Scientific Research Publishing

Journal of Applied Mathematics and Physics, 2022, 10, 688-702
https://www.scirp.org/journal/jamp
ISSN Online: 2327-4379
ISSN Print: 2327-4352

(4 + 4)-Dimensional Space-Time as a Dual Scenario for Quantum Gravity and Dark Matter

Brandon Mariano Martínez-Olivas, Juan Antonio Nieto, Andrés Sandoval-Rodríguez

Facultad de Ciencias Fsico-Matemáticas de la Universidad Autónoma, Culiacán Sinaloa, México
Email: brandon.fcfm@uas.edu.mx, niet@uas.edu.mx, janieto1@asu.edu, andres.fcfm@uas.edu.mx

How to cite this paper: Martínez-Olivas, B.M., Nieto, J.A. and Sandoval-Rodríguez, A. (2022) (4 + 4)-Dimensional Space-Time as a Dual Scenario for Quantum Gravity and Dark Matter. *Journal of Applied Mathematics and Physics*, **10**, 688-702.
https://doi.org/10.4236/jamp.2022.103049

Received: January 14, 2022
Accepted: March 6, 2022
Published: March 9, 2022

Abstract

In this work, we make a number of proposals to explain how a world of (4 + 4)-dimensions can be useful for a better understanding of both dark matter and quantum gravity. The key idea is to look for some advantage of considering self-dual invariants in (4 + 4)-dimensions rather than in a separate context of (1 + 3)-dimensions or (3 + 1)-dimensions. In fact, we show that by considering the duality concept in (4 + 4)-dimensions we may provide an alternative meaning of a framework for loop quantum gravity. Moreover, considering the Dirac equation in (4 + 4)-dimensions for a particle without electric charge and mass, we show that when it is projected into the (1 + 3) and (3 + 1)-worlds may describe a system with electric charge and mass. We also discuss the relation between the three physical scenarios; (4 + 4)-world, black-holes and dark matter.

Keywords

(4 + 4)-Dimensions, Dark Matter, Quantum Gravity

1. Introduction

Dark matter (see Refs. [1] and [2] references therein) and quantum gravity (see Refs. [3] [4] [5] and references therein) are two of the main open problems in physics. In fact, they do not have a totally consistent solution [4]. There have been multiple attempts to solve them, including loop quantum gravity [5] and string theory [6], but they do not fully explain everything. We believe that this might be a consequence of the asymmetry in space-time. There is no fundamental reason that explains why the background of the universe may be described with 3 spatial coordinates and just 1 time real coordinate. In this paper, in order to search for a mathematical solution of dark matter and quantum gravity, we explore the idea that taking a more symmetric universe with (4 + 4)-dimensions (four time and four space coordinates) might provide an alternative solution. In principle, the origin of (4 + 4)-signature may arises from *M*-theory via type II *A* and *B* strings which predicts a (5 + 5)-signature (see Refs. [7] and references therein), but here we try to promote the idea that such a signature is independent of *M*-theory.

Moreover, with heuristic physical reasoning in [8] was proposed another route for becoming interested in world of (4 + 4)-dimensions. Roughly speaking the main idea is to assume that the 2-sphere determined by the Schwarzschild radius associated with black-holes separate two worlds: 1) the exterior in (1 + 3)-dimensions where ordinary matter lives and moves with velocities less than the light velocity and 2) the interior in (3 + 1)-dimensions where tachyons move with velocities greater than the light velocity. It turns out that this idea provides with an alternative explanation of the strange gravitational behavior of the rotation curves of spiral galaxies, and therefore can be seen as a candidate for a solution of dark matter origin.

It is worth mentioning that it has been proved that massless Dirac equation formulated in flat (4 + 4) (or (5 + 5)) dimensions may lead to massive spinors in (1 + 3)-dimensions [9]. In turn, (4 + 4)-dimensions have an interesting connection with qubits and chirotopes (see Refs. [10]-[17] and references therein).

An illustration of how this $(4+4)\sim(1+3)+(3+1)$ construction might be useful in the development of quantum gravity we take into account, as guide, the Euler characteristic in graph theory. For this purpose, let us start describing the idea with a simple graph (see Refs [18] and [19]). An edge with 2 vertices, in a graphic space *G*. The dual of this line, will be a loop with 1 vertex in the dual graphic space G^*. The idea is to consider that in general, the dual of a graph G corresponds to a graph in G^*. Now, the Euler characteristic of a general graph gives us a truly wonderful equation

$$V-E+F=2, \tag{1}$$

where V denotes number of vertices, E the number edges and F the number faces of *G*. Considering that $F=V^*$ the vertices on G^* and $E=E^*$ we see that (1) becomes

$$V-E+V^*=2. \tag{2}$$

This equation implies that

$$V-1+V^*-1=E \tag{3}$$

and therefore defining the so called rank $R=V-1$ we discover the formula

$$R+R^*=E. \tag{4}$$

By the way, starting from (4) we may be able to obtain (2). Let us try to see the behavior of (4), according to 2-vertices in *G*. In this case, $V=2$, $V^*=1$ and $E=1$, so $2-1+1=2$. This provides an easy relationship between two different but complementary graphic spaces G and G^*. Our proposal is that something similar is what may happen in our Universe at different scales and circumstances. Assuming we "live" in the world *G*, and we have made every calculation on *G*, in particular quantum gravity and dark matter structures, without taking into account the world G^* we expect to find all kinds of problems since we are not using the complete background framework $\mathcal{G} = G \bigcup G^*$. This means that we might have been ignoring the dual space-time G^*, and therefore without considering it, the gravitational physical theories are just incomplete. Our conjecture is that it might not be possible to get a solution on quantum gravity without considering the dual of our Universe.

2. Towards Loop Quantum Gravity in (4 + 4)-Dimensions

Consider the metric

$$g_{\mu\nu} = e_\mu^a e_\nu^b \eta_{ab}, \tag{5}$$

where we require that the vielbein e_μ^a satisfies

$$\partial_\mu e_\nu^a - \Gamma_{\mu\nu}^\alpha e_\alpha^a + \omega_\mu^{ab} e_{\nu b} = 0. \tag{6}$$

Here, $\Gamma_{\mu\nu}^\alpha$ is the Christoffel symbol and ω_μ^{ab} spin connection.

It is well known that the Riemann tensor in terms of ω_μ^{ab} is given by

$$R_{\mu\nu}^{ab} = \partial_\mu \omega_\nu^{ab} - \partial_\nu \omega_\mu^{ab} + \omega_\mu^{ac} \omega_{\nu c}^b - \omega_\mu^{bc} \omega_{\nu c}^a. \tag{7}$$

It turns out convenient to define the extended Riemann tensor

$$\mathcal{R}_{\mu\nu}^{ab} = R_{\mu\nu}^{ab} + e_\mu^a e_\nu^b - e_\mu^b e_\nu^a. \tag{8}$$

A self-duality of $\mathcal{R}_{\mu\nu}^{ab}$ in (1 + 3)-dimensions is determined by

$${}^{\pm}\mathcal{R}_{\mu\nu}^{ab} = \frac{1}{2}\left(\mathcal{R}_{\mu\nu}^{ab} \mp i\varepsilon_{cd}^{ab} \mathcal{R}_{\mu\nu}^{cd}\right). \tag{9}$$

It is worth mentioning that this expression can be used as starting point in the development of loop quantum gravity. In fact, we can verify that

$$\frac{1}{2}\varepsilon_{cd}^{ab}\, {}^{\pm}\mathcal{R}_{\mu\nu}^{cd} = \pm i\, {}^{\pm}\mathcal{R}_{\mu\nu}^{cd}. \tag{10}$$

Observe that (9) may be considered as the analogue of the version of the Euler characteristic (4) in graph theory. If we choose that instead of (1 + 3)-signature we choose the (0 + 4)-signature, then we must have

$${}^{\pm}\mathcal{R}_{\mu\nu}^{ab} = \frac{1}{2}\left(\mathcal{R}_{\mu\nu}^{ab} \pm \varepsilon_{cd}^{ab} \mathcal{R}_{\mu\nu}^{cd}\right). \tag{11}$$

In fact, we can verify that

$$\frac{1}{2}\varepsilon_{cd}^{ab}\, {}^{\pm}\mathcal{R}_{\mu\nu}^{cd} = \pm\, {}^{\pm}\mathcal{R}_{\mu\nu}^{cd}. \tag{12}$$

It turns out that the extension of (9) and (11) to any signature is not so straightforward. This is because these relations depend strongly in the properties of the ε-symbol. First, recall that the number of indices of ε-symbol determines the dimension of the space. Secondly, combination of the properties of the ε-symbol and the flat metric $\eta_{ab} = diag(-1,\cdots,-1,+1,\cdots,+1)$ plays a central role. For instance, if the space is of the form $(0+n)$-signature then

$$\frac{1}{n!}\varepsilon^{a_1\cdots a_n}\varepsilon^{b_1\cdots b_n}\eta_{a_1b_1}\cdots\eta_{a_nb_n}=1. \tag{13}$$

Moreover, it is not difficult to see that we also obtain (13) in a $(m+n)$-dimensions, with $m=2s$ and $s=0,1,2,\cdots etc$. On the other hand if $m=2s+1$ then

$$\frac{1}{n!}\varepsilon^{a_1\cdots a_n}\varepsilon_{a_1\cdots a_n}=-1 \tag{14}$$

and therefore as in the case of (1 + 3)-dimensions we require a complex structure for self-duality as (10). While in the case of the signatures $(0+n)$ and $(2s+m)$ we still accomplish self-duality in a real scenario.

Another, important problem in an extension of self-duality to higher dimensions emerges from the fact that in any dimension, the extended curvature $\mathcal{R}_{\mu\nu}^{ab}$ always contains the same number of indices. Thus, if we want to define its dual we must have a quantity with different indices that the original curvature $\mathcal{R}_{\mu\nu}^{ab}$. For instance, in 8-dimensions the ε-symbol contains eight indices and therefore we obtain

$${}^{*}\mathcal{R}_{\mu\nu}^{a_1a_2a_3a_4a_5a_6}=\frac{1}{2!}\varepsilon^{a_1a_2a_3a_4a_5a_6}_{a_7a_8}\mathcal{R}_{\mu\nu}^{a_7a_8}. \tag{15}$$

This looks so asymmetric that we are forced to look for an alternative definition of a new extended curvature. This idea was developed in even dimension in Refs. [20] [21] in which a new tensor was proposed, namely

$$\Omega_{\mu_1\cdots\mu_n}^{a_1\cdots a_n}=\frac{1}{n!}\delta_{b_1\cdots b_n}^{a_1\cdots a_n}\mathcal{R}_{\mu_1\mu_2}^{b_1b_2}\cdots\mathcal{R}_{\mu_{n-1}\mu_n}^{b_{n-1}b_n}, \tag{16}$$

where $\delta_{b_1\cdots b_n}^{a_1\cdots a_n}$ is a generalized δ-symbol. In 8-dimensions we get

$$\Omega_{\mu_1\mu_2\mu_3\mu_4}^{a_1a_2a_3a_4}=\frac{1}{4!}\delta_{b_1b_2b_3b_4}^{a_1a_2a_3a_4}\mathcal{R}_{\mu_1\mu_2}^{b_1b_2}\mathcal{R}_{\mu_3\mu_4}^{b_3b_4}. \tag{17}$$

Now, self-duality of $\Omega_{\mu_1\mu_2\mu_3\mu_4}^{a_1a_2a_3a_4}$ is given by

$${}^{*}\Omega_{\mu_1\mu_2\mu_3\mu_4}^{a_1a_2a_3a_4}=\frac{1}{4!}\varepsilon_{b_1b_2b_3b_4}^{a_1a_2a_3a_4}\Omega_{\mu_1\mu_2\mu_3\mu_4}^{b_1b_2b_3b_4} \tag{18}$$

and therefore we have the same number of Latin indices in $\Omega_{\mu_1\mu_2\mu_3\mu_4}^{b_1b_2b_3b_4}$ and ${}^{*}\Omega_{\mu_1\mu_2\mu_3\mu_4}^{a_1a_2a_3a_4}$. This means that in 8-dimensions the self-dual can be defined as

$${}^{\pm}\Omega_{\mu_1\mu_2\mu_3\mu_4}^{a_1a_2a_3a_4}=\frac{1}{2}\left(\Omega_{\mu_1\mu_2\mu_3\mu_4}^{a_1a_2a_3a_4}\pm\frac{1}{4!}\varepsilon_{b_1b_2b_3b_4}^{a_1a_2a_3a_4}\Omega_{\mu_1\mu_2\mu_3\mu_4}^{b_1b_2b_3b_4}\right). \tag{19}$$

Of course, the tensor ${}^{\pm}\Omega_{\mu_1\mu_2\mu_3\mu_4}^{a_1a_2a_3a_4}$ satisfies the self-duality formula

$${}^{*\pm}\Omega_{\mu_1\mu_2\mu_3\mu_4}^{a_1a_2a_3a_4}=\pm{}^{\pm}\Omega_{\mu_1\mu_2\mu_3\mu_4}^{a_1a_2a_3a_4}. \tag{20}$$

If we observe that in (4 + 4)-dimensions both (19) and (20) are structures in a real scenario we may expect that, in the route of quantization, key differences with the (1 + 3)-dimensional case may emerge. In particular, for self-duality we have that in (1 + 3)-dimensions complex structure is required, while (4 + 4)-dimensions we can develop quantization in a real background.

3. Dirac Equation in (1 + 3) and (3 + 1) Dimensions

The Dirac equation for $\frac{1}{2}$-spin [22] provides one of the most remarkable theories of modern physics. Among its features one can cite the unification of special relativity and quantum mechanics and the prediction of antiparticles. Moreover, such an equation can be considered as main source of supersymmetry, superstrings and supergravity. The aim of this section is to revise the Dirac equation from the point of view of the (1 + 3) and (3 + 1) signatures. We argue that our analysis may help to have a better understanding of the Dirac equation in (4 + 4)-dimensions, which we shall discuss in the next section. Roughly speaking this is due to the fact that in vacuum one may expect the symmetry braking

$$(4+4) \to (1+3)+(3+1). \tag{21}$$

Let us start defining the flat metrics

$$\eta^{\mu\nu}_{(+)} = diag(-1,+1,+1,+1) \tag{22}$$

and

$$\eta^{\mu\nu}_{(-)} = diag(+1,-1,-1,-1), \tag{23}$$

corresponding to (1 + 3) and (3 + 1) dimensions, respectively. Here the indices $\mu,\nu \in \{0,1,2,3\}$. Notice from the beginning that one has

$$\eta^{\mu\nu}_{(+)} = -\eta^{\mu\nu}_{(-)}. \tag{24}$$

With these two kinds of flat metrics one can write two constraints equations associated with a point test particle of rest mass m_0 and of linear momentum p_μ, namely

$$p_\mu p_\nu \eta^{\mu\nu}_{(+)} + m_0^2 c^2 = 0 \tag{25}$$

and

$$p_\mu p_\nu \eta^{\mu\nu}_{(-)} + m_0^2 c^2 = 0, \tag{26}$$

where c is the light velocity. If we consider the energy definition $E \equiv cp_0$ from (25) we may derive the well known relation $E = \pm\sqrt{\overline{p}^2 c^2 + m_0^2 c^4}$. It turns out that the two constraints (25) and (26) are unique in the sense that any other possible constraint involving quadratic linear momentum p_μ may arise from these two constraints by simply multiplying the whole constraints by minus sign. Of course, only in the case of $m_0 = 0$ the constrains (25) and (26) collapses to just one constraint

$$p_\mu p_\nu \eta^{\mu\nu}_{(+)} = 0. \tag{27}$$

In order to clarify the meaning of the constraints (25), (26) and (27) we shall write

$$p^\mu = m_0 \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}. \tag{28}$$

Substituting this expression into (25), (26) and (27) we learn that these constraints lead to

$$\frac{\mathrm{d}x^{\mu}}{\mathrm{d}\tau}\frac{\mathrm{d}x^{\nu}}{\mathrm{d}\tau}\eta_{(+)\mu\nu} + c^2 = 0, \tag{29}$$

$$\frac{\mathrm{d}x^{\mu}}{\mathrm{d}\tau}\frac{\mathrm{d}x^{\nu}}{\mathrm{d}\tau}\eta_{(-)\mu\nu} + c^2 = 0 \tag{30}$$

and

$$\frac{\mathrm{d}x^{\mu}}{\mathrm{d}\tau}\frac{\mathrm{d}x^{\nu}}{\mathrm{d}\tau}\eta_{(+)\mu\nu} = 0, \tag{31}$$

respectively. Therefore, in the case of (29) and (30) τ admits the meaning of proper time, while in the case of (31) is just a null parameter. Developing these three constrains it is straightforward to verify that (29) corresponds to $v < c$, (30) refers to $v > c$ and (31) leads to the case $v = c$. Here, v is the magnitude of the usual velocity of the particle, that is $v^i = \frac{\mathrm{d}x^i}{\mathrm{d}t}$, with the index i running from 1 to 3. Thus, with the constraints (29), (30) and (31) we cover all possibilities; $v < c$, $v > c$ and $v = c$. Of course, $v < c$ refers to ordinary matter, $v > c$ describes superluminal particles (tachyons) and $v = c$ light like particles.

In order to quantize the system we promote the linear momentum p_μ to an operator;

$$p_\mu \to \hat{p}_\mu = -i\hbar\frac{\partial}{\partial x^\mu}. \tag{32}$$

Thus, the constraints (25) and (26) lead to the field equations

$$\left(\hat{p}_\mu \hat{p}_\nu \eta_{(+)}^{\mu\nu} + m_0^2 c^2\right)\psi = 0 \tag{33}$$

and

$$\left(\hat{p}_\mu \hat{p}_\nu \eta_{(-)}^{\mu\nu} + m_0^2 c^2\right)\psi = 0, \tag{34}$$

respectively. Of course, by using a Fourier transform method we may find that (33) and (34) will lead back to (25) and (26).

It is well known that (33) admit a square root of the form

$$\left(\gamma^\mu \hat{p}_\mu + m_0 c\right)\psi = 0. \tag{35}$$

This can be proved by multiplying (33) in the left side by the operator

$$\left(\gamma^\mu \hat{p}_\mu - m_0 c\right) \tag{36}$$

and requiring that the γ^μ satisfy

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = -2\eta_{(+)}^{\mu\nu}. \tag{37}$$

We may think that another possible square root of (13) is given by

$$\left(\gamma^\mu \hat{p}_\mu - m_0 c\right)\psi = 0. \tag{38}$$

However, by multiplying this expression by (−1) and considering that (37) is invariant under the change $\gamma^{\mu} \to -\gamma^{\mu}$ we learn that (35) and (38) are indeed equivalents.

On the other hand the square root of (34) is exactly the same Equation (35) but with the condition that now the γ^{μ} satisfy

$$\gamma^{\mu}\gamma^{\nu} + \gamma^{\nu}\gamma^{\mu} = -2\eta_{(-)}^{\mu\nu}. \tag{39}$$

This seems to be an intriguing result, because the constraints (33) and (34) describe completely different systems; ordinary matter and tachyons. However, both constraints have exactly the same square root (35).

4. Dirac Equation in (4 + 4)-Dimensions

In this section we discuss a number of features of the Dirac equation in four time and four space dimensions [9]. Let us start assuming that we can make the split

$$(4+4) \to (3+1) + (1+3). \tag{40}$$

First, we note that a signature duality emerges because (3 + 1)-world is mere a changing signature of our ordinary world in (1 + 3)-dimensions. Suppose an electron "lives" in (1 + 3)-world. We can ask: What could be the corresponding dual electron in (3 + 1)-dimensions? A partial answer to this question may be obtained from the observation that since in (4 + 4)-dimensions there exist Majorana-Weyl spinors (see Ref. [9] and references therein) 16 spinors complex components of the Dirac equation can be reduced to 4-complex spinor components: the same number than an ordinary $\frac{1}{2}$-fermion in 4-dimensions.

Consider the Dirac equation in any $(t+s)$-signature, namely

$$\left(\gamma^{\hat{\mu}} \hat{p}_{\hat{\mu}} + m_0\right)\psi = 0, \tag{41}$$

where $\gamma^{\hat{\mu}}$ are the gamma matrices satisfying the Clifford algebra

$$\gamma^{\hat{\mu}}\gamma^{\hat{\nu}} + \gamma^{\hat{\nu}}\gamma^{\hat{\mu}} = -2\eta^{\hat{\mu}\hat{\nu}}. \tag{42}$$

Here, $\eta^{\hat{\mu}\hat{\nu}}$ is a $(t+s)$-dimensional flat diagonal metric which depends on the signature $(t+s)$ (t times and s space dimensions). Note that (41) depends on the signature via the expression (42). Moreover, the $\gamma^{\hat{\mu}}$ are matrices of $2^{\frac{D}{2}} \times 2^{\frac{D}{2}}$, with $D = t + s$.

It turns out convenient to mention that in (1 + 3)-dimensions, the three more communes representations of the gamma matrices γ^{μ} are the Weyl (γ_W^{μ}), Dirac (γ_D^{μ}) and Majorana (γ_M^{μ}) representations. Explicitly, considering the Pauli matrices

$$\sigma_1 \equiv \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \sigma_2 \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 \equiv \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \tag{43}$$

we have

$$\gamma_W^1 \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_W^2 \equiv \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \tag{44}$$

$$\gamma_D^1 \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma_D^2 \equiv \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \tag{45}$$

and

$$\gamma_M^1 \equiv \begin{pmatrix} 0 & -\sigma^2 \\ -\sigma^2 & 0 \end{pmatrix}, \quad \gamma_M^2 \equiv \begin{pmatrix} i\sigma^3 & 0 \\ 0 & i\sigma^3 \end{pmatrix},$$
$$\gamma_M^3 \equiv \begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix}, \quad \gamma_M^4 \equiv \begin{pmatrix} -i\sigma^1 & 0 \\ 0 & -i\sigma^1 \end{pmatrix}. \tag{46}$$

By considering the unitary transformations

$$V = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \tag{47}$$

and

$$W = \begin{pmatrix} 1 & \sigma^2 \\ -\sigma^2 & 1 \end{pmatrix}, \tag{48}$$

we can determine different relations between γ_W^μ, γ_D^μ and γ_M^μ. In fact, a connection between γ_W^μ and γ_D^μ is given by $\gamma_D^\mu = V\gamma_W^\mu V^{-1}$. Also we have $\gamma_M^\mu = W\gamma_D^\mu W^{-1}$ and $\gamma_M^\mu = WV\gamma_W^\mu V^{-1}W^{-1}$. Moreover the corresponding spinors ψ_D, ψ_W and ψ_M are linked by $\psi_D = V\psi_W$, $\gamma_M^\mu = W\gamma_D^\mu W^{-1}$, $\psi_M = W\psi_D$ and $\psi_M = WV\psi_W$.

In $D = 8$ the $\gamma^{\hat{\mu}}$ in (42) are 16×16-matrices. This leads to column spinor with 16-complex components. The Majorana condition shall reduce number to just 16-real components and the Weyl condition shall reduce this number to just 8-real components, surprisingly the same number of complex components of the Dirac spinor in (1 + 3)-dimensions. This observation allows to suggest [9] that massless Majorana-Weyl fermion in (4 + 4)-dimensions is equivalent to massive fermion in (1 + 3)-dimensions.

In order to clarify this observation let us write the massless Dirac Equation (1) as

$$\left(\gamma^\mu \hat{p}_\mu + \gamma^a \hat{p}_a\right)\psi = 0. \tag{49}$$

Here, the terms $\gamma^\mu \hat{p}_\mu$ and $\gamma^a \hat{p}_a$ refer to (1 + 3)- and (3 + 1)-signature, respectively. Now, note that if $\gamma^a \hat{p}_a$ may determine a mass m_0 in the form

$$\gamma^a \hat{p}_a \psi - m_0\psi = 0, \tag{50}$$

then (49) becomes the massive Dirac equation in (1 + 3)-dimensions

$$\left(\gamma^\mu \hat{p}_\mu + m_0\right)\psi = 0. \tag{51}$$

This means that in the world of (1 + 3)-signature one has massive fermions. While according to (50) in the (3 + 1)-world we have tachyons, that is we discover that the in the mirow (3 + 1)-world we also have massive fermions: but with opposite signed mass. However, again since (42) is invariant under the change $\gamma^a \to -\gamma^a$ we learn that (50) describe a system with positive mass which seems to be unexpected result for tachyons. A possible mechanism to solve this problem is to add a new constraint

$$m_0 + m_0^* = 0, \tag{52}$$

where $m_0 > 0$ denotes the mass of a particle in the (1 + 3)-world and $m_0^* < 0$ refers to a system in the (3 + 1)-world. This means that we can apply the transform $\gamma^a \to -\gamma^a$ in (50) but according to (52) we reintroduce the mass m_0^* and we end up with the equation

$$\gamma^a \hat{p}_a \psi + m_0^* \psi = 0, \tag{53}$$

with m_0^* always satisfying the inequality $m_0^* < 0$.

Our goal now it is to see what are the consequences of introducing minimal coupling in our system of (4 + 4)-world. In this case the Dirac Equation (49) reads as

$$\left(\gamma^\mu \left(\hat{p}_\mu - eA_\mu\right) + \gamma^a \left(\hat{p}_a - e^* A_a\right)\right)\psi = 0. \tag{54}$$

Here, we try to associate with the (1 + 3)-world a particle with mass $m_0 > 0$ and charge $e < 0$ and with the (3 + 1)-world a particle with mass $m^* < 0$ and charge $e^* > 0$. Of course, just as the that masses of the two dual worlds satisfy the relation (52) we assume that the charges are forced to satisfy the relation

$$e + e^* = 0, \tag{55}$$

Note that both (52) and (55) are in agreement with the idea in the Euler characteristic expressed in (4) with 0 as the self-dual quantity.

The Equation (54) can be understood as massless

$$\text{particle} + \text{anti-particle} = \text{self-dual} \tag{56}$$

system, again in agreement with (4) in graph theory. Something even more interesting arises if in the Majorana representation for γ_M^μ we impose the Majorana reality condition

$$\psi = \bar{\psi}. \tag{57}$$

This is because taking the complex conjugate of (54) leads to

$$\left(-\bar{\gamma}_M^\mu \left(\hat{p}_\mu + eA_\mu\right) - \bar{\gamma}_M^a \left(\hat{p}_a + e^* A_a\right)\right)\bar{\psi} = 0. \tag{58}$$

But, in this case $\bar{\gamma}_M^\mu$ and $\bar{\gamma}_M^a$ are pure imaginary and therefore (58) becomes

$$\left(\gamma_M^\mu \left(\hat{p}_\mu + eA_\mu\right) + \gamma_M^a \left(\hat{p}_a + e^* A_a\right)\right)\bar{\psi} = 0. \tag{59}$$

Thus, if (57) it is satisfied then (59) yields to

$$\left(\gamma_M^\mu \left(\hat{p}_\mu + eA_\mu\right) + \gamma_M^a \left(\hat{p}_a + e^* A_a\right)\right)\psi = 0. \tag{60}$$

A comparison with (54), we learn that if we assume (55) we still have a consistent theory with $e \neq 0$ and therefore with $e^* \neq 0$. This must be compare with the usual case of the Dirac in (1 + 3)-dimensions in which the condition (57) implies that $e = 0$, meaning that Majorana fermions are uncharged.

5. Black-Holes in (4 + 4)-Dimensions

In terms of (4 + 4)-dimensions the flat line element reads as

$$\mathrm{d}\mathcal{S}^2 = \mathrm{d}\mathcal{S}_{(+)}^2 + \mathrm{d}\mathcal{S}_{(-)}^2 = \eta_{\mu\nu}^{(+)} \mathrm{d}x^\mu \mathrm{d}x^\nu + \eta_{ab}^{(-)} \mathrm{d}x^a \mathrm{d}x^b. \tag{61}$$

This is a theory formulated with $x^M = \left(x^\mu, x^a\right)$ coordinates corresponding to the double space $R^{2n} \times R^{2n}$. In (4 + 4)-dimensions flat background leads to (61). The relevant group in this case is $O(4,4)$ which is associated with the manifold R^8. It turns out that R^8 can be compactified in such a way that becomes the product $S^7 \times S^7$ modulo the group G_2. We may expect the transition $\eta_{MN} \to g_{MN}\left(x, \tilde{x}\right)$ in a curved spacetime of (4 + 4)-dimensions.

It may be helpful to make some remarks about the so-called Kruskal--Szekeres transformations of black-holes. In the case of the Schwarzschild solution in (1 + 3)-dimensions such transformation covers the entire spacetime manifold and are well-behaved everywhere outside the central physical singularity. In fact, we may consider that the Kruskal--Szekeres transformations provide with maximal extended Schwarzschild solution, giving an alternative description of the event horizon of a black-hole. In Ref. [23] it was shown that an extended Kruskal-Szekeres transformation implies a (4 + 4)-dimensional spacetime as predicted by the flat line element (61). In fact, we recall that a surprising result of the Kruskal-Szekeres transformation is that predicts four regions (4-region) instead of only two regions (2-region) as in the case of a Schwarzschild black-hole (interior and exterior regions, which are determined by the event horizon). However as it was shown in [23] more general Kruskal-Szekeres transformations describe 8-regions (instead of only 4-region), which can be associated with (4 + 4)-dimensions. Let us briefly explain this result. Such extended Kruskal-Szekeres transformations can be written as

$$\begin{aligned} X &= \varepsilon\left[\eta\left(\frac{r}{r_s}-1\right)\right]^{1/2} \mathrm{e}^{\frac{r}{2r_s}} \cosh\left(\frac{t}{2r_s}\right), \\ T &= \varepsilon\left[\eta\left(\frac{r}{r_s}-1\right)\right]^{1/2} \mathrm{e}^{\frac{r}{2r_s}} \sinh\left(\frac{t}{2r_s}\right), \end{aligned} \tag{62}$$

or in the alternative form

$$\begin{aligned} X &= \varepsilon\left[\eta\left(\frac{r}{r_s}-1\right)\right]^{1/2} \mathrm{e}^{\frac{r}{2R_s}} \sinh\left(\frac{t}{2r_s}\right), \\ T &= \varepsilon\left[\eta\left(\frac{r}{r_s}-1\right)\right]^{1/2} \mathrm{e}^{\frac{r}{2R_s}} \cosh\left(\frac{t}{2r_s}\right). \end{aligned} \tag{63}$$

Here, the quantities ε and η are parameters that take values in the set $\{\pm 1\}$. Notice that since $\varepsilon = \pm 1$ and $\eta = \pm 1$ the transformations (35) and (36) describe 8-regions, instead of 4-regions. We find that when one takes $\eta = \pm 1$ one must have values of r such that $r > r_s$, while when $\eta = -1$ one must have values of r such that $r < r_s$. This give us 4-regions structure; 2-regions with $\eta = +1$ and 2-regions for $\eta = -1$. Assuming now the two values $\varepsilon = 1$ and $\varepsilon = -1$ one ends up with 8-regions. Let us split these 8-regions in the form; 8-regions→4-regions and 4^*-regions. The 4-regions can be understood as the interior and exterior of a black-hole and also the interior and exterior of a white-hole, both living in a (1 + 3)-dimensional world, while the 4^*-regions can be understood as the interior and exterior of a mirror black-hole and the interior and exterior of a mirror white-hole living in (3 + 1)-dimensional world. This means that the total 8-regions can be associated with (4 + 4)-world.

Now a simply generalization of (61) is the curved space

$$\mathrm{d}\mathcal{S}^2 = \mathrm{d}\mathcal{S}^2_{(+)} + \mathrm{d}\mathcal{S}^2_{(-)} = g^{(+)}_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu + g^{(-)}_{ab}\mathrm{d}x^a \mathrm{d}x^b, \tag{64}$$

where the metrics $g^{(+)}_{\mu\nu} = e^{(+)(\alpha)}_\mu e^{(+)(\beta)}_\nu \eta^{(+)}_{\alpha\beta}$ and $g^{(-)}_{ab} = e^{(-)(c)}_a e^{(-)(d)}_b \eta^{(-)}_{ab}$ are written in terms of the tetrads $e^{(+)(\alpha)}_\mu$ and $e^{(-)(\alpha)}_a$, respectively. The simplest possibility is to assume that $g^{(+)}_{\mu\nu} = g^{(+)}_{\mu\nu}\left(x^\lambda\right)$ and $g^{(-)}_{ab} = g^{(-)}_{ab}\left(x^c\right)$ This means the (4 + 4)-metric $g_{AB} = g_{AB}\left(x^8\right)$ can be written in the form

$$g_{AB}\left(x^\lambda, x^c\right) = \begin{pmatrix} g^{(+)}_{\mu\nu}\left(x^\lambda\right) & 0 \\ 0 & g^{(-)}_{ab}\left(x^c\right) \end{pmatrix}. \tag{65}$$

However, a more general Kaluza-Klein type metric is

$$\gamma_{AB}\left(x^C\right) = g_{AB}\left(x^C\right) = \begin{pmatrix} g_{\mu\nu} + A^i_\mu A^j_\nu g_{ij} & A^j_\mu g_{ij} + g_{\mu\nu} B^\nu_i \\ A^k_\nu g_{kj} + g_{\nu\mu} B^\mu_j & g_{ij} + B^\mu_i B^\nu_j g_{\mu\nu} \end{pmatrix}. \tag{66}$$

(see Ref. [23] for details).

From the current experiments one knows that our world is a (1 + 3)-dimensional spacetime. But theoretically one may ask why is our world (1 + 3)-dimensional? As far as we know until now there is non satisfactory answer to this question. But at least we are proposing a consistent theoretical (4 + 4)-dimensional scenario with a more symmetric structure between space and time which eventually may give us a satisfactory answer to quantum gravity and dark matter. In fact, a clue in this direction comes from the result that massless Majorana-Weyl fermion in (4 + 4)-dimensions [20] can be considered as the electron in (1 + 3)-dimensions.

It is worth mentioning that other physical topics in which (4 + 4)-dimensional world is a central concept are oriented matroid theory and qubit theory (see Refs. [10]-[17] and references therein). In this context, division algebras and the Hopf maps are the mathematical notions that restrict the dimensionality of the space to the only values 1, 2, 4 and 8. It turns out that these dimensions are linked to the parallelizable spheres S^0, S^1, S^3 and S^7 which in turn are closely related to the real numbers, the complex numbers, the quaternions and the octonions, respectively.

6. Final Remarks: Dark Matter and the (4 + 4)-World?

One may ask whether the (4 + 4)-world is related to dark matter (see Refs. [1] [2] and references therein). As we saw in the previous section (4 + 4)-dimensional world is closely related to fermions and black-holes. So, if we are able to connect black-holes with dark matter we may provide a solution to our problem. In Ref. [8] it was assumed that a black-hole in the center of a spiral galaxy with associated mass M and a star with mass m in this galaxy satisfy the heuristic formulae

$$m+\frac{r_s}{r}m^*=0 \tag{67}$$

and

$$r^*=\frac{r_s^2}{r}. \tag{68}$$

Here, r denotes the start distance from the center of the black-hole and it is assumed that at the interior of the black-hole, determined by an event horizon at r_s, there is a tachyon of mass m^* at distance r^*. From these assumptions the energy relation

$$E=\frac{mv^2}{2}-\frac{GMm}{r}-\frac{Gm^2r_s}{2r^2}+\frac{Gm^2r_s}{2\left(r^2-r_s^2\right)} \tag{69}$$

can be obtained, which leads to the expression for the velocity v

$$v=\sqrt{\frac{2E}{m}+\frac{2GM}{r}+\frac{Gmr_s}{r^2}-\frac{Gmr_s}{r^2-r_s^2}}. \tag{70}$$

Surprisingly, a plot of v *vs* r of (70) leads to a graphic which resembles the rotation curve of spiral galaxies (see Ref. [8]). Such rotations curves have been studied from different routes, but perhaps the proposed by Matos and collaborators [1] [2], which is referred to as the so-called dark matter scalar field theory, is one of the most interesting. What is important for us, is that the idea behind (67) and (68) is precisely a (4 + 4)-dimensional world, in the sense that (1 + 3)-world is associated with the exterior of a black-hole and (3 + 1)-world is linked to a superluminal particle which lives in the interior of such black-hole. Notice that (67) can be understood as a generalization of duality relation (52). Moreover, observe that r and r^* are dual in the sense of product operation in (68).

Finally, it is worth mentioning an alternative Dirac equation for massive $\frac{1}{2}$-spin particles which may be interesting for further work. Let us first clarify that two spinors ψ_1 an ψ_2 in (4 + 4)-dimensions lead to 32 complex components. Thus, a Majorana condition must reduce to 32 real components. This is still too much to be associated with the let say with an electron. But we still have the freedom to impose in ψ_1 and ψ_2 the Weyl conditions which will reduce to only 16 real components which may be identified with 8 real components of the (1 + 3)-world and 8 real components of the (3 + 1)-world. Let us further clarify these comments.

Let us propose the two equations

$$\left(\gamma^\mu\left(\hat{p}_\mu-eA_\mu\right)+\gamma^a\left(\hat{p}_a-e^*A_a\right)\right)\psi_1=0 \tag{71}$$

and

$$\left(\gamma^\mu\left(\hat{p}_\mu+eA_\mu\right)+\gamma^a\left(\hat{p}_a+e^*A_a\right)\right)\psi_2=0. \tag{72}$$

Of course, following similar procedure as in section 3, we can prove that (71) and (72) are consistent with the condition (55) and therefore with $e \neq 0$. Now assume that

$$\gamma^{a}\left(\hat{p}_{a}-e^{*}A_{a}\right)\psi_{1}+m_{0}^{*}\psi_{2}=0 \tag{73}$$

and

$$\gamma^{a}\left(\hat{p}_{a}+e^{*}A_{a}\right)\psi_{2}+m_{0}^{*}\psi_{1}=0. \tag{74}$$

Thus, from (73) and (71) we find

$$\gamma^{\mu}\left(\hat{p}_{\mu}-eA_{\mu}\right)\psi_{1}+m_{0}\psi_{2}=0 \tag{75}$$

and from (74) and (72) we get

$$\gamma^{\mu}\left(\hat{p}_{\mu}+eA_{\mu}\right)\psi_{2}+m_{0}\psi_{1}=0, \tag{76}$$

where we require that the condition (52) is satisfied. If we now impose the Majorana conditions in ψ_1 and ψ_2, namely

$$C\bar{\psi}_{1}=\psi_{1} \tag{77}$$

and

$$C\bar{\psi}_{2}=\psi_{2}, \tag{78}$$

where C is the charge conjugation matrix such that

$$C\bar{\gamma}C^{-1}=-\gamma. \tag{79}$$

In the Majorana representation in which $C=I$ and, γ_{M}^{μ} and γ_{M}^{a} are pure imaginary, we have now the reality conditions

$$\psi_{1}=\bar{\psi}_{1} \tag{80}$$

and

$$\psi_{2}=\bar{\psi}_{2}. \tag{81}$$

But, this procedure applied to (72) and (75) implies that $e=0$. It turns out that in our case the constraints (80) and (81) no necessarily must be imposed in ψ_1 and ψ_2. Let us clarify this comment recalling that usually the Majorana condition is necessary in order to reduce the number of components of ψ_1 an ψ_2. As we mentioned before, the 32 complex components of ψ_1 and ψ_2 are reduced by imposing the Majorana and Weyl conditions reduce to only 16 real components which may be identified with 8 real components of the (1 + 3)-world and 8 real components of the (3 + 1)-world. But in the usual case, we get a system with $e=0$. However in our formalism things are different because in addition to the Elko Equations (75) and (76) we have the Elko Equations (73) and (74) of the dual-world. This suggested that non-longer its necessary to impose the Majorana conditions (80) and (81) in order to reduce the degrees of freedom of ψ_1 and ψ_2. This means that without imposing (80) and (81) we may still have a system with 4-complex components in the (1 + 3)-world and 4-complex components in the (3 + 1)-world, with particles with $m_0 \neq 0$ and $e \neq 0$, which in may be identified with the electron in (1 + 3)-world and dual electron in the (3 + 1)-world.

Of course, open issues that emerge from our proposal of (4 + 4)-world deals with possible causality violations. In the usual case of (1 + 3)-world, closed time-like curves violate in principle causality. However, although in a (4 + 4)-world can be a proliferation of four closed time-like curves this must be carefully analyzed since according to the view of this work the (3 + 1)-world cannot be ignored and therefore closed time-like curves in (1 + 3)-world must be dually compensated by corresponding time-like curves in (3 + 1)-world. We leave for further work a more detailed analysis of this interesting subject.

Acknowledgements

J. A. Nieto thanks the Mathematical Department of the Guadalajara University and the Mathematical, Computational & Modeling Sciences Center of the Arizona State University where part of this work was developed. This work was partially supported by PROFAPI-UAS/2013.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

References

[1] Martinez-Medina, L.A. and Matos, T. (2014) Scalar Field Dark Matter Mass Model and Evolution of Rotation Curves for Low Surface Brightness Galaxies. *Monthly Notices of the Royal Astronomical Society*, **444**, 185-191. https://doi.org/10.1093/mnras/stu1453

[2] Matos, T. and Urena-Lopez, L.A. (2001) Further Analysis of a Cosmological Model with Quintessence and Scalar Dark Matter. *Physical Review D*, **63**, Article ID: 063506.

[3] Strominger, A. (2009) Five Problems in Quantum Gravity. *Nuclear Physics B: Proceedings Supplements*, **192-193**, 119-125. https://doi.org/10.1016/j.nuclphysbps.2009.07.049

[4] Kiefer, C. (2013) Conceptual Problems in Quantum Gravity and Quantum Cosmology. *International Scholarly Research Notices*, **2013**, Article ID: 509316. https://doi.org/10.1155/2013/509316

[5] Rovelli, C. (2004) Quantum Gravity (Cambridge Monographs on Mathematical Physics). Cambridge University Press, Cambridge. https://doi.org/10.1017/CBO9780511755804

[6] Green, M., Schwarz, J. and Witten, E. (1987) Superstring Theory I and II. Cambridge University Press, Cambridge.

[7] Hull, C.M. (1998) Duality and the Signature of Space-Time. *Journal of High Energy Physics*, **11**, 17. https://doi.org/10.1088/1126-6708/1998/11/017

[8] Avila, C., Nieto-Marn, P.A. and Nieto, J.A. (2020) Towards Exterior/Interior Correspondence of Black-Holes. *International Journal of Geometric Methods in Modern Physics*, **17**, Article ID: 2050180. https://doi.org/10.1142/S0219887820501807

[9] Nieto, J.A. and Espinoza, M. (2016) Dirac Equation in Four Time and Four Space Dimensions. *International Journal of Geometric Methods in Modern Physics*, **14**, Article ID: 1750014. https://doi.org/10.1142/S0219887817500141

[10] Nieto, J.A. (2004) Matroids and P-Branes. *Advances in Theoretical and Mathemat-*

ical Physics, **8**, 177-188. https://doi.org/10.4310/ATMP.2004.v8.n1.a4

[11] Nieto, J.A. (2006) Oriented Matroid Theory as a Mathematical Framework for M-Theory. *Advances in Theoretical and Mathematical Physics*, **10**, 747-757. https://doi.org/10.4310/ATMP.2006.v10.n5.a5

[12] Nieto, J.A. (2004) Searching for a Connection between Matroid Theory and String Theory. *Journal of Mathematical Physics*, **45**, 285.

[13] Nieto, J.A. (2014) Phirotopes, Super p-Branes and Qubit Theory. *Nuclear Physics B*, **883**, 350-372. https://doi.org/10.1016/j.nuclphysb.2014.04.001

[14] Nieto, J.A. and Marn, M.C. (2000) Matroid Theory and Chern-Simons. *Journal of Mathematical Physics*, **41**, 7997.

[15] Nieto, J.A. (2013) Qubits and Oriented Matroids in Four Time and Four Space Dimensions. *Physics Letters B*, **718**, 1543-1547. https://doi.org/10.1016/j.physletb.2012.12.034

[16] Nieto, J.A. (2010) Qubits and Chirotopes. *Physics Letters B*, **692**, 43-46. https://doi.org/10.1016/j.physletb.2010.07.010

[17] Nieto, J.A. (2018) Cosmological Duality in Four Time and Four Space Dimensions. *Frontiers in Physics*, **6**, Article No. 106.

[18] Whitney, H. (1935) On the Abstract Properties of Linear Dependence. *American Journal of Mathematics*, **57**, 509-533. https://doi.org/10.2307/2371182

[19] Monk, D. (1973) Introduction to Graph Theory, (Oliver and Boyd). *The Mathematical Gazette*, **57**, 348-348. https://doi.org/10.2307/3616070

[20] Nieto, J.A. (2005) Towards an Ashtekar formalism in eight dimensions. *Classical and Quantum Gravity*, **22**, 947-955. https://doi.org/10.1088/0264-9381/22/6/004

[21] Nieto, J.A. (2016) Alternative Self-Dual Gravity in Eight Dimensions. *Modern Physics Letters A*, **31**, Article ID: 1650147. https://doi.org/10.1142/S0217732316501479

[22] Dirac, P.A.M. (1928) The Quantum Theory of the Electron. *Proceedings of the Royal Society A*, **117**, 610-624. https://doi.org/10.1098/rspa.1928.0023

[23] Nieto, J.A. and Madriz-Aguilar, J.E. (2019) Aspects of (4+4)-Kaluza-Klein Type Theory. *Physica Scripta*, **94**, Article ID: 115303. https://doi.org/10.1088/1402-4896/ab2d96