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Multiple Solutions for a Class of Variable-Order Fractional Laplacian Equations with Concave-Convex Nonlinearity

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Abstract

This paper is concerned with the following variable-order fractional Laplacian equations $\begin{cases} (-\Delta)^{s(\cdot)}u + \lambda V(x)u = f(x,u) + \mu|u|^{q(x)-2}u, & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$ where $N \geq 1$ and $N > 2s(x,y)$ for $(x,y) \in \Omega \times \Omega$, Ω is a bounded domain in $\mathbb{R}^N$, $s(\cdot) \in C(\mathbb{R}^N \times \mathbb{R}^N, (0,1))$, $(-\Delta)^{s(\cdot)}$ is the variable-order fractional Laplacian operator, $\lambda, \mu > 0$ are two parameters, $V : \Omega \to [0,\infty)$ is a continuous function, $f \in C(\Omega \times \mathbb{R})$ and $q \in C(\Omega)$. Under some suitable conditions on *f*, we obtain two solutions for this problem by employing the mountain pass theorem and Ekeland's variational principle. Our result generalizes the related ones in the literature.

Keywords

Concave-Convex Nonlinearity, Variable-Order Fractional Laplacian, Variational Methods, Fractional Elliptic Equation

1. Introduction

In this paper, we consider the following variable-order fractional Laplacian equations

$$\begin{cases} (-\Delta)^{s(\cdot)}u + \lambda V(x)u = f(x,u) + \mu|u|^{q(x)-2}u, & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1.1}$$

where $N \geq 1$ and $N > 2s(x,y)$ for $(x,y) \in \Omega \times \Omega$, Ω is a bounded domain in $\mathbb{R}^N$, $s(\cdot) \in C(\mathbb{R}^N \times \mathbb{R}^N, (0,1))$, $(-\Delta)^{s(\cdot)}$ is the variable-order fractional

Laplacian operator, $\lambda,\mu>0$ are two parameters, $V:\Omega\to[0,\infty)$ is a continuous function, $f\in C(\Omega\times\mathbb{R})$ and $q\in C(\Omega)$. The Laplacian operator $(-\Delta)^{s(\cdot)}$ is defined by

$$(-\Delta)^{s(\cdot)}\xi(x)=2P.V.\int_{\mathbb{R}^N}\frac{\xi(x)-\xi(y)}{|x-y|^{N+2s(x,y)}}\mathrm{d}y,$$

for each $x\in\mathbb{R}^N$ and any $\xi\in C_0^\infty(\Omega)$, where $P.V.$ denotes the Cauchy principal value.

When $s(\cdot)\equiv\text{constant}$ and $q(x)\equiv\text{constant}$, $(-\Delta)^{s(\cdot)}$ becomes to the usual fractional Laplacian operator and problem (1.1) reduces to fractional Schrödinger equation. This kind of equation is introduced by Laskin [1] [2] as a result of expanding the Feynman path integral from the Brownian-like to the Lévy-like quantum mechanical paths. The fractional Schrödinger equation is studied by many researchers. For example, Zhang *et al.* [3] investigated fractional Schrödinger equation with critical exponents by using variational methods. They used Pohožaev identity and Jeanjean's monotonicity trick to obtain a radially symmetric weak solution. Another example is [4], the multiplicity and concentration of solutions for fractional Schrödinger equation with concave-convex nonlinearity are studied by Gao *et al.* For more results about fractional Schrödinger equation, please see [5] [6] [7] and the references therein. Particularly, when $s(\cdot)\equiv 1$ and $q(x)\equiv\text{constant}$, problem (1.1) becomes to the Schrödinger equation with concave-convex nonlinearity.

If $s(\cdot)\equiv 1$, (1.1) becomes the following second order elliptic equation with variable growth nonlinearity

$$\begin{cases}-\Delta u+\lambda V(x)u=f(x,u)+\mu|u|^{q(x)-2}u & \text{in }\Omega\\ u=0 & \text{in }\mathbb{R}^N\setminus\Omega.\end{cases}\tag{1.2}$$

Some interesting phenomena can be described by this type of model. For example, Ružička [8] showed the application in the modeling of electrorheological fluids involving variable exponent Laplacian operator. It happens that there is a similar case, Ayazoglu and Ekincioglu [9] obtained a positive solution for electrorheological fluids equations with variable exponent via mountain pass technique. For other applications of these similar models, we refer the readers to [10] [11] [12] [13] [14].

In this paper, we consider the variable-order fractional Laplacian operator case with variable growth. The fractional variable order derivatives are introduced by Lorenzo and Hartley [15] to better describe some diffusion processes reacting to temperature changes. In fact, the literature involving the variable-order fractional Laplacian operator cases is few. Specially, Xiang *et al.* [16] obtained multiple solutions for the following elliptic equations with variable-order fractional Laplacian operator involving variable exponents by using variational methods,

$$\begin{cases}(-\Delta)^{s(\cdot)}u+\lambda V(x)u=\alpha|u|^{p(x)-2}u+\mu|u|^{q(x)-2}u & \text{in }\Omega\\ u=0 & \text{in }\mathbb{R}^N\setminus\Omega,\end{cases}\tag{1.3}$$

where $p, q \in C(\Omega)$. Another example is [17], Biswas and Tiwari studied a type of Choquard problem with variable-order nonlocal term and variable exponents and obtained some results for the above mention problem by employing Hardy-Sobolev-Littlewood-type inequality. Very recently, Xiang *et al.* [18] investigated variable-order fractional Kirchhoff equations with nonstandard growth and obtained multiple solutions for these equations by applying the Nehari manifold approach. For other results on variable-order fractional Kirchhoff equations, please see [19] [20] and the references therein.

Inspired mainly by the aforementioned results, we proved the existence of solutions for (1.1) with concave-convex nonlinearity. Compared to [16], we deal with a general case, *i.e.*, the general nonlinearity *f* with variable growth conditions. To show our result, we make the following assumptions first:

(H1) $0 < s^- := \min_{(x,y)\in\mathbb{R}^N\times\mathbb{R}^N} s(x,y) \le s^+ := \max_{(x,y)\in\mathbb{R}^N\times\mathbb{R}^N} s(x,y) < 1$.

(H2) $s(x,y) = s(y,x)$, $\forall (x,y) \in \mathbb{R}^N \times \mathbb{R}^N$.

(H3) $J = int\left(V^{-1}(0)\right) \subset \Omega$ *is a nonempty bounded domain and* $\tilde{J} = V^{-1}(0)$.

(H4) *there exists a nonempty open domain* $\Omega_0 \subset J$ *such that* $V(x) \equiv 0$ *for all* $x \in \bar{\Omega}_0$.

For the nonlinearity term *f* and the variable exponents *q*, we assume that $q \in C(\bar{\Omega})$ and the following assumptions hold:

(H5) $q : \bar{\Omega} \to (1,2)$.

(H6) $f \in C(\Omega \times \mathbb{R})$ *and there exist a positive constant c and a continuous function* $p \in C(\bar{\Omega})$ *with* $2 < p(x) < \dfrac{2N}{N-2s(x,x)}$ *such that*

$\left|f(x,u)\right| \le c\left(1+|u|^{p(x)-1}\right)$.

(H7) $f(x,u) = o(|u|)$ *as* $|u| \to 0$ *uniformly for* $x \in \Omega$.

(H8) *there exists* $p^- > 2$ *such that* $0 < p^- F(x,u) \le f(x,u)u$ *for every* $x \in \Omega$, *where* $F(x,t) = \int_0^t f(x,t)\,\mathrm{d}t$ *and*

$2 < p^- := \operatorname{essinf}_{x\in\bar{\Omega}} p(x) \le p^+ := \operatorname{esssup}_{x\in\bar{\Omega}} p(x) < \dfrac{2N}{N-2s(x,x)}$.

Let η_p and η_q be the Sobolev embedding constants which will be defined in the next section, set

$$A = \frac{\max\left\{\eta_p^{p^+}, \eta_p^{p^-}\right\}}{p^-}, \quad B = \frac{\max\left\{\eta_q^{q^+}, \eta_q^{q^-}\right\}}{q^-}.$$

We assume that μ is a positive parameter satisfying the following assumption:

(H9) *there holds*

$$\mu < \left(\frac{2-q^+}{2Ac_\varepsilon\left(p^+-q^+\right)}\right)^{\frac{2-q^+}{p^+-2}} \frac{\left(1-\eta_2^2\varepsilon\right)p^+ + \eta_2^2\varepsilon q^+ - 2}{2B\left(p^+-q^+\right)}, \quad c_\varepsilon \le \frac{2-q^+}{2A\left(p^+-q^+\right)}.$$

where c_ε is a positive constant depending on ε.

Based on the hypothesis (H2), we can give the following definition of weak solutions for problem (1.1).

Definition 1.1. *We say that* $u \in E_\lambda$ *is a* (*weak*) *solution of problem* (1.1), *if for any* $v \in E_\lambda$, *there holds*

$$\iint_{\mathbb{R}^{2N}} \frac{\left(u(x)-u(y)\right)\left(v(x)-v(y)\right)}{|x-y|^{N+2s(x,y)}} \mathrm{d}x\mathrm{d}y + \lambda \int_\Omega V(x) uv \mathrm{d}x$$
$$= \int_\Omega \left(f(x,u)v + \mu |u|^{q(x)-2} uv \right) \mathrm{d}x,$$

where E_λ is a variable exponent Banach space which will be defined in the next section.

Theorem 1.2. *Suppose* (*H1*)-(*H9*) *hold. Let* $N > 2s^+$, *then problem* (1.1) *admits at least two distinct solutions for all* $\lambda > 0$.

Remark 1.3. In fact, the multiple solutions for variable-order fractional Laplacian equation involving general nonlinearity with critical growth of variable exponent have not been investigated. It is very interesting and full of challenge for us to deal with this problem.

This paper is organized as follows. In Section 1, we give some reviews on the topic of variable-order fractional Lapla-cian equations and give the main result of this paper. In Section 2, some preliminary results are presented. In Section 3, we give the proof of Theorem 1.2.

2. Preliminaries

Some preliminary results of variable exponent Lebesgue spaces will be given in this section which come from [21].

A variable exponent is a measurable function $p : \overline{\Omega} \to [1, \infty)$. The exponent p is said to be bounded if p^+ is finite. Let

$$L^{p(x)}(\Omega) = \left\{ w : \Omega \to \mathbb{R} \text{ is a measurable function}; \phi_{p(x)}(w) = \int_\Omega |w(x)|^{p(x)} \mathrm{d}x < \infty \right\},$$

then $L^{p(x)}(\Omega)$ is a variable exponent Banach space with the following Luxemburg norm

$$\|w\|_{L^{p(\cdot)}(\Omega)} = \inf\left\{ \theta > 0 : \phi_{p(x)}\left(\theta^{-1} w\right) \le 1 \right\}.$$

If p is bounded, there holds

$$\min\left\{ \|w\|^{p^-}_{L^{p(\cdot)}(\Omega)}, \|w\|^{p^+}_{L^{p(\cdot)}(\Omega)} \right\} \le \phi_{p(\cdot)}(w) \le \max\left\{ \|w\|^{p^-}_{L^{p(\cdot)}(\Omega)}, \|w\|^{p^+}_{L^{p(\cdot)}(\Omega)} \right\}. \quad (2.1)$$

From (2.1), we know that the norm convergence and the convergence is equivalent with respect to $\phi_{p(x)}$ when p is bounded. Moreover, the dual space of $L^{p(\cdot)}(\Omega)$ can be written as $L^{p'(\cdot)}(\Omega)$ for bounded exponent, where $1/p'(x) + 1/p(x) = 1$. It is obvious that $L^{p(\cdot)}(\Omega)$ is separable and reflexive since $1 < p^- \le p^+ < \infty$.

The following inequality is Hölder's inequality in variable exponent Lebesgue space

$$\int_\Omega |uv| \mathrm{d}x \le \left(\frac{1}{p^-} + \frac{1}{(p')^-} \right) \|u\|_{L^{p(\cdot)}(\Omega)} \|v\|_{L^{p'(\cdot)}(\Omega)} \le 2 \|u\|_{L^{p(\cdot)}(\Omega)} \|v\|_{L^{p'(\cdot)}(\Omega)},$$

for all $u \in L^{p(\cdot)}(\Omega)$, $v \in L^{p'(\cdot)}(\Omega)$ and $p(x) \in (1,\infty)$.

Next, the variational setting for problem (1.1) will be given. Let Ω be a non-empty open subset of $\mathbb{R}^N$ and let $s(\cdot)$ be a measurable function, there are two constants $s_0, s_1 \in (0,1)$ with $s_0 < s_1$ such that

$$s_0 \le s(x,y) \le s_1,\ \forall (x,y) \in \mathbb{R}^N \times \mathbb{R}^N.$$

Set

$$H^{s(\cdot)}(\Omega) = \left\{ w \in L^2(\Omega) : [w]_{s(\cdot),\Omega} < \infty \right\},$$

where

$$[w]_{s(\cdot),\Omega} = \left(\int_\Omega \int_\Omega \frac{|w(x) - w(y)|^2}{|x-y|^{N+2s(x,y)}} \mathrm{d}x\mathrm{d}y \right)^{1/2}.$$

Let $H^{s(\cdot)}(\Omega)$ be equipped with the norm

$$\|w\|_\Omega = \left(\|w\|^2_{L^2(\Omega)} + [w]^2_{s(\cdot),\Omega} \right)^{1/2}.$$

Especially, $H^{s(\cdot)}(\Omega)$ becomes to the usual fractional Sobolev space if $s(\cdot)$ is a constant function.

The space $H_0^{s(\cdot)}(\Omega)$ is defined as

$$H_0^{s(\cdot)}(\Omega) = \Big\{ w : \mathbb{R}^N \to \mathbb{R} \text{ is a measurable function, } w|_{\mathbb{R}^N \setminus \Omega} = 0,$$
$$w \in L^2(\Omega) \text{ and } [w]_{s(\cdot)} < \infty \Big\},$$

where

$$[w]_{s(\cdot)} := \left(\iint_{\mathbb{R}^{2N}} \frac{|w(x) - w(y)|^2}{|x-y|^{N+2s(x,y)}} \mathrm{d}x\mathrm{d}y \right)^{1/2}.$$

The norm on $H_0^{s(\cdot)}(\Omega)$ is given as

$$\|w\| = \left(\|w\|^2_{L^2(\Omega)} + [w]^2_{s(\cdot)} \right)^{1/2}.$$

The following lemma implies that $[\cdot]_{s(\cdot)}$ is an equivalent norm of $H_0^{s(\cdot)}(\Omega)$.

Lemma 2.1. [16]. *The embedding* $H_0^{s_1}(\Omega) \hookrightarrow H_0^{s(\cdot)}(\Omega) \hookrightarrow H_0^{s_0}(\Omega)$ *are continuous. Moreover, if* $N > 2s_0$, *for any fixed constant exponent* $p \in \left[1, \frac{2N}{N-2s_0}\right]$, $H_0^{s(\cdot)}(\Omega)$ *can be continuously embedded into* $L^p(\Omega)$.

Using the same argument as the proof of ([22], Lemma 7), one can easily prove that $\left(H_0^{s(\cdot)}(\Omega), [\cdot]_{s(\cdot)}\right)$ is a Hilbert space. From ([22], Theorem 2.1), we know that the embedding $H_0^{s(\cdot)}(\Omega) \hookrightarrow L^{p(x)}(\Omega)$ is continuous and compact. Moreover, there exists $\eta_p = \eta(N, s^+, s^-, p^+) > 0$ such that

$$\|w\|_{L^{p(x)}(\Omega)} \le \eta_p \|u\|,\ \forall\, w \in H_0^{s(\cdot)}(\Omega), \tag{2.2}$$

and

$$\|w\|_{L^{p(x)}(\Omega)} \le \eta_p [w]_{s(\cdot)},\ \forall\, w \in H_0^{s(\cdot)}(\Omega). \tag{2.3}$$

Let

$$E=\left\{w\in H_0^{s(\cdot)}(\Omega):[w]_{s(\cdot)}^2+\int_\Omega V(x)|w|^2\,\mathrm{d}x<\infty\right\},$$

the inner product on E is defined as

$$(v,w)_E=\iint_{\mathbb{R}^{2N}}\frac{(v(x)-v(y))(w(x)-w(y))}{|x-y|^{N+2s(x,y)}}\mathrm{d}x\mathrm{d}y$$
$$+\int_\Omega V(x)vw\mathrm{d}x\quad\forall\, v,w\in E,$$

and the corresponding norm is $\|w\|_E=(w,w)_E^{1/2}$. The following inner product

$$(v,w)_\lambda=\iint_{\mathbb{R}^{2N}}\frac{(v(x)-v(y))(w(x)-w(y))}{|x-y|^{N+2s(x,y)}}\mathrm{d}x\mathrm{d}y+\lambda\int_\Omega V(x)vw\mathrm{d}x,$$

and the corresponding norm $\|w\|_\lambda=(w,w)_\lambda^{1/2}$ are also used in this paper. Obviously, for all $\lambda\ge 1$, one has $\|w\|_E\le\|u\|_\lambda$. Set $E_\lambda=(E,\|\cdot\|_\lambda)$. Moreover, for $p(x)\in(1,2N/(N-2s(x,x)))$, from (2.3), one has

$$\int_\Omega|w(x)|^{p(x)}\,\mathrm{d}x\le\max\left\{\|w\|_{L^{p(x)}(\Omega)}^{p^+},\|w\|_{L^{p(x)}(\Omega)}^{p^-}\right\}\le\max\left\{\eta_p^{p^+}[w]_{s(\cdot)}^{p^+},\eta_p^{p^-}[w]_{s(\cdot)}^{p^-}\right\}.$$

Evidently, problem (1.1) has a variational formulation and the corresponding functional is defined in E_λ by

$$\begin{aligned}\Phi_\lambda(u)&=\frac{1}{2}\iint_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s(x,y)}}\mathrm{d}x\mathrm{d}y+\frac{\lambda}{2}\int_\Omega V(x)|u|^2\,\mathrm{d}x\\&\quad-\int_\Omega F(x,u)\mathrm{d}x-\frac{\mu}{q(x)}\int_\Omega|u|^{q(x)}\,\mathrm{d}x\\&=\frac{1}{2}\|u\|_\lambda^2-\int_\Omega F(x,u)\mathrm{d}x-\frac{\mu}{q(x)}\int_\Omega|u|^{q(x)}\,\mathrm{d}x,\ \forall\, u\in E_\lambda.\end{aligned}\tag{2.4}$$

Actually, $\Phi_\lambda\in C^1(E_\lambda,\mathbb{R})$ is well-defined and

$$\langle\Phi'(u),v\rangle=(u,v)_\lambda-\mu\int_\Omega|u|^{q(x)-2}uv\mathrm{d}x-\int_\Omega f(x,u)v\mathrm{d}x,\ \forall\, u,v\in E_\lambda.\tag{2.5}$$

Hence, u is a solution to problem (1.1) if $u\in E_\lambda$ is a critical point of Φ_λ.

3. Proof of Theorem 1.2

It is well known that a C^1 functional Φ_λ satisfying Palais-Smale ((*PS*) for short) condition at level c if for any sequence $\{u_n\}\subset E_\lambda$ such that $\Phi_\lambda(u_n)\to c$ and $\Phi'_\lambda(u_n)\to 0$, there exists a convergent subsequence in E_λ, which is called a $(PS)_c$ sequence.

First, we shall verify the mountain pass geometry of Φ_λ.

Lemma 3.1. *Assume that* (*H1*) *and* (*H3*)-(*H9*) *hold. Then for all* $\lambda>0$, *the functional* Φ_λ *satisfies*

1) There exists $\beta,\rho>0$ such that $\Phi_\lambda(u)\ge\beta$ if $\|u\|_\lambda=\rho$;

2) There exists $u\in E_\lambda$ with $\|u\|_\lambda>\rho$ such that $\Phi_\lambda(u)<0$.

Proof. For any $\varepsilon>0$, it follows from the condition (H5) and (H6) that there exists $c_\varepsilon>0$ such that

$$\left|F(x,t)\right| \le \frac{\varepsilon}{2}|t|^2 + \frac{c_\varepsilon}{p^-}|t|^{p(x)}, \quad \forall\, t \in \mathbb{R}. \tag{3.1}$$

Thus, from (3.1) and the fractional Sobolev inequality, one has

$$\begin{aligned} \int_\Omega F(x,u)\,\mathrm{d}x &\le \frac{\varepsilon}{2}\int_\Omega |u|^2\,\mathrm{d}x + \frac{c_\varepsilon}{p^-}\int_\Omega |u|^{p(x)}\,\mathrm{d}x \\ &\le \frac{C_2^2\varepsilon}{2}\|u\|_\lambda^2 + \frac{c_\varepsilon \max\left\{\eta_p^{p^+},\eta_p^{p^-}\right\}}{p^-}\|u\|_\lambda^{p^+}, \ \forall\, u \in E_\lambda \text{ with } \|u\|_\lambda \ge 1. \end{aligned} \tag{3.2}$$

For $u \in E_\lambda$ and $\|u\|_\lambda \ge 1$, it follows from (2.4) and (3.2) that

$$\begin{aligned} \Phi_\lambda(u) &= \frac{1}{2}\|u\|_\lambda^2 - \int_\Omega F(x,u)\,\mathrm{d}x - \frac{\mu}{q(x)}\int_\Omega |u|^{q(x)}\,\mathrm{d}x \\ &\ge \frac{1}{2}\|u\|_\lambda^2 - \frac{C_2^2\varepsilon}{2}\|u\|_\lambda^2 - \frac{c_\varepsilon \max\left\{\eta_p^{p^+},\eta_p^{p^-}\right\}}{p^-}\|u\|_\lambda^{p^+} - \frac{\mu \max\left\{\eta_q^{q^+},\eta_q^{q^-}\right\}}{q^-}\|u\|_\lambda^{q^+} \\ &= \|u\|_\lambda^{q^+}\left[\frac{1}{2}\left(1-C_2^2\varepsilon\right)\|u\|_\lambda^{2-q^+} - \frac{c_\varepsilon \max\left\{\eta_p^{p^+},\eta_p^{p^-}\right\}}{p^-}\|u\|_\lambda^{p^+-q^+} - \frac{\mu \max\left\{\eta_q^{q^+},\eta_q^{q^-}\right\}}{q^-}\right]. \end{aligned}$$

Let

$$g(t) = h(t)t^{q^+}, \quad \forall\, t \ge 0,$$

where

$$h(t) = \frac{1-\eta_2^2\varepsilon}{2}t^{2-q^+} - Ac_\varepsilon t^{p^+-q^+} - B\mu.$$

By (H9) and an easy computation, for $t = t^* = \left[\frac{2-q^+}{2Ac_\varepsilon\left(p^+-q^+\right)}\right]^{\frac{1}{p^+-2}}$, one has

$$h\left(t^*\right) = \max_{t\ge 0} h(t) > 0.$$

Since $c_\varepsilon \le \frac{2-q^+}{2A\left(p^+-q^+\right)}$, we have

$$t^* = \left[\frac{2-q^+}{2Ac_\varepsilon\left(p^+-q^+\right)}\right]^{\frac{1}{p^+-2}} \ge 1.$$

Let $\rho = t^*$ and $\beta = g\left(t^*\right)$, then (1) of Lemma 3.1 is satisfied by.

By (3.1) and (H7), there exists $c_1 > 0$ such that

$$F(x,u) \ge c_1\left(|u|^{p^-} - |u|^2\right), \quad \forall\,(x,u) \in \Omega\times\mathbb{R}.$$

Then we choose a function $v_0 \in E_\lambda$ such that

$$\|v_0\|_\lambda = 1 \quad \text{and} \quad \int_\Omega \left|v_0(x)\right|^{p(x)}\mathrm{d}x > 0.$$

By (2.4), for all $t_0 \ge 1$, we obtain

$$\begin{aligned} \Phi_\lambda(t_0v_0) &= \frac{t_0^2}{2}\|v_0\|_\lambda^2 - \int_\Omega F(x,t_0v_0)\,\mathrm{d}x - \frac{\mu}{q(x)}\int_\Omega |t_0v_0|^{q(x)}\,\mathrm{d}x \\ &\le \frac{t_0^2}{2}\|v_0\|_\lambda^2 - c_1t_0^{p^-}\int_\Omega |v_0|^{p(x)}\,\mathrm{d}x + c_1t_0^2\int_\Omega |v_0|^2\,\mathrm{d}x - \frac{\mu t_0^{q^-}}{q^+}\int_\Omega |v_0|^{q(x)}\,\mathrm{d}x. \end{aligned}$$

Since $p^{-}>2$ and $q^{-}>1$, there exists $t_0\geq 1$ large enough such that $\Phi_\lambda(t_0v_0)<0$. The proof is completed.

Let

$$c_\lambda=\inf_{\rho\in\Gamma}\max_{t\in[0,1]}\Phi_\lambda(\rho(t)),$$

$$c(\Omega_0)=\inf_{\rho\in\tilde{\Gamma}}\max_{t\in[0,1]}\Phi_\lambda\big|_{H_0^{s(\cdot)}(\Omega_0)}(\rho(t)),$$

$$\Gamma=\{\rho\in C([0,1],E_\lambda);\rho(0)=0,\rho(1)=e\}$$

and

$$\tilde{\Gamma}=\{\rho\in C([0,1],H_0^{s(\cdot)}(\Omega_0));\rho(0)=0,\rho(1)=e\}.$$

Notice that

$$\Phi_\lambda\big|_{H_0^{s(\cdot)}(\Omega_0)}(u)=\frac{1}{2}[u]_{s(\cdot)}^2-\int_{\Omega_0}\left(F(x,u)+\frac{\mu}{q(x)}|u|^{q(x)}\right)\mathrm{d}x,\quad\forall\, u\in H_0^{s(\cdot)}(\Omega_0).$$

Obviously, $c(\Omega_0)$ is independent of λ. It is clear that the mountain pass geometry of $\Phi_\lambda\big|_{H_0^{s(\cdot)}(\Omega_0)}$ is proved by Lemma 3.1. Since $H_0^{s(\cdot)}(\Omega_0)\subset E_\lambda$ for all $\lambda>0$, we have $0\leq c_\lambda\leq c(\Omega_0)$ for all $\lambda>0$. Evidently, for all $t\in[0,1]$, $te\in\tilde{\Gamma}$. Hence, it follows from $p^{-}>2$ that there exists $C_0>0$ such that

$$c(\Omega_0)\leq\max_{t\in[0,1]}\Phi_\lambda(te)\leq C_0<\infty.$$

Then, for all $\lambda>0$, we have

$$0<\beta\leq c_\lambda\leq c(\Omega_0)<C_0. \tag{3.3}$$

By Lemma 3.1 and the mountain pass theorem, we derive that there exists $\{u_k\}_k\subset E_\lambda$ such that

$$\Phi_\lambda(u_k)\to c_\lambda>0 \text{ and } \Phi'_\lambda(u_k)\to 0,\quad \text{as } k\to\infty. \tag{3.4}$$

Lemma 3.2. *Assume that* (*H1*) *and* (*H3*)-(*H9*) *hold. Then the* $(PS)_c$ *sequence* $\{u_k\}_k\subset E_\lambda$ *given in* (3.4) *is bounded for all* $\lambda>0$.

Proof. By (2.4), (2.5), (3.4) (H7) and the Hölder inequality, one obtains

$$\begin{aligned}
c_\lambda+o(1)&\geq\Phi_\lambda(u_k)-\frac{1}{p^{-}}\langle\Phi'_\lambda(u_k),u_k\rangle\\
&=\left(\frac{1}{2}-\frac{1}{p^{-}}\right)\|u_k\|_\lambda^2+\int_\Omega\left[\frac{1}{p^{-}}f(x,u_k)u_k-F(x,u_k)\right]\mathrm{d}x\\
&\quad-\int_\Omega\mu\left(\frac{1}{q(x)}-\frac{1}{p^{-}}\right)|u_k|^{q(x)}\,\mathrm{d}x\\
&\geq\frac{p^{-}-2}{2p^{-}}\|u_k\|_\lambda^2-\left(\frac{1}{q^{-}}-\frac{1}{p^{-}}\right)\int_\Omega\mu|u_k|^{q(x)}\,\mathrm{d}x\\
&\geq\frac{p^{-}-2}{2p^{-}}\|u_k\|_\lambda^2-\mu\left(\frac{1}{q^{-}}-\frac{1}{p^{-}}\right)\max\left\{C_q^{q^{+}}\|u_k\|^{q^{+}},C_q^{q^{-}}\|u_k\|^{q^{-}}\right\}.
\end{aligned}\tag{3.5}$$

Arguing indirectly, we assume that $\{u_k\}_k$ is not bounded in $H_0^{s(\cdot)}(\Omega)$. Then there exists a subsequence still denoted by $\{u_k\}_k$ such that $\|u_k\|_\lambda\to\infty$ as

$k \to \infty$. Then, by (3.5), we obtain

$$\frac{c_\lambda + o(1)}{\|u_k\|_\lambda^2} \geq \frac{p^- - 2}{2p^-} - \max\left\{C_q^{q^+}\|u_k\|^{q^+-2}, C_q^{q^-}\|u_k\|^{q^--2}\right\}, \tag{3.6}$$

which contradicts to $p^- > 2$. Hence, the boundedness of $\{u_k\}_k$ in E_λ is obtained for all $\lambda > 0$.

Lemma 3.3. *Assume that* (*H1*) *and* (*H3*)-(*H9*) *hold. Then* Φ_λ *satisfies the* $(PS)_c$ *condition in* E_λ *for all* $c \in \mathbb{R}$ *and* $\lambda > 0$.

Proof. Let $\{u_k\}_k$ be a $(PS)_c$ sequence with $c < C_0$. From Lemma 3.2, we know that $\{u_k\}_k$ is bounded in E_λ and there exists $C > 0$ such that $\|u_k\|_\lambda \leq C$. Hence, there exists a subsequence of $\{u_k\}_k$ still denoted by $\{u_k\}_k$ and u_0 in E_λ such that

$$\begin{aligned} &u_k \rightharpoonup u_0 \text{ weakly in } E_\lambda, \\ &u_k \to u_0 \text{ a.e. in } \mathbb{R}, \\ &|u_k|^{p(x)-2} u_k \rightharpoonup |u_0|^{p(x)-2} u_0 \text{ weakly in } L^{p'(x)}(\Omega). \end{aligned} \tag{3.7}$$

Now we prove that $u_k \to u_0$ in E_λ. From ([22], Theorem 2.1), we have $u_k \to u_0$ in $L^{p(x)}(\Omega)$ and $L^{q(x)}(\Omega)$, respectively. Hence,

$$\lim_{k\to\infty} \int_\Omega |u_k - u_0|^{q(x)} \mathrm{d}x = 0. \tag{3.8}$$

By (H5) and (H6), we have

$$\begin{aligned} &\int_\Omega f(x, u_k - u_0)(u_k - u_0)\mathrm{d}x \\ &\leq \int_\Omega \left(\varepsilon|u_k - u_0| + c_\varepsilon |u_k - u_0|^{p(x)-1}\right)(u_k - u_0)\mathrm{d}x \\ &= \int_\Omega \left(\varepsilon|u_k - u_0|^2 + c_\varepsilon |u_k - u_0|^{p(x)}\right)\mathrm{d}x \end{aligned}$$

Thus,

$$\lim_{k\to\infty} \int_\Omega f(x, u_k - u_0)(u_k - u_0)\mathrm{d}x = 0. \tag{3.9}$$

It follows from (2.5) and (3.4) that

$$\begin{aligned} o(1) &= \langle \Phi_\lambda'(u_k) - \Phi_\lambda'(u_0), u_k - u_0 \rangle \\ &= (u_k - u_0, u_k - u_0)_\lambda - \int_\Omega f(x, u_k - u_0)(u_k - u_0)\mathrm{d}x \\ &\quad - \mu\int_\Omega \left(|u_k|^{q(x)-2} u_k - |u_0|^{q(x)-2} u_0\right)(u_k - u_0)\mathrm{d}x. \end{aligned} \tag{3.10}$$

By (3.8), (3.9) and (3.10), we have

$$\lim_{k\to\infty} \|u_k - u_0\|_\lambda = 0.$$

The proof is complete.

Proof of Theorem 1.2. By Lemmas 3.1 - 3.2 and the mountain pass theorem, for all $\lambda > 0$, there exists a $(PS)_{c_\lambda}$ sequence $\{u_k\}_k$ for Φ_λ on E_λ. From Lemma 3.2 and $0 < c_\lambda < c(\Omega_0) < C_0$, there exists a subsequence of $\{u_k\}_k$ still denoted by $\{u_k\}_k$ and $u_\lambda^0 \in E_\lambda$ such that $u_k \to u_\lambda^0$ in E_λ. Moreover, $\Phi_\lambda(u_n) = c_\lambda \geq \beta$ and u_λ^0 is a solution to problem (1.1).

Next, we verify that problem (1.1) has another solution. Let

$$\tilde{c}_\lambda = \inf\left\{\Phi_\lambda(w) : w \in \overline{B_\rho}\right\},$$

where $B_\rho = \left\{w \in E_\lambda : \|w\|_\lambda < \rho\right\}$ and $\rho > 0$ is given by Lemma 3.1. Then $\tilde{c}_\lambda < 0$ for all $\lambda > 0$. Let $w_0 \in H_0^{s(\cdot)}(\Omega)$ such that $\int_\Omega |w_0|^{q(x)} > 0$. By (2.4), (H7) and (H8), one has that

$$\begin{aligned}\Phi_\lambda(tw_0) &= \frac{t^2}{2}\|w_0\|_\lambda^2 - \int_\Omega F(x, tw_0)\,\mathrm{d}x - \frac{\mu}{q(x)}\int_\Omega |tw_0|^{q(x)}\,\mathrm{d}x \\ &\le \frac{t^2}{2}\|w_0\|_\lambda^2 - c_1 t^{p^-}\int_\Omega |w_0|^{p(x)}\,\mathrm{d}x + c_1 t^2 \int_\Omega |w_0|^2\,\mathrm{d}x - \frac{\mu t_0^{q^-}}{q^+}\int_\Omega |w_0|^{q(x)}\,\mathrm{d}x \\ &< 0\end{aligned} \quad (3.11)$$

for $t \in (0,1)$ small enough.

Hence, there is $w_0 \in E_\lambda$ such that $\Phi_\lambda(tw_0) < 0$ for all $t > 0$ small enough.

It follows from Lemma 3.1 and the Ekeland's variational principle that there exists a sequence $\{u_k\}_k$ such that

$$\tilde{c}_\lambda \le \Phi_\lambda(u_k) \le \tilde{c}_\lambda + \frac{1}{k}, \quad (3.12)$$

and

$$\Phi_\lambda(v) \ge \Phi_\lambda(u_k) - \frac{\|u_k - v\|_\lambda}{k}, \ \forall\, v \in \overline{B_\rho}. \quad (3.13)$$

Now we should show that $\|u_k\|_\lambda < \rho$ for k large enough. Indirectly, we suppose that $\|u_k\|_\lambda = \rho$ for infinitely many k. Without loss of generality, one may assume that $\|u_k\|_\lambda = \rho$ for any $k \in \mathbb{N}$. By Lemma 3.1, one has

$$\Phi_\lambda(u_k) \ge \beta > 0. \quad (3.14)$$

From (3.12) and (3.14), we have $\tilde{c}_\lambda \ge \beta > 0$, which contradicts to $\tilde{c}_\lambda < 0$. Now we prove that $\Phi'_\lambda(u_k) \to 0$ in E_λ^*. Let

$$w_k = u_k + tv, \quad \forall\, v \in B_1,$$

where $B_1 := \left\{v \in E_\lambda : \|v\|_\lambda = 1\right\}$, $t > 0$ is small enough such that $\rho^2 - \|u_k\|_\lambda^2 - 2t\rho - t^2 \ge 0$ for fixed k large. Thus

$$\|w_k\|_\lambda^2 = \|u_k\|_\lambda^2 + 2t\rho\langle u_k, v\rangle_\lambda + t^2 \le \|u_k\|_\lambda^2 + 2\rho t + t^2 \le \rho^2,$$

which implies that $w_k \in \overline{B_\rho}$. Hence, from (3.13), one has

$$\Phi_\lambda(w_k) \ge \Phi_\lambda(u_k) - \frac{t}{n}\|u_k - w_k\|,$$

that is,

$$\frac{\Phi_\lambda(u_k + tv) - \Phi_\lambda(u_k)}{t} \ge -\frac{1}{k}.$$

Letting $t \to 0^+$, one has $\langle \Phi'_\lambda(u_k), v\rangle \ge -\frac{1}{k}$ for any fixed k large. Similarly, by repeating the process above, choosing $t < 0$ and $|t|$ small enough, one gets

$$\left\langle \Phi'_\lambda\left(u_k\right), v\right\rangle \le \frac{1}{k}, \quad \text{for any fixed } k \text{ large.}$$

Hence, we obtain

$$\lim_{n\to\infty} \sup_{v\in B_1} \left|\left\langle \Phi_\lambda\left(u_k\right), v\right\rangle\right| = 0,$$

which implies that $\Phi_\lambda\left(u_k\right) \to 0$ in E_λ^* as $k \to \infty$. Thus, $\{u_n\}_n$ is a $(PS)_{\tilde{c}_\lambda}$ sequence for the functional Φ_λ. Using the same argument as Lemma 3.3, there exists $u_\lambda^{(2)}$ such that $u_n \to u_\lambda^{(2)}$ in E_λ. Hence, one obtains a nontrivial solution $u_\lambda^{(2)}$ of (1.1) satisfying

$$\Phi_\lambda\left(u_\lambda^{(2)}\right) \le \zeta < 0 \quad \text{and} \quad \left\|u_\lambda^{(2)}\right\|_\lambda < \rho.$$

Hence, we conclude that

$$\Phi_\lambda\left(u_\lambda^{(2)}\right) = \tilde{c}_\lambda \le \zeta < 0 < \beta < c_\lambda = \Phi_\lambda\left(u_\lambda^{(1)}\right), \quad \forall \lambda > 0,$$

which completes the proof.

4. Conclusion

This paper considers the existence of solutions for a kind of variable-order fractional Laplacian equations. By employing the mountain pass theorem and Ekeland's variational principle, two solutions are obtained under some suitable conditions on *f*. Specially speaking, we first prove the mountain pass geometry of the function for this kind of variable-order fractional Laplacian equations. Secondly, we verify the boundedness of $(PS)_c$ sequences. Finally, we prove that Φ_λ satisfies the $(PS)_c$ condition in E_λ for all $c \in \mathbb{R}$ and $\lambda > 0$. The result obtained in this paper generalizes the related ones in the literature, which can be used in similar kinds of variable-order fractional Laplacian equations. We hope that this result can be widely used in variable-order fractional Laplacian equations and other systems.

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Conflicts of Interest

All authors declare no conflicts of interest in this paper.

Authors' Contributions

The authors have the same contributions in writing this paper.

References

[1] Laskin, N. (2000) Fractional Quantum Mechanics and Lévy Path Integrals. *Physics Letters A*, **268**, 298-305. https://doi.org/10.1016/S0375-9601(00)00201-2

[2] Laskin, N. (2002) Fractional Schrödinger Equation. *Physical Review E*, **66**, 56-108. https://doi.org/10.1103/PhysRevE.66.056108

[3] Zhang, X., Zhang, B. and Repovš, D. (2016) Existence and Symmetry of Solutions for Critical Fractional Schrödinger Equations with Bounded Potentials. *Nonlinear Analysis*, **142**, 48-68. https://doi.org/10.1016/j.na.2016.04.012

[4] Gao, Z., Tang, X. and Zhang, W. (2017) Multiplicity and Concentration of Solutions for Fractional Schrödinger Equations. *Taiwanese Journal of Mathematics*, **21**, 187-210. https://doi.org/10.11650/tjm.21.2017.7147

[5] Zhang, W., Tang, X. and Zhang, J. (2016) Infinitely Many Radial and Non-Radial Solutions for a Fractional Schrödinger Equation. *Computers & Mathematics with Applications*, **71**, 737-747. https://doi.org/10.1016/j.camwa.2015.12.036

[6] Teng, K. (2015) Multiple Solutions for a Class of Fractional Schrödinger Equations in R^N. *Nonlinear Analysis*, **21**, 76-86.

[7] Wang, Z. and Zhou, H. (2016) Radial Sign-Changing Solution for Fractional Schrödinger Equation. *Discrete and Continuous Dynamical Systems*, **36**, 499-508. https://doi.org/10.3934/dcds.2016.36.499

[8] Ružička, M. (2002) Electrorheological Fluids: Modeling and Mathematical Theory. Springer-Verlag, Berlin.

[9] Ayazoglu, R. and Ekincioglu, I. (2016) Electrorheological Fluids Equations Involving Variable Exponent with Dependence on the Gradient via Mountain Pass Techniques. *Numerical Functional Analysis and Optimization*, **37**, 1144-1157. https://doi.org/10.1080/01630563.2016.1205088

[10] Chen, Y., Levine, S. and Rao, M. (2006) Variable Exponent, Linear Growth Functionals in Image Restoration. *SIAM Journal on Applied Mathematics*, **66**, 1383-1406. https://doi.org/10.1137/050624522

[11] Mihăilescu, M., Pucci, P. and Rădulescu, V. (2008) Eigenvalue Problems for Anisotropic Quasilinear Elliptic Equations with Variable Exponent. *Journal of Mathematical Analysis and Applications*, **340**, 687-698. https://doi.org/10.1016/j.jmaa.2007.09.015

[12] Shen, Z. and Li, H. (2021) Standing Waves for Quasilinear Schrödinger Equations with Indefinite Nonlinearity. *Journal of Applied Mathematics and Physics*, **9**, 1003-1010. https://doi.org/10.4236/jamp.2021.95069

[13] Chen, H. (2019) The Existence of Solution of a Critical Fractional Equation. *Journal of Applied Mathematics and Physics*, **7**, 243-253. https://doi.org/10.4236/jamp.2019.71020

[14] Simsen, J., Simsen, M. and Primo, M. (2016) Reaction-Diffusion Equations with Spatially Variable Exponents and Large Diffusion. *Communications on Pure & Applied Analysis*, **15**, 495-506. https://doi.org/10.3934/cpaa.2016.15.495

[15] Lorenzo, C. and Hartley, T. (2000) Initialized Fractional Calculus. *International Journal of Applied Mathematics*, **3**, 249-265.

[16] Xiang, M., Zhang, B. and Yang, D. (2019) Multiplicity Results for Variable-Order Fractional Laplacian Equations with Variable Growth. *Nonlinear Analysis*, **178**, 190-204. https://doi.org/10.1016/j.na.2018.07.016

[17] Biswas, R. and Tiwari, S. (2020) Variable Order Nonlocal Choquard Problem with Variable Exponents. *Complex Variables and Elliptic Equations*, **66**, 853-875. https://doi.org/10.1080/17476933.2020.1751136

[18] Xiang, M., Hu, D., Zhang, B. and Wang, Y. (2020) Multiplicity of Solutions for Variable-Order Fractional Kirchhoff Equations with Nonstandard Growth. *Journal of Mathematical Analysis and Applications*, **501**, Article ID: 124269. https://doi.org/10.1016/j.jmaa.2020.124269

[19] Wang, L. and Zhang, B.L. (2019) Infinitely Many Solutions for Kirchhoff-Type Variable-Order Fractional Laplacian Problems Involving Variable Exponents. *Applicable Analysis*, **100**, 1-18.

[20] Xiang, M., Yang, D. and Zhang, B. (2020) Homoclinic Solutions for Hamiltonian Systems with Variable-Order Fractional Derivatives. *Complex Variables and Elliptic Equations*, **65**, 1412-1432. https://doi.org/10.1080/17476933.2019.1652281

[21] Diening, L., Harjulehto, P., Hästö, P. and Ružička, M. (2011) Lebesgue and Sobolev Spaces with Variable Exponents. Springer-Verlag, Heidelberg. https://doi.org/10.1007/978-3-642-18363-8

[22] Servadei, R. and Valdinoci, E. (2012) Mountain Pass Solutions for Non-Local Elliptic Operators. *Journal of Mathematical Analysis and Applications*, **389**, 887-898. https://doi.org/10.1016/j.jmaa.2011.12.032