_0-V_0. \tag{3.13}$$

To verify that the Formulas (1.1)-(1.3) has an exponential attractor, it is necessary to prove that the dynamical system $S(t)$ is Lipschitz continuous on E_k.

Lemma 3.2. (Lipschitz property). For any $U_0,V_0\in E_k, T\geq 0$, there is

$$\left\|S(t)U_0-S(t)V_0\right\|_{E_k}^2\leq \mathrm{e}^{bt}\left\|U_0-V_0\right\|_{E_k}^2.$$

Proof. We can get the inner product of $W(t)$ and Formula (3.4) in E_k space,

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|W(t)\right\|^2+\left(HU-HV,W(t)\right)+\left((-\Delta)^{2m+\frac{k}{2}}w,\nabla^k z(t)\right)\\&+\left(M\left(\left\|\nabla^m u\right\|_p^p\right)(-\Delta)^{2m+\frac{k}{2}}u-M\left(\left\|\nabla^m \tilde{u}\right\|_p^p\right)(-\Delta)^{2m+\frac{k}{2}}\tilde{u},\nabla^k z(t)\right)\\&+\left(\nabla^k\left(g(u_t)-g(\tilde{u}_t)\right),\nabla^k z(t)\right)=0.\end{aligned} \tag{3.14}$$

A proof similar to lemma 3.1, obtained

$$\left(HU-HV,W(t)\right)_{E_k}\geq k_1\left\|W(t)\right\|_{E_k}^2+k_2\left\|\nabla^{2m+k}z(t)\right\|_{E_k}^2. \tag{3.15}$$

From Young's inequality and Poincare's inequality and the mean value Theorem, we can get

$$\begin{aligned}-\left((-\Delta)^{2m+\frac{k}{2}}w,\nabla^k z(t)\right)&\geq-\left\|\nabla^{2m+k}w\right\|\left\|\nabla^{2m+k}z\right\|\\&\geq-\frac{1}{2}\left\|\nabla^{2m+k}w\right\|^2-\frac{1}{2}\left\|\nabla^{2m+k}z\right\|^2.\end{aligned} \tag{3.16}$$

let

$$s=\left\|\nabla^m u\right\|_p^p,\tilde{s}=\left\|\nabla^m\tilde{u}\right\|_p^p,$$

then

$$\begin{aligned}&\left(M\left(\left\|\nabla^m u\right\|_p^p\right)(-\Delta)^{2m+\frac{k}{2}}u-M\left(\left\|\nabla^m \tilde{u}\right\|_p^p\right)(-\Delta)^{2m+\frac{k}{2}}\tilde{u},\nabla^k z(t)\right)\\&\leq\left|\left(M(\tilde{s})(-\Delta)^{2m+\frac{k}{2}}w,\nabla^k z(t)\right)\right|+\left|\left(M'(\vartheta)(\tilde{s}-s)(-\Delta)^{2m+\frac{k}{2}}u,\nabla^k z(t)\right)\right|\\&\leq\frac{\mu_1}{2}\left\|\nabla^{2m+k}w\right\|^2+\frac{\mu_1}{2}\left\|\nabla^{2m+k}z\right\|^2+C_0\left\|\nabla^{2m+k}w\right\|\left\|\nabla^{2m+k}z\right\|\\&\leq\frac{\mu_1+C_0}{2}\left\|\nabla^{2m+k}w\right\|^2+\frac{\mu_1+C_0}{2}\left\|\nabla^{2m+k}z\right\|^2.\end{aligned} \tag{3.17}$$

$$\begin{aligned}&\left|\left(\nabla^k\left(g(u_t)-g(\tilde{u}_t)\right),\nabla^k z(t)\right)\right|=\left|\left(\nabla^k g'(s)w_t,\nabla^k z(t)\right)\right|\\&=g'(s)\left|\left(\nabla^k w_t(t),\nabla^k z(t)\right)\right|\leq C_2\left|\left(\nabla^k\left(z(t)-\varepsilon w(t)\right),\nabla^k z(t)\right)\right|\\&\leq C_2\left|\left(\nabla^k z(t),\nabla^k z(t)\right)\right|-\varepsilon C_2\left|\left(\nabla^k w(t),\nabla^k z(t)\right)\right|\\&\leq C_2\left\|\nabla^k z(t)\right\|^2+\frac{\varepsilon C_2\lambda_1^{-2m}}{2}\left\|\nabla^{2m+k}w(t)\right\|^2+\frac{\varepsilon C_2}{2}\left\|\nabla^k z(t)\right\|^2.\end{aligned} \tag{3.18}$$

then

$$\frac{\mathrm{d}}{\mathrm{d}t}\|W(t)\|^2+2k_1\|W(t)\|^2+(2k_2-1-\mu_1-C_0)\|\nabla^{2m+k}z(t)\|$$
$$\le(1+\mu_1+C_0+\varepsilon C_2\lambda_1^{-2m})\|\nabla^{2m+k}w(t)\|^2+(2C_2+\varepsilon C_2)\|\nabla^k z(t)\|^2 \tag{3.19}$$
$$\le C_5\|W(t)\|^2.$$

where

$$C_5=\max\{1+\mu_1+C_0+\varepsilon C_2\lambda_1^{-2m},2C_2+\varepsilon C_2\}.$$

By Gronwall's inequality, we can get

$$\|W(t)\|^2\le \mathrm{e}^{2C_5t}\|W(0)\|_{E_k}^2=\mathrm{e}^{bt}\|W(0)\|_{E_k}^2, \tag{3.20}$$

where $b=2C_5$.

Then

$$\|S(t)U_0-S(t)V_0\|_{E_k}^2\le \mathrm{e}^{bt}\|U_0-V_0\|_{E_k}^2.$$

Apparently, $-\Delta$ is an unbounded self-adjoint closed positive operator, and $(-\Delta)^{-1}$ is compact, we know that there is an Orthonormal basis of H through the basic theory of spectral spacing, it is made up of the eigenvector w_j of $-\Delta$, so that

$$-\Delta w_j=\lambda_j w_j, 0<\lambda_1\le\lambda_2\le\cdots\le\lambda_j\to+\infty, j\to\infty.$$

P_N is an orthogonal projection in E_k. $Q_N=I-P_N$.

Next, were going to use

$$\|((-\Delta)^{2m}u)\|\ge\lambda_{n+1}^{2m}\|u\|,\forall u\in Q_n(H^{4m}(\Omega)\cap H_0^1(\Omega)),\|Q_nu\|\le\|u\|,u\in H.$$

Lemma 3.3. For any $U_0,V_0\in E_k$, $Q_{n_0}(t)=Q_{n_0}(U(t)-V(t))=Q_{n_0}W(t)=(w_{n_0},z_{n_0})^{\mathrm{T}}$, then

$$\|W_{n_0}(t)\|_{E_k}^2\le\left(\mathrm{e}^{-2k_1t}+\frac{C_3\lambda_{1+n_0}^{-m}}{2k_1+b}\mathrm{e}^{bt}\right)\|W(0)\|^2$$

Proof. Apply $Q_{n_0}(t)$ to Equation (3.4), we get

$$W_{n_0}(t)+Q_{n_0}(HU-HV)+Q_{n_0}(F(V)-F(U))=0. \tag{3.21}$$

The sum of (3.11) and $W_{n_0}(t)$ is the inner product of E_k, we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|W_{n_0}(t)\|^2+k_1\|W_{n_0}(t)\|^2+k_2\|\nabla^{2m+k}z_{n_0}(t)\|^2$$
$$+(Q_{n_0}(F(V)-F(U)),\nabla^k z_{n_0}(t))=0. \tag{3.22}$$

Known by Young's inequality and hypothesis condition

$$-\left(Q_{n_0}(-\Delta)^{2m+\frac{k}{2}}w,\nabla^k z_{n_0}(t)\right)\ge-\|\nabla^{2m+k}w_{n_0}\|\|\nabla^{2m+k}z_{n_0}\|$$
$$\ge-\frac{1}{2}\|\nabla^{2m+k}w_{n_0}\|^2-\frac{1}{2}\|\nabla^{2m+k}z_{n_0}\|^2. \tag{3.23}$$

$$\begin{aligned}
&\left(Q_{n_0}\left(M\left(\left\|\nabla^m u\right\|_p^p\right)(-\Delta)^{2m+\frac{k}{2}}u - M\left(\left\|\nabla^m \tilde{u}\right\|_p^p\right)(-\Delta)^{2m+\frac{k}{2}}\tilde{u}\right),\nabla^k z_{n_0}(t)\right)\\
&\le\left|\left(M(\tilde{s})(-\Delta)^{2m+\frac{k}{2}}w_{n_0},\nabla^k z_{n_0}(t)\right)\right|+\left|\left(M'(\vartheta)(\tilde{s}-s)(-\Delta)^{2m+\frac{k}{2}}u,\nabla^k z_{n_0}(t)\right)\right|\\
&\le\frac{\mu_1}{2}\left\|\nabla^{2m+k}w_{n_0}\right\|^2+\frac{\mu_1}{2}\left\|\nabla^{2m+k}z_{n_0}\right\|^2+C_0\left\|\nabla^{2m+k}w_{n_0}\right\|\left\|\nabla^{2m+k}z_{n_0}\right\|\\
&\le\frac{\mu_1+C_0}{2}\left\|\nabla^{2m+k}w_{n_0}\right\|^2+\frac{\mu_1+C_0}{2}\left\|\nabla^{2m+k}z_{n_0}\right\|^2.
\end{aligned}\tag{3.24}$$

$$\begin{aligned}
&\left|Q_{n_0}\nabla^k\left(g(u_t)-g(\tilde{u}_t)\right),\nabla^k z_{n_0}(t)\right|=\left|\left(\nabla^k g'(s)w_{tn_0},\nabla^k z_{n_0}(t)\right)\right|\\
&=g'(s)\left|\left(\nabla^k w_{tn_0}(t),\nabla^k z_{n_0}(t)\right)\right|\le C_3\left|\left(\nabla^k\left(z_{n_0}(t)-\varepsilon w_{n_0}(t)\right),\nabla^k z_{n_0}(t)\right)\right|\\
&\le C_3\left|\left(\nabla^k z_{n_0}(t),\nabla^k z_{n_0}(t)\right)\right|-\varepsilon C_3\left|\left(\nabla^k w_{n_0}(t),\nabla^k z_{n_0}(t)\right)\right|\\
&\le C_3\left\|\nabla^k z_{n_0}(t)\right\|^2+\frac{\varepsilon C_3\lambda_{1+n_0}^{-2m}}{2}\left\|\nabla^{2m+k}w_{n_0}(t)\right\|^2+\frac{\varepsilon C_3}{2}\left\|\nabla^k z_{n_0}(t)\right\|^2\\
&\le\left(1+\mu_1+C_0+\varepsilon C_3\lambda_{1+n_0}^{-2m}\right)\left\|\nabla^{2m+k}w_{n_0}(t)\right\|^2+\left(2C_3+\varepsilon C_3\right)\left\|\nabla^k z_{n_0}(t)\right\|^2.
\end{aligned}\tag{3.25}$$

Replace (3.13) with (3.12) to get

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left\|W_{n_0}(t)\right\|^2+2k_1\left\|W_{n_0}(t)\right\|^2+\left(2k_2-1-\mu_1-C_0\right)\left\|\nabla^{2m+k}z_{n_0}(t)\right\|^2\\
&\le\left(1+\mu_1+C_0+\varepsilon C_3\lambda_{1+n_0}^{-2m}\right)\left\|\nabla^{2m+k}w_{n_0}(t)\right\|^2+\left(2C_3+\varepsilon C_3\right)\left\|\nabla^k z_{n_0}(t)\right\|^2.
\end{aligned}\tag{3.26}$$

By Gronwall's inequality, we can get

$$\begin{aligned}
\left\|W_{n_0}(t)\right\|^2&\le\left\|W(0)\right\|^2\mathrm{e}^{-2k_1t}+\frac{C_3\lambda_{1+n_0}^{-2m}}{2k_1+b}\mathrm{e}^{bt}\left\|W(0)\right\|^2\\
&=\left(\mathrm{e}^{-2k_1t}+\frac{C_3\lambda_{1+n_0}^{-2m}}{2k_1+b}\mathrm{e}^{bt}\right)\left\|W(0)\right\|^2
\end{aligned}\tag{3.27}$$

Thus lemma 3.3 is proved.

Lemma 3.4. (Discrete squeezing property). For any $U_0,V_0\in E_k$, if

$$\left\|P_{n_0}\left(S(T^*)U_0-S(T^*)V_0\right)\right\|_{E_k}\le\left\|\left(I-P_{n_0}\right)\left(S(T^*)U_0-S(T^*)V_0\right)\right\|_{E_k},$$

then

$$\left\|S(T^*)U_0-S(T^*)V_0\right\|_{E_k}\le\frac{1}{8}\left\|U_0-V_0\right\|_{E_k}$$

Proof. If

$$\left\|P_{n_0}\left(S(T^*)U_0-S(T^*)V_0\right)\right\|_{E_k}\le\left\|\left(I-P_{n_0}\right)\left(S(T^*)U_0-S(T^*)V_0\right)\right\|_{E_k},$$

then

$$\begin{aligned}
&\left\|S(T^*)U_0-S(T^*)V_0\right\|_{E_k}^2\\
&\le\left\|\left(I-P_{n_0}\right)\left(S(T^*)U_0-S(T^*)V_0\right)\right\|_{E_k}^2+\left\|P_{n_0}\left(S(T^*)U_0-S(T^*)V_0\right)\right\|_{E_k}^2\\
&\le2\left\|\left(I-P_{n_0}\right)\left(S(T^*)U_0-S(T^*)V_0\right)\right\|_{E_k}^2\\
&\le2\left(\mathrm{e}^{-2k_1T^*}+\frac{C_4\lambda_{1+n_0}^{-2m}}{2k_1+b}\mathrm{e}^{bT^*}\right)\left\|U_0-V_0\right\|_{E_k}^2.
\end{aligned}\tag{3.28}$$

Let T^* be big enough

$$e^{-2k_1T^*} \le \frac{1}{256}. \tag{3.29}$$

and let n_0 be big enough

$$\frac{C_4\lambda_{1+n_0}^{-2m}}{2k_1+b} e^{bT^*} \le \frac{1}{256}. \tag{3.30}$$

Replace the Formulas (3.17) and (3.18) into the Formula (3.16), we get

$$\left\| S(T^*)U_0 - S(T^*)V_0 \right\|_{E_k} \le \frac{1}{8}\left\| U_0 - V_0 \right\|_{E_k}. \tag{3.31}$$

Theorem 3.3. Under the appropriate assumptions above, $(u_0, v_0) \in E_k$, $k = 1, 2, \cdots, 2m$, $f \in H$, $v = u_t + \varepsilon u$,

$$\frac{\lambda_1^{-2m}}{2-\beta} \le \varepsilon \le \min\left\{ \frac{\beta\lambda_1^{2m}}{2}, \frac{1+\sqrt{1-\beta\lambda_1^{-2m}}}{\beta}, \sqrt[4]{\frac{\beta}{2\lambda_1^{-2m}}} \right\},$$

then the solution semigroup of the initial boundary value problem (1.1)-(1.3) has a family of (E_k, E_0) exponential attractors on E_k,

$$M_k = \bigcup_{0\le t\le T^*} S(t)\left(A_k \cup \left(\bigcup_{j=1}^{\infty}\bigcup_{i=1}^{\infty} S(T^*)^j \left(E^{(i)} \right) \right) \right),$$

and the fractal dimension is satisfied $d_f(M_k) \le cN_0 + 1$.

Proof: According to Theorem 3.1, Lemma 3.2, Lemma 3.4, Theorem 3.3 is easy to prove.

4. Inertial Manifolds

Definition 4.1. [18] Assum $S = S(t)_{t\ge 0}$ is a solution semigroup of Banach space $E_k = H_0^{2m+k}(\Omega) \times H_0^k(\Omega)$, $k = 1, 2, \cdots, 2m$, a subset $\mu_k \subset E_k$ satisfies the following three properties:

1) μ_k is finite dimensional Lipschitz manifold;

2) μ_k is positively invariant, $S(t)\mu_0 \subset \mu_k, \forall t \ge 0$;

3) μ_k attracts exponentially all the orbits of the solution, and $u \in E_k$, there are constants $\eta > 0, \gamma > 0$, then

$$dist\left(S(t)u, \mu_k \right) \le \gamma e^{-\eta t}, t \ge 0.$$

It is said that μ_k is an inertial manifold of $\{S(t)\}_{t\ge 0}$.

Definition 4.2. [19] Let $\Lambda : E_k \to E_k$ be an operator and assume that $F \in C_b(E_k, E_k)$ satisfies the Lipschitz condition

$$\left\| F(U) - F(V) \right\|_{E_k} \le l_F \left\| U - V \right\|_{E_k}, \quad U, V \in E_k. \tag{4.1}$$

If the point spectrum of the operator Λ can be divided into two parts σ_1 and σ_2, where σ_1 is finite,

$$\Lambda_1 = \sup\{ Re\lambda \mid \lambda \in \sigma_1 \}, \quad \Lambda_2 = \inf\{ Re\lambda \mid \lambda \in \sigma_2 \}, \tag{4.2}$$

$$E_{k_i} = span\left\{ \omega_j \mid \lambda_j \in \sigma_i, i = 1, 2 \right\}. \tag{4.3}$$

and satisfies the condition

$$\Lambda_2 - \Lambda_1 > 4l_F, \tag{4.4}$$

and the orthogonal decomposition

$$E_k = E_{k_1} \oplus E_{k_2}, \tag{4.5}$$

set $P_1 : E_k \to E_{k_1}$ and $P_2 : E_k \to E_{k_2}$ are both continuous orthogonal projections, then the operator Λ is said to satisfy the spectral interval condition.

Lemma 4.1. [18] Let the eigenvalues $\mu_j^{\pm}(j \ge 1)$ be non-decreasing, and $m \in N^*$, there exists $N \ge m$, such that μ_{N+1}^- and μ_N^- are consecutive adjacent values.

Equation (1.1) is equivalent to the following first order evolution equation

$$U_t + \Lambda U = F(U), \tag{4.6}$$

where

$$U \in E_k, U = (u,v)^{\mathrm{T}} = (u,u_t)^{\mathrm{T}},$$

$$\Lambda = \begin{pmatrix} 0 & -I \\ M\left(\left\|\nabla^m u\right\|_p^p\right)(-\Delta)^{2m} & \beta(-\Delta)^{2m} \end{pmatrix},$$

$$F(U) = \begin{pmatrix} 0 \\ f(x) - g(u_t) \end{pmatrix}$$

a graph defined on E_k by the quantity product:

$$\langle U,V\rangle_{E_k} = \left(M \cdot \nabla^{2m+k} u, \nabla^{2m+k} \overline{y}\right) + (v, \overline{z}). \tag{4.7}$$

where $U = (u,v)^{\mathrm{T}}, V = (y,z)^{\mathrm{T}} \in E_k$, $\overline{y}, \overline{z}$ respectively represents the conjugation of y and z $v,z \in H_0^{2m+k}(\Omega)$, $u,y \in H_0^{2m+k}(\Omega)$. $\forall U \in E_k$, there is

$$\begin{aligned} \langle \Lambda U,U\rangle_{E_k} &= -\left(M \cdot \nabla^{2m+k} u_t, \nabla^{2m+k} \overline{u}\right) + \left(M\left(\left\|\nabla^m u\right\|_p^p\right)(-\Delta)^{2m} u + \beta(-\Delta)^{2m} u_t, \overline{v}\right) \\ &\ge -\left(M \cdot \nabla^{2m+k} u_t, \nabla^{2m+k} \overline{u}\right) + M\left(\nabla^{2m+k} u, \nabla^{2m+k} v\right) + \beta\left(-\Delta^m v, -\Delta^m \overline{v}\right) \\ &\ge \beta\left\|\nabla^{2m} v\right\|^2 > 0. \end{aligned} \tag{4.8}$$

Therefore, the operator Λ is monotonically increasing, and $\langle \Lambda U,U\rangle_{E_k}$ is a nonnegative real number.

The characteristic equation $\Lambda U = \lambda U$, $U = (u,v)^{\mathrm{T}} \in E_k$ is equivalent to

$$-v = \lambda u \tag{4.9}$$

$$M\left(\left\|\nabla^m u\right\|_p^p\right)(-\Delta)^{2m} u + \beta(-\Delta)^{2m} v = \lambda v. \tag{4.10}$$

Therefore, λ satisfies the following eigenvalue problem

$$\begin{cases} \lambda^2 u + M\left(\left\|\nabla^m u\right\|_p^p\right)(-\Delta)^{2m} u - \beta\lambda(-\Delta)^{2m} u = 0, \\ u|_{\partial\Omega} = (-\Delta)^{2m} u\big|_{\partial\Omega} = 0. \end{cases} \tag{4.11}$$

Given by Formulas (4.11) and (4.12), the corresponding eigenvectors take the form

$$U_j^{\pm} = \left(u_j, -\lambda_j^{\pm} u_j\right), \mu_j = \lambda_1 j^{\frac{2m}{n}}. \tag{4.12}$$

where $\mu_j\left(j \geq 1\right)$ is the eigenvecroot of $\left(-\Delta\right)^{2m}$ in $H_0^{2m}\left(\Omega\right)$.

For $\forall j \geq 1$, there is

$$\left\|\nabla^{2m+k} u_j\right\| = \sqrt{\mu_j}, \left\|\nabla^k u_j\right\| = 1, \left\|\nabla^{-2m-k} u_j\right\| = \frac{1}{\sqrt{\mu_j}}, k = 1, 2, \cdots, 2m. \tag{4.13}$$

Take the position of (4.12) u in u_j and use $\left(-\Delta\right)^k u$ to get the inner product

$$\lambda^2 \left\|\nabla^k u\right\|^2 + M\left(\left\|\nabla^m u\right\|_p^p\right)\left\|\nabla^{2m+k} u\right\|^2 - \beta\lambda\left\|\nabla^{2m+k} u\right\|^2 = 0. \tag{4.14}$$

Consider the Formula (4.16) as the quadratic equation of λ, as follows:

$$\lambda_j^{\pm} = \frac{\beta\mu_j \pm \sqrt{\beta^2\mu_j^2 - 4M\mu_j}}{2}. \tag{4.15}$$

Theorem 4.1. If $N_1 \in N$ is large enough, when $N \geq N_1$, the following inequality holds

$$\frac{1}{8}\left(\left(\mu_{N+1} - \mu_N\right)\left(\beta - \sqrt{\beta^2\mu_j^2 - 4M\left(s\right)}\right) - 1\right) \geq l_F. \tag{4.16}$$

then the operator Λ is said to satisfy the spectral interval condition.

Proof. Because all the eigenvalues of Λ are positive real numbers, and the known sequence $\left\{\lambda_N^-\right\}_{N\geq 1}$ and $\left\{\lambda_N^+\right\}_{N\geq 1}$ is incremented.

This theorem is then proved in four steps.

Step 1: Known non-subtractive sequence of $\lambda_N^{\pm}$, according to lemma 4.1, for $\forall m \in N$, $\exists N \geq m$ makes λ_N^- and λ_{N+1}^- adjacent, the eigenvalues of the operator Λ can be decomposed into

$$\sigma_1 = \left\{\lambda_j^-, \lambda_k^+ \mid \max\left\{\lambda_j^-, \lambda_k^+\right\} \leq \lambda_N^-\right\}, \tag{4.17}$$

$$\sigma_2 = \left\{\lambda_j^+, \lambda_k^{\pm} \mid \lambda_j^- \leq \lambda_N^- \leq \min\left\{\lambda_j^+, \lambda_k^{\pm}\right\}\right\}. \tag{4.18}$$

Step 2: The corresponding E_k can be decomposed into

$$E_{k_1} = span\left\{U_j^-, U_k^+ \mid \lambda_j^-, \lambda_k^+ \in \sigma_1\right\}, \tag{4.19}$$

$$E_{k_2} = span\left\{U_j^-, U_k^{\pm} \mid \lambda_j^-, \lambda_k^{\pm} \in \sigma_2\right\}. \tag{4.20}$$

In order to make the two subspace orthogonal and satisfy the interspectral Formula (4.4).

$\Lambda_1 = \lambda_N^-, \Lambda_2 = \lambda_{N+1}^-$. further decomposition $E_{k_2} = E_C + E_R$, *i.e.*

$$E_C = span\left\{U_j^- \mid \lambda_j^- \leq \lambda_N^- \leq \lambda_j^+\right\}, \tag{4.21}$$

$$E_R = span\left\{U_R^{\pm} \mid \lambda_N^- \leq \lambda_k^{\pm}\right\}. \tag{4.22}$$

and let $E_N = E_{k_1} \oplus E_C$.

Next, we specify the quantity product of the eigenvalues over E_k, makes E_{k_1} and E_{k_2} orthogonal, here are two functions

$$\Phi: E_N \to R,\ \ \psi: E_R \to R.$$

$$\begin{aligned}\Phi(U,V) &= \beta\left(\nabla^{2m+k}u,\nabla^{2m+k}\overline{y}\right)+2\beta\left(\nabla^{-2m-k}\overline{z},\nabla^{2m}u\right)\\ &\quad+2\beta\left(\nabla^{-2m-k}v,\nabla^{2m}\overline{y}\right)+4\left(\nabla^{-2m-k}v,\nabla^{-2m-k}z\right)\\ &\quad-4M\left(\left\|\nabla^m u\right\|_p^p\right)\left(\nabla^k\overline{u},\nabla^k y\right)+\left(2\beta^2-\beta\right)\left(\nabla^{2m+k}\overline{u},\nabla^{2m+k}y\right).\end{aligned} \tag{4.23}$$

$$\begin{aligned}\Psi(U,V) &= \left(\nabla^{2m+k}u,\nabla^{2m+k}\overline{y}\right)+\left(\nabla^{-2m-k}\overline{z},\nabla^{2m+k}u\right)-\left(\nabla^{-2m-k}v,\nabla^{2m+k}\overline{y}\right)\\ &\quad-4M\left(\left\|\nabla^m u\right\|_p^p\right)\left(\nabla^k\overline{u},\nabla^k y\right)+\left(\beta^2-1\right)\left(\nabla^{2m+k}\overline{u},\nabla^{2m+k}y\right).\end{aligned} \tag{4.24}$$

where $U=(u,v)^{\mathrm{T}}, V=(y,z)^{\mathrm{T}} \in E_N$, $\overline{y},\overline{z}$ respectively represents the conjugation of y and z.

Set $\forall U=(u,v)\in E_N$, then

$$\begin{aligned}\Phi(U,U) &= \beta\left(\nabla^{2m+k}u,\nabla^{2m+k}\overline{u}\right)+2\beta\left(\nabla^{-2m-k}\overline{v},\nabla^{2m}u\right)\\ &\quad+2\beta\left(\nabla^{-2m-k}v,\nabla^{2m}\overline{u}\right)+4\left(\nabla^{-2m-k}v,\nabla^{-2m-k}v\right)\\ &\quad-4M\left(\left\|\nabla^m u\right\|_p^p\right)\left(\nabla^k\overline{u},\nabla^k u\right)+\left(2\beta^2-\beta\right)\left(\nabla^{2m+k}\overline{u},\nabla^{2m+k}u\right)\\ &\geq \beta\left\|\nabla^{2m+k}u\right\|^2-4\left\|\nabla^{-2m-k}\overline{v}\right\|^2-\beta^2\left\|\nabla^{2m+k}u\right\|^2-4M(s)\left\|\nabla^{m+k}u\right\|^2\\ &\quad+\left(2\beta^2-\beta\right)\left\|\nabla^{2m+k}u\right\|^2+4\left\|\nabla^{-2m-k}\overline{v}\right\|^2\\ &= \beta^2\left\|\nabla^{2m+k}u\right\|^2-4\mu_1\left\|\nabla^k u\right\|^2\geq\left(\beta^2\mu_j^2-4M(s)\right)\left\|\nabla^k u\right\|^2.\end{aligned} \tag{4.25}$$

Since β is sufficiently large, can Be obtained $\Phi(U,U)\geq 0$, $\forall U\in E_N$, thus Φ is positive definite.

In the same way, $\forall U=(u,v)\in E_R$, there is

$$\begin{aligned}\Psi(U,U) &= \left(\nabla^{2m+k}u,\nabla^{2m+k}\overline{u}\right)+\left(\nabla^{-2m-k}\overline{v},\nabla^{2m+k}u\right)-\left(\nabla^{-2m-k}v,\nabla^{2m+k}\overline{u}\right)\\ &\quad-4M\left(\left\|\nabla^m u\right\|_p^p\right)\left(\nabla^k\overline{u},\nabla^k u\right)+\left(\beta^2-1\right)\left(\nabla^{2m+k}\overline{u},\nabla^{2m+k}u\right)\\ &\geq\left\|\nabla^{2m+k}u\right\|^2-4M(s)\left\|\nabla^{m+k}u\right\|^2+\left(\beta^2-1\right)\left\|\nabla^{2m+k}u\right\|^2\\ &\geq\left(\beta^2\mu_j^2-4M(s)\right)\left\|\nabla^k u\right\|^2.\end{aligned} \tag{4.26}$$

$\Psi(U,U)\geq 0$, Ψ are positive definite.

Specifies the inner product of E_k:

$$\langle\langle U,V\rangle\rangle_{E_k}=\Phi\left(P_N U,P_N V\right)+\Psi\left(P_R U,P_R V\right). \tag{4.27}$$

where P_N and P_R are maps of $E_k\to E_N$ and $E_k\to E_R$ respectively.

The above formula can be abbreviated to

$$\langle\langle U,V\rangle\rangle_{E_k}=\Phi(U,V)+\Psi(U,V). \tag{4.28}$$

In the inner product of E_k, E_{k_1} and E_{k_2} orthogonal. If E_N and E_C orthogonal, just have to prove it

$$\begin{aligned}
&\left\langle\left\langle U_j^+,U_j^-\right\rangle\right\rangle_{E_k}=\Phi\left(U_j^+,U_j^-\right)\\
&=\beta\left(\nabla^{2m+k}u_j,\nabla^{2m+k}\overline{u}_j\right)+2\beta\left(-\lambda_j^-\nabla^{-2m-k}\overline{u}_j,\nabla^{2m}u_j\right)\\
&\quad+2\beta\left(-\lambda_j^+\nabla^{-2m-k}u_j,\nabla^{2m}\overline{u}_j\right)+4\left(-\lambda_j^+\nabla^{-2m-k}u_j,-\lambda_j^-\nabla^{-2m-k}u_j\right)\\
&\quad-4M\left(\left\|\nabla^m u\right\|_p^p\right)\left(\nabla^k\overline{u}_j,\nabla^k u_j\right)+\left(2\beta^2-\beta\right)\left(\nabla^{2m+k}\overline{u}_j,\nabla^{2m+k}u_j\right)\\
&=\beta\left\|\nabla^{2m+k}u_j\right\|^2-2\beta\left(\lambda_j^-+\lambda_j^+\right)\left\|\nabla^{-k}u_j\right\|^2+4\lambda_j^-\lambda_j^+\left\|\nabla^{-2m-k}u_j\right\|^2\\
&\quad-4M(s)\left\|\nabla^k u_j\right\|^2+\left(2\beta^2-\beta\right)\left\|\nabla^{2m+k}u_j\right\|^2\\
&=2\beta^2\mu_j-2\beta\left(\lambda_j^-+\lambda_j^+\right)+4\lambda_j^-\lambda_j^+\frac{1}{\mu_j}-4M(s)=0.
\end{aligned}\tag{4.29}$$

According to (4.13), and

$$\lambda_j^-+\lambda_j^+=\beta\mu_j.\tag{4.30}$$

$$\lambda_j^-\lambda_j^+=M\mu_j.\tag{4.31}$$

Step 3: Further estimating the Lipschitz constant l_F of *F*, where $F(U)=\left(0,f(x)-g(u_t)\right)^{\mathrm{T}}$, $g:H_0^{2m}(\Omega)\to L^2(\Omega)$. If $\forall U(u,v)^{\mathrm{T}}\in E_k$, $U=(u,v)^{\mathrm{T}}$, $V=(\tilde{u},\tilde{v})^{\mathrm{T}}=(y,z)^{\mathrm{T}}\in E_k$, then

$$\begin{aligned}
\left\|F(U)-F(V)\right\|_{E_k}&\le\left\|D^k\left(g(u_t)-g(v_t)\right)\right\|\\
&\le C_0\left\|g^{\infty}(u_t)\right\|_{\infty}\left\|D^k(u_t-v_t)\right\|\\
&\le C_0\left\|g^{(k)}(\xi)\right\|_{\infty}\left\|u_t-v_t\right\|_k\\
&\le l_F\left\|U-V\right\|_{E_k}.
\end{aligned}\tag{4.32}$$

Step 4: Now verify that the spectral interval condition $\Lambda_2-\Lambda_1>4l_F$ holds. Then

$$\Lambda_2-\Lambda_1=\lambda_{N+1}^--\lambda_N^-=\frac{\beta}{2}\left(\mu_{N+1}-\mu_N\right)+\frac{1}{2}\left(\sqrt{R(N)}-\sqrt{R(N+1)}\right).\tag{4.33}$$

where $R(N)=\beta^2\mu_N^2-4M\mu_N^2$.

There exists $N_1\ge 0$, such that for $\forall N\ge N_1$,

$R_1(N)=1-\sqrt{\dfrac{\beta^2}{\beta^2\mu_j^2-4M(s)}-\dfrac{4M}{\beta^2\mu_j^2-4M(s)}}$. We can get

$$\begin{aligned}
&\sqrt{R(N)}-\sqrt{R(N+1)}+\sqrt{\beta^2\mu_j^2-4M(s)}\left(\mu_{N+1}-\mu_N\right)\\
&=\sqrt{\beta^2\mu_j^2-4M(s)}\left(\mu_{N+1}R_1(N+1)-\mu_N R_1(N)\right),
\end{aligned}\tag{4.34}$$

According to assumption (H_3), we can easily see that

$$\lim_{N\to+\infty}\left(\sqrt{R(N)}-\sqrt{R(N+1)}+\sqrt{\beta^2\mu_j^2-4M(s)}\left(\mu_{N+1}-\mu_N\right)\right)=0,\tag{4.35}$$

Then according to (4.16) and (4.32)-(4.35), we have

$$\Lambda_2-\Lambda_1\ge\frac{1}{2}\left(\left(\mu_{N+1}-\mu_N\right)\left(\beta-\sqrt{\beta^2\mu_j^2-4M(s)}\right)-1\right)\ge 4l_F.\tag{4.36}$$

So the spectral interval condition holds.

Theorem 4.2. Under the assumption of Theorem 4.1, the initial boundary value problem (1.1)-(1.3) has the family of inertial manifolds h_k on the space E_k, and the form is

$$h_k = graph(m) := \left\{ \zeta_k + m(\zeta_k) : \zeta_k \in E_{k_1} \right\} \tag{4.37}$$

where E_{k_1}, E_{k_2} is defined in Formulas (4.19)-(4.20), $m : E_{k_1} \to E_{k_2}$ is a Lipschitz continuous function.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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