Journal of Applied Mathematics and Physics, 2022, 10, 703-713
https://www.scirp.org/journal/jamp
ISSN Online: 2327-4379
ISSN Print: 2327-4352

On Two New Groups of Non-Elementary Functions That Are Giving Solutions to Some Second-Order Nonlinear Autonomous ODEs

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How to cite this paper: Stensland, M. (2022) On Two New Groups of Non-Elementary Functions That Are Giving Solutions to Some Second-Order Nonlinear Autonomous ODEs. *Journal of Applied Mathematics and Physics*, **10**, 703-713.
https://doi.org/10.4236/jamp.2022.103050

Received: February 1, 2022
Accepted: March 6, 2022
Published: March 9, 2022

Abstract

In this paper, we define a group of solutions $x(t)$ that are sine and cosine to the upper limit of integration in a non-elementary integral that can be arbitrary. We will also define a group of solutions $x(t)$ that are equal to the amplitude. This is a generalized amplitude function. We are using Abel's methods, described by Armitage and Eberlein. And finally, we define an exponential function whose exponent is the product of a complex number and the upper limit of integration in a non-elementary integral that can be arbitrary. At least three groups of non-elementary functions are special cases of this complex function.

Keywords

Non-Elementary Functions, Second-Order Nonlinear Autonomous ODE

1. Introduction

On page 1 in the book [1], we find the sentences: Very few ordinary differential equations have explicit solutions expressible in finite terms. This is not because ingenuity fails, but because the repertory of standard functions (polynomials, exp, sin and so on) in terms of which solutions may be expressed is too limited to accommodate the variety of differential equations encountered in practice.

This is the main reason for this work. It should be possible to do something about this problem. If we don't have enough tools in our mathematical toolbox, we must make the tools first. For this problem, we will attempt to define some new functions, and groups of functions. The numbers I have given the ODEs and the integral functions (IF) in the text are the numbers they have in my collection.

Wolfram Math World describes three nonlinear second-order ODEs that have the Jacobi elliptic functions sn, cn and dn as solutions. Define a solution $x(t) = cn(t)$ and differentiate twice, and you will obtain the ODE:

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = \left(2k^2 - 1\right)x - 2k^2 x^3, \ 0 \le k < 1 \tag{1}$$

And if we use the Jacobi amplitude function $am(t,k)$ as a solution $x(t)$ and differentiate twice, we will obtain the ODE:

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = -k^2 \sin(x)\cos(x) \tag{2}$$

This causes us to think that other second-order nonlinear ODEs have functions made by the same methods as Jacobi elliptic functions, as their solutions. It should be possible to make more non-elementary functions by changing the non-elementary integral. In this paper, we will work in the same way: First, define some non-elementary functions, and then differentiate them twice in order to see what kind of ODEs these functions are giving solutions to.

We will use the methods described by Armitage and Eberlein [2] in their book Elliptic Functions, especially Sections 1.6 and 1.7. They apply what they call the Abel's methods. "Eberlein sought to relate the ideas of Abel to the later work of Jacobi." Here is a brief summary of how they define the Jacobi elliptic functions:

In Section 1.6 they define a function x:

$$1)\quad x = x(\psi) = \int_0^{\psi} \frac{\mathrm{d}u}{\sqrt{1 - k^2 \sin^2 u}}, \quad 0 \le k < 1, \quad -\infty < \psi < \infty \tag{3}$$

Here is x a function of ψ. The positive derivative of x is:

$$2)\quad \frac{\mathrm{d}x}{\mathrm{d}\psi} = \frac{1}{\sqrt{1 - k^2 \sin^2 \psi}} \text{ and then inverting } \frac{\mathrm{d}x}{\mathrm{d}\psi}, \text{ so that} \tag{4}$$

$$3)\quad \frac{\mathrm{d}\psi}{\mathrm{d}x} = \sqrt{1 - k^2 \sin^2 \psi} \text{ here is } \psi \text{ a function of } x. \tag{5}$$

And then in Section 1.7 they define a set of functions sn, cn and dn, so that

$$sn(x) = \sin\psi, \quad cn(x) = \cos\psi \text{ and } dn(x) = \sqrt{1 - k^2 \sin^2 \psi} \tag{6}$$

Armitage and Eberlein are using x as variable to the functions sn, cn and dn. In order to avoid misunderstandings with the solution $x(t)$, we will use the variable u instead of x, and the amplitude φ instead of ψ.

During the last 30 years there have been done a lot of progress in finding solutions to nonlinear ODEs and PDEs. The progress is mostly made by using different methods like the Prelle-Singer method [3], Abel's equations [4] [5], the new Jacobi elliptic functions [6] [7], and the old Jacobi elliptic functions [8] [9], a new method [10], revised methods [11], Jacobi elliptic function expansion method [12], expo-elliptic functions [13].

With exception of the new Jacobi elliptic functions and the expo-elliptic functions, it seems to me that nobody has tried to make new non-elementary functions that can give solutions to second-order nonlinear ODEs. In this paper, we will attempt to take a step further.

The functions defined in this paper are new to the literature, at least to my knowledge. This paper is a continuation of [13], which is the first part.

2. The Jef-Family

I have named this group of functions The Jef-Family, after the Jacobi elliptic functions (Jef). Many of the functions in this group have Jacobi elliptic functions as a special case. We will define a general set of functions for this group. Common for all members of the Jef-Family are that they contain $\sin\varphi$ or $\cos\varphi$ or both, where $\varphi=\varphi(u)$ is the upper limit of integration in a non-elementary integral. The applications to these functions are as solutions to some nonlinear ODEs.

In order to work with functions, we must give them some symbols. I've used the notations of Jacobi elliptic functions, sn, cn and dn and added a letter in front, as for example psn, pcn and pdn.

2.1. Definition

Define a set of three functions asn, acn and adn, so that $asn(u)=\sin\varphi$, $acn(u)=\cos\varphi$ and $adn(u)=\frac{\mathrm{d}\varphi}{\mathrm{d}u}=h(\varphi)$, where the function h can be arbitrary, with or without square root, with fraction, and even $\sin(\sin\varphi)$ or $\mathrm{e}^{\sin\varphi}$. $\varphi=\varphi(u)$ is the upper limit of integration in a non-elementary integral. When $\frac{\mathrm{d}\varphi}{\mathrm{d}u}=\sqrt{1-k^2\sin^2\phi}$, is

$$asn(u)=sn(u,k),\quad acn(u)=cn(u,k) \quad \text{and} \quad adn(u)=dn(u,k) \tag{7}$$

2.2. Some Connections between These Functions

$$asn^2(u)+acn^2(u)=1 \tag{8}$$

The connection between the functions $adn(u), asn(u)$ and $acn(u)$ depends on how the function h is. They are continuous and differentiable on the whole R, for the limitations of the parameters.

2.3. The Derivatives to These Functions

$$\frac{\mathrm{d}}{\mathrm{d}u}asn(u)=acn(u)adn(u) \tag{9}$$

$$\frac{\mathrm{d}}{\mathrm{d}u}acn(u)=-asn(u)adn(u) \tag{10}$$

$$\frac{\mathrm{d}}{\mathrm{d}u}adn(u)=\frac{\mathrm{d}h}{\mathrm{d}\varphi}\frac{\mathrm{d}\varphi}{\mathrm{d}u} \tag{11}$$

2.4. One Example of the Jef-Family

The functions psn, pcn and pdn.

Define an integral function u (IF 16):

$$u = u(\varphi) = \int_0^{\phi} \frac{\mathrm{d}\theta}{\left(b + p\sin\theta + v\sin^2\theta\right)\sqrt{\left(n + h\sin\theta + g\sin^2\theta\right)\left(d + l\sin\theta + f\sin^2\theta\right)}} \tag{12}$$

$$-1 < h, g, l, f, p, v < 1,\quad b, n, d \ge 2$$

$$\frac{\mathrm{d}u}{\mathrm{d}\varphi} = \frac{1}{\left(b + p\sin\phi + v\sin^2\phi\right)\sqrt{\left(n + h\sin\phi + g\sin^2\phi\right)\left(d + l\sin\phi + f\sin^2\phi\right)}} \tag{13}$$

Inverting:

$$\frac{\mathrm{d}\varphi}{\mathrm{d}u} = \left(b + p\sin\phi + v\sin^2\phi\right)\sqrt{\left(n + h\sin\phi + g\sin^2\phi\right)\left(d + l\sin\phi + f\sin^2\phi\right)} \tag{14}$$

Define a set of three functions psn, pcn and pdn, so that

$$psn(u) = \sin\varphi,\quad pcn(u) = \cos\varphi \quad \text{and} \quad pdn(u) = \frac{\mathrm{d}\varphi}{\mathrm{d}u} \tag{15}$$

The connection between the functions psn, pcn and pdn:

$$psn^2(u) + pcn^2(u) = 1 \tag{16}$$

$$\begin{aligned} pdn^2(u) = {} & \left(b + p\,psn(u) + v\,psn^2(u)\right)^2 \left[\left(n + h\,psn(u) + g\,psn^2(u)\right)\right. \\ & \left. \times \left(d + l\,psn(u) + f\,psn^2(u)\right)\right] \end{aligned} \tag{17}$$

The derivatives to these functions:

$$\frac{\mathrm{d}}{\mathrm{d}u} psn(u) = pcn(u)\,pdn(u) \tag{18}$$

$$\frac{\mathrm{d}}{\mathrm{d}u} pcn(u) = -psn(u)\,pdn(u) \tag{19}$$

$$\begin{aligned} & \frac{\mathrm{d}}{\mathrm{d}u} pdn(u) = pcn(u)\left(b + p\,psn(u) + v\,psn^2(u)\right) \\ & \times \Big[\left(n + h\,psn(u) + g\,psn^2(u)\right)\left(d + l\,psn(u) + f\,psn^2(u)\right)\left(p + 2v\,psn(u)\right) \\ & + \left(b + p\,psn(u) + v\,psn^2(u)\right)\left(d + l\,psn(u) + f\,psn^2(u)\right)\left(\frac{1}{2}h + g\,psn(u)\right) \\ & + \left(b + p\,psn(u) + v\,psn^2(u)\right)\left(n + h\,psn(u) + g\,psn^2(u)\right)\left(\frac{1}{2}l + f\,psn(u)\right)\Big] \end{aligned} \tag{20}$$

When $-1 < h, g, l, f < 1$ and $n, d \ge 2$, are the functions psn, pcn and pdn continues and differentiable on the whole R.

We can make a lot of special cases of the functions psn, pcn and pdn by defining one or more of the parameters = 0. For example when $p = v = h = g = l = 0$ and $b = d = n = 1$ and $f = -k^2$, $0 \le k < 1$. Then is $psn(u) = sn(u,k)$, $pcn(u) = cn(u,k)$ and $pdn(u) = dn(u,k)$. Jacobi elliptic functions (Jef) are a special case of the functions psn, pcn and pdn.

Periods.

Consider the integral function (IF 16). $\sin\theta$ increases from −1 to 1 when θ

increases from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$. And $\sin\theta$ decreases from 1 to −1 when θ increases from $\frac{\pi}{2}$ to $\frac{3}{2}\pi$. The period of the function, that we may name *P*, has a maximum value at $\varphi=\frac{\pi}{2}$. P is then a quarter-period.

The period

$$P=\int_0^{\frac{\pi}{2}}\frac{\mathrm{d}\theta}{\left(b+p\sin\theta+v\sin^2\theta\right)\sqrt{\left(n+h\sin\theta+g\sin^2\theta\right)\left(d+l\sin\theta+f\sin^2\theta\right)}} \tag{21}$$

The period P depends on the values of the parameters.

$$psn(u+4P)=\sin(\varphi+2\pi)=\sin\varphi=psn(u) \tag{22}$$

$$pcn(u+4P)=\cos(\varphi+2\pi)=\cos\varphi=pcn(u) \tag{23}$$

$$\begin{aligned} pdn(u+4P)&=\left(b+p\sin(\varphi+2\pi)+v\sin^2(\varphi+2\pi)\right)\\ &\times\sqrt{\left(n+h\sin(\varphi+2\pi)+g\sin^2(\varphi+2\pi)\right)\left(d+l\sin(\varphi+2\pi)+f\sin^2(\varphi+2\pi)\right)}\\ &=pdn(u)\end{aligned} \tag{24}$$

The functions $psn(u), pcn(u)$ and $pdn(u)$ have a real period of 4*P*.

$$psn(P)=1,\ \ pcn(P)=0,\ \ pdn(P)=(b+p+v)\sqrt{(n+h+g)(d+l+f)} \tag{25}$$

$$psn(0)=0,\ \ pcn(0)=1,\ \ pdn(0)=b\sqrt{nd} \tag{26}$$

Define a solution

$$x(t)=psn(t),\ \ \frac{\mathrm{d}x}{\mathrm{d}t}=pcn(t)\,pdn(t) \tag{27}$$

$$\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2=\left(1-x^2\right)\left(b+px+vx^2\right)^2\left(n+hx+gx^2\right)\left(d+lx+fx^2\right) \tag{28}$$

The general expression of this equation:

$$\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2=Ax^{10}+Bx^9+Cx^8+Dx^7+Ex^6+Fx^5+Gx^4+Hx^3+Ix^2+Jx+K \tag{29}$$

According to Schwalm [8] this kind of differential equations occurs in the study of mechanical systems, at least to the degree of 4. Some of the parameters may be zero. The function $psn(t)$ give a solution to this kind of equations.

Let us go back to $\frac{\mathrm{d}x}{\mathrm{d}t}=pcn(t)\,pdn(t)$ and differentiating one more time, and we obtain the Equation (2082):

$$\begin{aligned}\frac{\mathrm{d}^2x}{\mathrm{d}t^2}&=-x\left(b+px+vx^2\right)^2\left(n+hx+gx^2\right)\left(d+lx+fx^2\right)\\ &+\left(1-x^2\right)\left(b+px+vx^2\right)\Big[\left(n+hx+gx^2\right)\left(d+lx+fx^2\right)\left(p+2vx\right)\\ &+\left(b+px+vx^2\right)\left(d+lx+fx^2\right)\left(\frac{1}{2}h+gx\right)\\ &+\left(b+px+vx^2\right)\left(n+hx+gx^2\right)\left(\frac{1}{2}l+fx\right)\Big]\end{aligned} \tag{2082}$$

The function $psn(t)$ give solution to a second-order nonlinear ODE of 9. degree. The equilibrium points are centers and saddles. See Figure 1.

In Figure 1 are the values of the parameters: $p=\frac{3}{4}$, $b=2$, $v=-\frac{1}{2}$, $n=2$, $h=-\frac{1}{2}$, $g=-\frac{2}{3}$, $d=3$, $l=-\frac{1}{4}$, $f=-\frac{2}{5}$.

3. The Amplitude Functions

This group of functions I have named *amplitude functions*, after the Jacobi amplitude function $am(u,k)$, that is a member of this group of functions. Common for the amplitude functions is the solution $x(t)=\varphi=\varphi(t)$, that is the amplitude or upper limit of integration in a non-elementary integral that can be arbitrary.

The function $am(u,k)$ is equal to the Jacobi amplitude φ.

$$am(u,k)=\int_0^u dn(u',k)\,\mathrm{d}u' \tag{30}$$

$$\frac{\mathrm{d}}{\mathrm{d}u}am(u,k)=dn(u,k) \tag{31}$$

The second-order nonlinear differential equation:

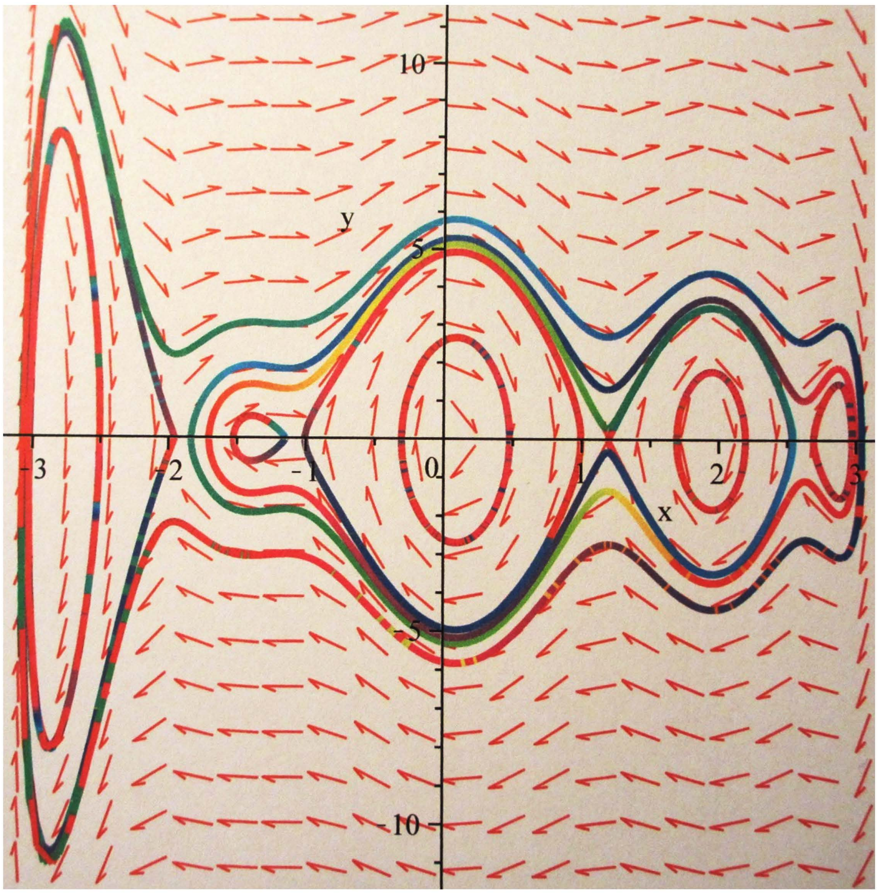

Figure 1. Five centers and four saddle-points.

$$\frac{d^2x}{dt^2} = -k^2 \sin x \cos x, \quad 0 \le k < 1 \tag{32}$$

has $am(t,k)$ as solution:

$$x(t) = am(t,k) = \varphi = \varphi(t) = \int_0^t dn(u,k)\,du \tag{33}$$

$$\frac{dx}{dt} = \frac{d\varphi}{dt} = dn(t) = \sqrt{1-k^2\sin^2\varphi} \tag{34}$$

One more differentiating give Equation (32).

3.1. Definition of the General Amplitude Function

Define a function

$$amp(u) = \varphi = \varphi(u) = \int_0^u adn(u')\,du', \quad -\infty < u, \varphi < \infty \tag{35}$$

where φ is the amplitude or upper limit of integration in a non-elementary integral.

$$\frac{d}{du} amp(u) = \frac{d\varphi}{du} = adn(u) = f(\varphi) \tag{36}$$

The function $adn(u)$ is the same function as in the Jef-Family (7).

The function f can be arbitrary. It may also be a square root or a fraction. We have named the derivative to $amp(u)$ for $adn(u)$ in order to show the relationship to $dn(u,k)$.

When $f(\varphi) = \sqrt{1-k^2\sin^2\varphi}$ is $adn(u) = dn(u,k)$ and $amp(u) = am(u,k)$ (37)

3.2. The *amp*-Function as Solution to Second-Order Nonlinear ODE

Consider the solution

$$x(t) = amp(t) = \varphi = \varphi(t) = \int_0^t adn(u)\,du \tag{38}$$

$$\frac{dx}{dt} = \frac{d\varphi}{dt} = adn(t) \tag{39}$$

When $\frac{d\varphi}{dt} = 1$, is

$$\varphi = \int 1 dt = t + C \tag{40}$$

We will give an example:

As for the Jacobi elliptic functions, we start with a non-elementary integral.

Define an integral function u (IF 105):

$$u = u(\varphi) = \int_0^\varphi \frac{d\theta}{\sqrt{c + ke^{b\theta}\sin(p\theta)}}, \quad b,c,k,p,\varphi \in R \tag{41}$$

$$b < 0, \ -1 < k < 1, \ -\infty < p < \infty, \ c \ge 1$$

$$\frac{du}{d\varphi} = \frac{1}{\sqrt{c + ke^{b\varphi}\sin(p\varphi)}} \tag{42}$$

Inverting:

$$\frac{\mathrm{d}\varphi}{\mathrm{d}u}=\sqrt{c+k\mathrm{e}^{b\varphi}\sin(p\varphi)} \tag{43}$$

Define an amplitude function as a solution $x(t)$:

$$x(t)=amp(t)=\varphi=\varphi(t)=\int_0^t adn(u)\,\mathrm{d}u \quad \text{Compare with (33)} \tag{44}$$

$$\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{\mathrm{d}\varphi}{\mathrm{d}t}=adn(t)=\sqrt{c+k\mathrm{e}^{bx}\sin(px)} \tag{45}$$

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2}=\frac{1}{2}k\mathrm{e}^{bx}\left(b\sin(px)+p\cos(px)\right) \tag{46}$$

Choose the parameter-values $p=1,b=-1,k=\frac{1}{2}$ and we obtain the ODE:

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2}=\frac{1}{4}\mathrm{e}^{-x}\left(\cos x-\sin x\right) \tag{47}$$

The second-order nonlinear ODE (46) has the amplitude function described above as solution:

$$x(t)=amp(t)$$

where in this case

$$adn(t)=\frac{\mathrm{d}x}{\mathrm{d}t}=\sqrt{c+k\mathrm{e}^{bx}\sin(px)} \tag{48}$$

We have gone forward the opposite of what is common. The usual way is to start with (46) and integrate. The purpose is to show that the solution $x(t)$ to (46) is a kind of amplitude function, where $x(t)=\varphi=\varphi(t)$. What might be new is to place the solution $x(t)$ in a group of functions, and to tell what kind of functions these solutions are. A lot of solutions to second-order nonlinear ODEs can be placed in the group of amplitude functions.

For example:

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2}=x-\sin x \tag{49}$$

We take it step by step:

$$\int\frac{\mathrm{d}^2x}{\mathrm{d}t^2}\mathrm{d}t=\int\left(x-\sin x\right)\mathrm{d}t \tag{50}$$

$$\frac{\mathrm{d}x}{\mathrm{d}t}+c_1=\sqrt{x^2+2\cos x+C} \tag{51}$$

Choose $c_1=0$ and define a solution: $x(t)=amp(t)=\varphi=\varphi(t)$

$$\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{\mathrm{d}\varphi}{\mathrm{d}t}=\sqrt{\varphi^2+2\cos\varphi+C} \tag{52}$$

Inverting:

$$\frac{\mathrm{d}t}{\mathrm{d}\varphi}=\frac{1}{\sqrt{\varphi^2+2\cos\varphi+C}} \tag{53}$$

$$t = t(\varphi) = \int_0^{\varphi} \frac{\mathrm{d}\theta}{\sqrt{\theta^2 + 2\cos\theta + C}} \tag{54}$$

Here we see that the solution $x(t) = \varphi = \varphi(t)$ is the amplitude, or upper limit of integration in a non-elementary integral. The solution $x(t)$ to Equation (49) is an amplitude function.

We can make this definition:

If a solution $x(t)$ is the upper limit of integration in a non-elementary integral, then this solution is an amplitude function.

The applications to the amplitude functions are as solutions to differential equations.

4. The Complex Expo-Elliptic Function

We have briefly looked at three groups of non-elementary functions, one group in the previous paper [13], the Jef-Family and the amplitude functions in this paper. They are very useful as solutions to second-order nonlinear ODEs. These solutions have elementary functions as special cases:

1) Expo-elliptic functions:

$$x(t) = \mu(t) = \mathrm{e}^{a\varphi}, \quad \frac{\mathrm{d}\varphi}{\mathrm{d}t} = 1 \to x(t) = \mathrm{e}^{at} \tag{55}$$

2) The Jef-Family:

$$x(t) = asn(t) = \sin\varphi, \quad \frac{\mathrm{d}\varphi}{\mathrm{d}t} = 1 \to x(t) = \sin t \tag{56}$$

$$x(t) = acn(t) = \cos\varphi, \quad \frac{\mathrm{d}\varphi}{\mathrm{d}t} = 1 \to x(t) = \cos t \tag{57}$$

3) Amplitude functions:

$$x(t) = amp(t) = \varphi, \quad \frac{\mathrm{d}\varphi}{\mathrm{d}t} = 1 \to x(t) = t + C \tag{58}$$

These solutions are functions of the amplitude φ, or is the amplitude φ, that is the upper limit of integration in a non-elementary integral. These functions are special cases of the complex expo-elliptic function. This function may also be named the complex μ-function:

$$\begin{aligned} &M = M(u) = \mathrm{e}^{\lambda\varphi}, \ \varphi = \varphi(u), \\ &\lambda = a + ib(\text{a complex number}), \ i = \sqrt{-1}, \ a, b \in R \end{aligned} \tag{59}$$

$$M = M(u) = \mathrm{e}^{\lambda\varphi} = \mathrm{e}^{a\varphi}\left(\cos(b\varphi) + i\sin(b\varphi)\right) \tag{60}$$

In Equation (60) we find the expo-elliptic function $\mathrm{e}^{a\varphi}$, the solutions in the Jef-Family: $\sin\varphi$ and $\cos\varphi$, and the amplitude function φ. Then we have combined three groups of non-elementary functions in one single function: $\mathrm{e}^{\lambda\varphi}$.

5. Conclusions

In this paper and in [13] we have defined three groups of non-elementary functions that are very useful as solutions to some second-order nonlinear autonomous ODEs: The expo-elliptic functions, where the solution $x(t) = e^{a\varphi}$, the Jef-Family, where the solution $x(t) = \sin\varphi$ or $x(t) = \cos\varphi$, and the amplitude functions, where the solution $x(t) = \varphi$, that is the amplitude or upper limit of integration in a non-elementary integral that can be arbitrary.

These groups of functions are containing many sets of non-elementary functions, where two examples of the expo-elliptic functions are studied in [13], and one example of the Jef-Family in this paper.

What is new in this paper:

1) The Jef-Family and the set of functions psn, pcn and pdn.

2) The group of amplitude functions, where the solution $x(t)$ is the amplitude or upper limit of integration in a non-elementary integral.

3) The complex expo-elliptic function $e^{\lambda\varphi}$ that has the solutions $e^{a\varphi}$, $\sin\varphi$ and $\cos\varphi$, and φ as special cases.

It is possible to make a lot of non-elementary functions using Abel's methods described by Armitage and Eberlein, by how they define the Jacobi elliptic functions. In the same way as Jacobi's functions sn, cn, dn and am give solutions to a few ODEs, the non-elementary functions described in this paper and a lot more, give solutions to many different kinds of ODEs. I don't see any limit for this subject. The only limit is our imagination.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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