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Generalized Trigonometric Power Sums Covering the Full Circle

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Abstract

The analytical calculation of the area moments of inertia used for special mechanical tests in materials science and further generalizations for moments of different orders and broader symmetry properties has led to a new type of trigonometric power sums. The corresponding generalized equations are presented, proven, and their characteristics discussed. Although the power sums have a basic form, their results have quite different properties, dependent on the values of the free parameters used. From these equations, a large variety of power reduction formulas can be derived. This is shown by some examples.

Keywords

Trigonometric Power Sum, Power Reduction Formula, Trigonometric Identity, Central Binomial Coefficient

1. Introduction

The original motivation, which ultimately led to this paper, was not based on pure mathematics, but had a technical background within the field of materials research. In order to obtain physical quantities, such as the elastic modulus and strength of materials by means of mechanical bending tests, the second (area) moment of inertia is needed, which depends on the shape of the cross section of the bending bars.

The used samples with special cross sections of fourfold rotational symmetry [1] allowed for the calculation by analytical integration. The next question was how the results would change if the cross section was rotated by an arbitrary angle, which led to the introduction of a phase shift in the arguments of the found trigonometric functions. Further expanding the range of validity of the equations for threefold and n-fold rotational symmetry and then to moments of higher order resulted in a step-by-step generalization of the formulas. Thus, the initial motivation was not the creation of generalized trigonometric power sums, but rather to learn more about certain physical moments. The new equations were found more or less accidentally. A detailed description of this development process is published as a technical report at the Hamburg University of Technology (TUHH) [2]. Since the power sums were not the initial aim of the report, they were proven only for some special cases. In this paper, we present the generalized equations in a different, compact form and provide detailed proof.

In the last decades, finite trigonometric power sums have become a subject of major interest. A brief overview is given, e.g., in a recent paper by da Fonseca *et al.* [3]. The main subject of their paper is new power sums concerning the sine and the cosine functions. Because we refer to these equations below, the main formulas in that paper are provided here.

With

$$S'(m,n) := \sum_{k=0}^{n-1} \sin^{2m}\left(\frac{k\pi}{n}\right)$$

and

$$C'(m,n) := \sum_{k=0}^{n-1} \cos^{2m}\left(\frac{k\pi}{n}\right),$$

da Fonseca *et al.* proved that

$$S'(m,n) = \begin{cases} 2^{1-2m} n \left(\binom{2m-1}{m-1} + \sum_{k=1}^{\lfloor m/n \rfloor} (-1)^{kn} \binom{2m}{m-kn} \right), & m \geq n, \\ 2^{1-2m} n \binom{2m-1}{m-1}, & m < n, \end{cases} \tag{1}$$

and

$$C'(m,n) = \begin{cases} 2^{1-2m} n \left(\binom{2m-1}{m-1} + \sum_{k=1}^{\lfloor m/n \rfloor} \binom{2m}{m-kn} \right), & m \geq n, \\ 2^{1-2m} n \binom{2m-1}{m-1}, & m < n, \end{cases} \tag{2}$$

with m and n being positive integers. The authors also noted that the results of the second sum (Equation (2)) were released in two former publications ([4]: Eq. 18.1.5; [5]: No. 4.4.2.11) using a similar representation. The apostrophes at S' and C' are added to distinguish these terms from the functions given in the main theorem below. As these sums comprise only half of the round angle, we have

$$S'(m,n) = \sum_{k=0}^{n-1} \sin^{2m}\left(\frac{k\pi}{n}\right) = \sum_{k=n}^{2n-1} \sin^{2m}\left(\frac{k\pi}{n}\right). \tag{3}$$

Adding the two sums yields

$$2S'(m,n) = \sum_{k=0}^{2n-1} \sin^{2m}\left(\frac{k\pi}{n}\right) = \sum_{k=1}^{N} \sin^{p}\left(\frac{2\pi k}{N}\right). \tag{4}$$

On the right, the exponent $2m$ is replaced by p and $2n$ by N. The shift of the summation bounds by 1 is allowed because the summands of $k = 0$ and $k = N$ are identical (both are zero). The right side of Equation (4) divided by 2 is equal to $S'(m,n)$, if N and p are both even. We now generalize Equations (1) and (2) by also allowing p and N to be odd numbers. Furthermore, we add a phase shift x to the argument of the trigonometric functions leading to the main theorem of this article. In fact, the full angle case together with an odd exponent p and even N yields zero if using Equation (4). Nevertheless, this is no longer valid for odd N when including the phase shift.

2. Main Result

The following theorem shows the main result of this paper, where the integer N is replaced by n:

Theorem 1. *Let p and n be positive integers and x any real number in the following trigonometric power sums*:

$$S(p,n,x) := \sum_{k=1}^{n} \sin^p\left(x + \frac{2\pi k}{n}\right)$$

and

$$C(p,n,x) := \sum_{k=1}^{n} \cos^p\left(x + \frac{2\pi k}{n}\right);$$

we then have

$$S(p,n,x) = \begin{cases} \dfrac{n}{2^p}\dbinom{p}{\frac{p}{2}} + \dfrac{n}{2^{p-1}} \displaystyle\sum_{m=0}^{p/2-1} s(-1)^{\frac{p}{2}-m}\binom{p}{m}\cos\left(\left(p-2m\right)x\right), & p \text{ even}, \\ \dfrac{n}{2^{p-1}} \displaystyle\sum_{m=0}^{(p-1)/2} s(-1)^{\frac{p-1}{2}-m}\binom{p}{m}\sin\left(\left(p-2m\right)x\right), & p \text{ odd}, \end{cases} \tag{5}$$

and

$$C(p,n,x) = \begin{cases} \dfrac{n}{2^p}\dbinom{p}{\frac{p}{2}} + \dfrac{n}{2^{p-1}} \displaystyle\sum_{m=0}^{p/2-1} s\binom{p}{m}\cos\left(\left(p-2m\right)x\right), & p \text{ even}, \\ \dfrac{n}{2^{p-1}} \displaystyle\sum_{m=0}^{(p-1)/2} s\binom{p}{m}\cos\left(\left(p-2m\right)x\right), & p \text{ odd}, \end{cases} \tag{6}$$

with $s = \delta_{(p-2m) \bmod n, 0}$, *in which* δ *is the Kronecker delta, and mod denotes the modulo operation.*

Remark 1. In the special case of p even, n even, and $x = 0$, $S(p, n, 0)$ and $C(p, n, 0)$ are equivalent to $S'(m, n)$ and $C'(m, n)$ in Equations (1) and (2), respectively, as stated above. Furthermore, Equation (5) for even p is equivalent to Equation (C10) in [2], with only the factor s calculated differently.

Proof. With Euler's formula, implying $\sin x = \left(e^{ix} - e^{-ix}\right)/(2i)$ and $\cos x = \left(e^{ix} + e^{-ix}\right)/2$, we find the following well-known identities [6]:

$$\sin^p x = \begin{cases} \frac{1}{2^p}\binom{p}{\frac{p}{2}} + \frac{1}{2^{p-1}}\sum_{m=0}^{p/2-1}(-1)^{\frac{p}{2}-m}\binom{p}{m}\cos\left((p-2m)x\right), & p \text{ even}, \\ \frac{1}{2^{p-1}}\sum_{m=0}^{(p-1)/2}(-1)^{\frac{p-1}{2}-m}\binom{p}{m}\sin\left((p-2m)x\right), & p \text{ odd}, \end{cases} \tag{7}$$

$$\cos^p x = \begin{cases} \frac{1}{2^p}\binom{p}{\frac{p}{2}} + \frac{1}{2^{p-1}}\sum_{m=0}^{p/2-1}\binom{p}{m}\cos\left((p-2m)x\right), & p \text{ even}, \\ \frac{1}{2^{p-1}}\sum_{m=0}^{(p-1)/2}\binom{p}{m}\cos\left((p-2m)x\right), & p \text{ odd}. \end{cases} \tag{8}$$

If we compare the right sides of Equations (5) and (6) with those of Equations (7) and (8), they appear very similar if the Kronecker delta s is ignored. For the derivation of Equations (5) and (6), we must distinguish two different cases concerning the fraction $(p - 2m)/n$, which means that the fraction can be an integer or not. This will become clear below.

Starting with Equation (7) and p even, we replace x by $x + 2\pi k/n$ and perform the summation $k = 1, \cdots, n$ on both sides. By shifting the summation over k to the right, we obtain

$$\begin{aligned} &\sum_{k=1}^{n}\sin^p\left(x+\frac{2\pi k}{n}\right) \\ &= \sum_{k=1}^{n}\left[\frac{1}{2^p}\binom{p}{\frac{p}{2}} + \frac{1}{2^{p-1}}\sum_{m=0}^{p/2-1}\left((-1)^{p/2-m}\binom{p}{m}\cos\left((p-2m)\left(x+\frac{2\pi k}{n}\right)\right)\right)\right] \\ &= \frac{n}{2^p}\binom{p}{\frac{p}{2}} + \frac{1}{2^{p-1}}\sum_{m=0}^{p/2-1}\left((-1)^{p/2-m}\binom{p}{m}\sum_{k=1}^{n}\cos\left((p-2m)\left(x+\frac{2\pi k}{n}\right)\right)\right). \end{aligned} \tag{9}$$

By applying an addition theorem, the last sum over k on the right side becomes

$$\begin{aligned} &\sum_{k=1}^{n}\cos\left((p-2m)x+(p-2m)\frac{2\pi k}{n}\right) \\ &= \cos\left((p-2m)x\right)\sum_{k=1}^{n}\cos\left((p-2m)\frac{2\pi k}{n}\right) \\ &\quad -\sin\left((p-2m)x\right)\sum_{k=1}^{n}\sin\left((p-2m)\frac{2\pi k}{n}\right). \end{aligned} \tag{10}$$

The arguments of the cosine and sine functions in the sums on the right side can be written as $k\theta$, where

$$\theta = \frac{p-2m}{n}\cdot 2\pi. \tag{11}$$

Thus, if $(p - 2m)/n$ is an integer, θ is an integral multiple of 2π, all of the cosine terms in the first sum on the right in Equation (10) are 1, and the sum is n. In the second sum, all the sine terms vanish. It follows that on the right side of Equation (9) the sum over k equals $n\cdot\cos((p - 2m)x)$. The number n can be shifted to the front of the sum over m and, thus, if we assume $s = 1$, the corresponding m-summands in Equations (9) and (5) for even p are identical. This is valid because the Kronecker delta s becomes 1 if $(p - 2m)/n$ is an integer.

In the other case of $(p - 2m)/n$ not being integer, equivalent to $s = 0$, we use the following two trigonometric identities from J. L. Lagrange, valid for positive integer n and $\theta \not\equiv 0 \bmod 2\pi$ [7]:

$$\sum_{k=1}^{n} \sin(k\theta) = \frac{1}{2}\cot\frac{\theta}{2} - \frac{\cos((n+1/2)\theta)}{2\sin(\theta/2)} \quad (12)$$

and

$$\sum_{k=1}^{n} \cos(k\theta) = -\frac{1}{2} + \frac{\sin((n+1/2)\theta)}{2\sin(\theta/2)}. \quad (13)$$

In order to evaluate the first sum on the right side of Equation (10), we expand the bracket $(n + 1/2)$ in Equation (13), replace θ by means of Equation (11), and find

$$\begin{aligned} &\sum_{k=1}^{n} \cos\left((p-2m)\frac{2\pi k}{n}\right) = -\frac{1}{2} + \frac{\sin(n\theta + (1/2)\theta)}{2\sin(\theta/2)} \\ &= -\frac{1}{2} + \frac{\sin(n(p-2m)2\pi/n + (1/2)(p-2m)2\pi/n)}{2\sin((p-2m)\pi/n)} = 0. \end{aligned} \quad (14)$$

The first summand in the argument of the sine function in the large numerator can be omitted because the factor $(p - 2m)$ is always integer, implying that the whole sum becomes zero. For the second sum on the right of Equation (10), we correspondingly obtain with Equation (12)

$$\begin{aligned} &\sum_{k=1}^{n} \sin\left((p-2m)\frac{2\pi k}{n}\right) = \frac{1}{2}\cot\frac{\theta}{2} - \frac{\cos(n\theta + (1/2)\theta)}{2\sin(\theta/2)} \\ &= \frac{1}{2}\cot\left(\frac{p-2m}{2n}2\pi\right) - \frac{\cos(n(p-2m)2\pi/n + (1/2)(p-2m)2\pi/n)}{2\sin((p-2m)\pi/n)} = 0. \end{aligned} \quad (15)$$

It follows that Equation (10) and the sum over k on the right side of Equation (9) yield zero, and, thus, the respective m-summands of Equations (9) and (5) (p even) both vanish. Because Equation (5) now holds for $s = 1$ and $s = 0$, it is generally valid for even p with s being the given Kronecker delta.

For even p, Equation (6) can be proven analogously on the basis of Equation (8). However, there is another simple argument for the correctness of Equation (6) for even p, as follows: Since x can be any real angle, we apply $\sin(x + \pi/2) = \cos x$ and, thus, by adding $\pi/2$ to x in Equation (5), the left side changes as desired. The cosine on the right side of Equation (5), after replacing x by $x + \pi/2$, can be modified in the following way:

$$\begin{aligned} &\cos((p-2m)(x+\pi/2)) = \cos((p-2m)x + (p-2m)\pi/2) \\ &= \cos((p-2m)x)\cos((p-2m)\pi/2) - \sin((p-2m)x)\sin((p-2m)\pi/2) \\ &= \cos((p-2m)x)\cdot(-1)^{p/2-m}. \end{aligned} \quad (16)$$

Because p and $p - 2m$ are even, the last sine term vanishes and the second cosine term in the same line is 1 or −1 and can be replaced by $(-1)^{p/2-m}$. When we further replace the cosine term in Equation (5) by the result of Equation (16), the product of the two identical factors $(-1)^{p/2-m}$ in Equation (5) is 1 and we are left with Equation (6) for even p. Accordingly, Equations (5) and (6) can be proven for odd p by starting again with Equations (7) and (8), respectively. This part does not contain any further difficulty and completes the proof.

Remark 2. It should be mentioned that with the conventions $0^0 = 1$, $\binom{0}{0} = 1$, and $\sum_{m=0}^{-1}(\ldots) = 0$, Equations (5) and (6) are even valid for $p = 0$, and the result is $S(0,n,x) = C(0,n,x) = n$.

Remark 3. The assumption that $(p - 2m)/n$ is an integer ($s = 1$) means that the angle θ in Lagrange's trigonometric identities would be an integral multiple of 2π, as mentioned previously (see Equation (11)). It follows that, on the right side of Equations (12) and (15), we would obtain an expression $-\infty + \infty$, and in the second fraction on the right side of Equations (13) and (14), a term 0/0. Although both cases are, by definition, not allowed, they are removable discontinuities and can be eliminated by replacing θ by $\theta + \varepsilon$ in either case and evaluating the limit $\varepsilon \to 0$. However, this procedure is not necessary because the case $s = 1$ has already been proven.

3. Discussion

The equations are universal in their nature because they comprise the proof of other corresponding theorems previously published. It is interesting that although in Equations (5) and (6) we have a summation on the left side, which does not exist in Equations (7) and (8), in many cases, the results of Equations (5) and (6) are much simpler than those of Equations (7) and (8). This has to do with the Kronecker delta.

We do not necessarily need to implement the value s. In [2], Equations (C5)-(C8) are equivalent to the given Equations (5) and (6), but are presented in a different form. Rather than using s, the number m is replaced by a function of p, N, and a summation index j, and the summation bounds are given as functions of p and N (In the report, we used the character N instead of n). In this way, the summation comprises only the terms of $s = 1$. Either way, this representation yields the same results and appears more complicated than the formulas given here.

For each combination of n and p, a power reduction formula can be deduced from the corresponding equation, in which only x is kept as a variable. In the case of higher values of n and p, for most summands, s is zero, implying that the number of terms on the right side of Equations (5) and (6) is usually less than the number of summands on the left side. Three examples are provided below.

For $n = 1$ and $p = 3$, we obtain the known equality [8]:

$$\sin^3 x = \frac{1}{4}\left(3\sin x - \sin(3x)\right). \tag{17}$$

For the next example of $n = 7$ and $p = 9$, we find

$$\sum_{k=1}^{7} \sin^9\left(x + \frac{2\pi k}{7}\right) = -\frac{63}{256}\sin(7x). \tag{18}$$

A visualization of this example with an arbitrary angle of $x = 20^\circ$ is provided in **Figure 1**. If we replace the terms in this sum by means of Equation (7), Equation (18) becomes a double sum and has $7 \times 5 = 35$ summands which are not zero, whereas the final result contains only one term. The third example of $n = 16$ and $p = 6$ with 64 corresponding summands does not depend on x and has the same result after replacing sine by cosine:

$$\sum_{k=1}^{16} \sin^6\left(x + \frac{2\pi k}{16}\right) = 5. \tag{19}$$

In many cases, the results of Equations (5) and (6) are independent of x. For an overview, these results are plotted as an n-p matrix in **Table 1**, revealing a relatively complex pattern.

The pattern of the white fields with fixed numbers and the gray fields with variable results is determined by the properties of n and p, being odd or even and specifying the Kronecker delta s. In the white fields, all s values are zero, and in the var. positions, one or more is 1. For odd p and $x = 0$, all the results are zero only in the case of the sine function (Equation (5)), whereas Equation (6) yields nonzero numbers in the gray fields also for $x = 0$. The gray fields with $n = 4, 8, 12, \ldots$, even p, and $p \geq n$ are marked by an asterisk (*). Here, the results are identical for both equations (5) and (6) if the same phase shift x is used. The reason for this is that the angles can be split into groups of four angles, each creating a square on the circle in **Figure 1**. These angles are separated by 90˚ and because of $\sin(x + 90^\circ) = \cos x$, the results are the same.

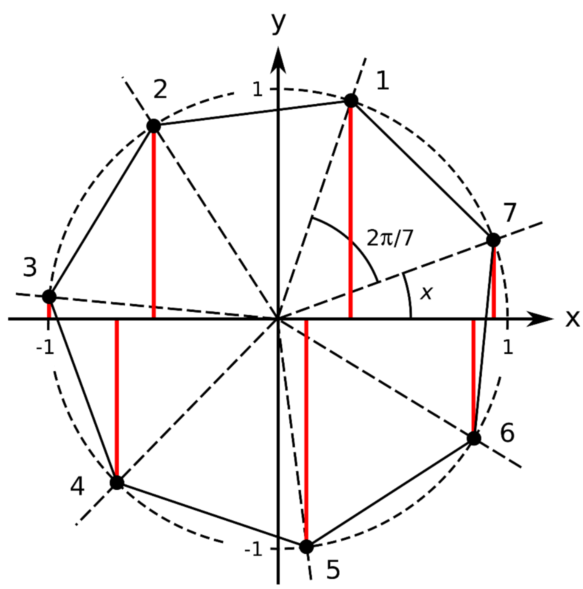

Figure 1. Points on the unit circle with sevenfold rotational symmetry. The vertical (red) lines represent the sine values of the angles according to $x = 20^\circ$ (= 0.349...). The numbers indicate the k-index.

Table 1. Results of Equations (5) and (6). The results in the white fields do not depend on x and on the equation used (sine or cosine). The results in the gray fields are variable (var.) and dependent on x. The asterisk (*) indicates results that depend on x but are the same for both Equations (5) and (6). Fixed positive numbers exist only to the right of the bold (red) polygon ($p < 2n$). For a clearer picture of the principles, the fractions are not reduced (This table corresponds to Table 3 in [2]).

	$n = 1$	2	3	4	5	6	7	8	9	10	11	12	13
$p = 1$	var.	0	0	0	0	0	0	0	0	0	0	0	0
2	var.	var.	$\frac{3}{2}$	$\frac{4}{2}$	$\frac{5}{2}$	$\frac{6}{2}$	$\frac{7}{2}$	$\boldsymbol{\frac{n}{2}}$	$\rightarrow$				
3	var.	0	var.	0	0	0	0	0	0	0	0	0	0
4	var.	var.	$\frac{9}{8}$	var.*	$\frac{15}{8}$	$\frac{18}{8}$	$\frac{21}{8}$	$\frac{24}{8}$	$\boldsymbol{\frac{3n}{8}}$	$\rightarrow$			
5	var.	0	var.	0	var.	0	0	0	0	0	0	0	0
6	var.	var.	var.	var.*	$\frac{50}{32}$	var.	$\frac{70}{32}$	$\frac{80}{32}$	$\frac{90}{32}$	$\boldsymbol{\frac{10n}{32}}$	$\rightarrow$		
7	var.	0	var.	0	var.	0	var.	0	0	0	0	0	0
8	var.	var.	var.	var.*	$\frac{175}{128}$	var.	$\frac{245}{128}$	var.*	$\frac{315}{128}$	$\frac{350}{128}$	$\boldsymbol{\frac{35n}{128}}$	$\rightarrow$	
9	var.	0	var.	0	var.	0	var.	0	var.	0	0	0	0
10	var.	var.	var.	var.*	var.	var.	$\frac{882}{512}$	var.*	$\frac{1134}{512}$	var.	$\frac{1386}{512}$	$\boldsymbol{\frac{126n}{512}}$	$\rightarrow$
11	var.	0	var.	0	var.	0	var.	0	var.	0	var.	0	0
12	var.	var.	var.	var.*	var.	var.	$\frac{3234}{2048}$	var.*	$\frac{4158}{2048}$	var.	$\frac{5082}{2048}$	var.*	$\boldsymbol{\frac{462n}{2048}}$
13	var.	0	var.	0	var.	0	var.	0	var.	0	var.	0	var.
14	var.	var.	var.	var.*	var.	var.	var.	var.*	$\frac{15444}{8192}$	var.	$\frac{18876}{8192}$	var.*	$\frac{1716n}{8192}$

In principle, the separation according to p even and p odd in Equations (5) and (6) can be avoided by introducing a second Kronecker delta q. This leads to:

$$S(p,n,x) = \frac{n}{2^p}\left(q\binom{p}{\frac{p}{2}} + 2\sum_{m=0}^{\lfloor (p-1)/2 \rfloor} s(-1)^{\lfloor p/2 \rfloor - m}\binom{p}{m}\sin\left((p-2m)x + \frac{q\pi}{2}\right)\right) \tag{20}$$

$$C(p,n,x) = \frac{n}{2^p}\left(q\binom{p}{\frac{p}{2}} + 2\sum_{m=0}^{\lfloor (p-1)/2 \rfloor} s\binom{p}{m}\cos\left((p-2m)x\right)\right) \tag{21}$$

with $q = \delta_{p \bmod 2, 0}$ and $s = \delta_{(p-2m) \bmod n, 0}$.

Anyway, this form does not necessarily appear more convenient than Equations (5) and (6), and should only be seen as an alternative representation. An obvious simplification can be obtained for $x = 0$ (or more precisely $x \equiv 0 \mod 2\pi$), which eliminates the trigonometric functions on the right sides of Equations (5) and (6) and reduces these formulas to:

$$\sum_{k=1}^{n} \sin^{p}\left(\frac{2\pi k}{n}\right) = \begin{cases} \frac{n}{2^{p}}\binom{p}{\frac{p}{2}} + \frac{n}{2^{p-1}} \sum_{m=0}^{p/2-1} s\left(-1\right)^{\frac{p}{2}-m}\binom{p}{m}, & p \text{ even}, \\ 0, & p \text{ odd}, \end{cases} \tag{22}$$

and

$$\sum_{k=1}^{n} \cos^{p}\left(\frac{2\pi k}{n}\right) = \begin{cases} \frac{n}{2^{p}}\binom{p}{\frac{p}{2}} + \frac{n}{2^{p-1}} \sum_{m=0}^{p/2-1} s\binom{p}{m}, & p \text{ even}, \\ \frac{n}{2^{p-1}} \sum_{m=0}^{(p-1)/2} s\binom{p}{m}, & p \text{ odd}, \end{cases} \tag{23}$$

with s as given above.

As a brief supplement, a special new equation is added. When trying to find a solution for the power sums, the following formula was found concerning the central binomial coefficient used in Equations (5) and (6). With the conventions $\prod_{k=2}^{1}(\ldots) = 1$ and $\sum_{j=2}^{1}(\ldots) = 0$, the equation is valid for any positive integer n. The proof is provided in ([2]: Appendix A), and we have

$$\binom{2n}{n} = 2\prod_{k=2}^{n}\left(3 + \sum_{j=2}^{k-1}\left(\sum_{i=1}^{j} i\right)^{-1}\right). \tag{24}$$

4. Summary

Two generalized trigonometric power sums (Equations (5) and (6)) are presented that can be illustrated by a circle with n points distributed regularly around the circumference. The key feature of the corresponding proof is the fact that two different cases concerning the fraction $(p - 2m)/n$ must be considered. If the parameter space is restricted to even n, even p, and $x = 0$, the equations are equivalent to the power sums recently published by da Fonseca *et al.* The new generalized formulas can be used, e.g., to check and derive similar equations. By inserting natural numbers for n and p and leaving x variable, a wide class of power reduction formulas can be created. Although these power reduction formulas as well as the power sums $S(p, n, x)$ and $C(p, n, x)$ can contain a large number of terms, in most cases the result is relatively simple. It is also interesting that we do not find similar properties for all the parameter combinations. Instead, the results show a complex but well-ordered pattern (Table 1).

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Availability of Data and Material

Supplemental data/information is published in a technical report [2] at the Hamburg University of Technology (open access).

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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