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Hyperbolic Reflections Leading to the Digits of ln(2)

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Abstract

We analyze a problem of interactions between elements of an ideal system which consists of two point masses and a wall in a hyperbolic setting. Thanks to a change of variables, the problem is reduced to a sequence of reflections on a hyperbola. For specific ratios of the two masses, the number of interactions is related to the first numerical digits of the logarithmic constant $\ln(2)$.

Keywords

Dynamical System, Hyperbolic Setting, Reflection, Rotation, Digits, Logarithmic Constant

1. Introduction

In [1], the author proposed the computation of the first digits of π by counting the number of collisions of a system consisting of two balls and a wall. Extensive analysis of this problem has been done, see [2] and references therein.

In the present work, we analyze a similar formulation for a hypothetical hyperbolic situation. Our analysis is based on an original decomposition of the velocity for given values of hyperbolic kinetic energy and hyperbolic momentum. Thanks to a useful transformation, the basic geometry of the problem is reduced to reflections on a hyperbola. It follows that a sequence of interactions becomes a sequence of reflections. Then, for the possible sequences of interactions, it is easy to determine the end of the process and the total number of interactions. Finally for two particular ratios of the masses, the analysis leads to the computation of the first digits of any logarithmic constant $\ln(p)$ (where p is a prime number) [3].

2. Ideal Physical System

Suppose two points of masses M and m ($M > m$), noted point$_M$ and point$_m$, interacting between each other in one dimension. Let us consider their respective speeds V and v. Suppose also that the point of mass m could interact with a wall. Two quantities will be considered: the h-momentum

$$Q = MV - mv,$$

and the h-kinetic energy

$$E_c = \frac{E}{2} = \frac{1}{2}\left(MV^2 - mv^2\right).$$

Let the matrix H and J be defined by

$$H = \begin{pmatrix} M & 0 \\ 0 & -m \end{pmatrix} \text{ and } J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The h-momentum of the two point mass system can be written as

$$Q = \begin{cases} \begin{pmatrix} V & v \end{pmatrix} H \begin{pmatrix} 1 \\ 1 \end{pmatrix} = (V, v) \bullet_H (1,1), \\ \begin{pmatrix} V & v \end{pmatrix} J \begin{pmatrix} M \\ m \end{pmatrix} = (V, v) \bullet_J (M, m), \end{cases}$$

and velocities (V, v) having the same h-momentum are on a straight line with direction (m, M) because $(m, M) \bullet_H (1,1) = 0$. The h-kinetic energy E_c of the two point mass system is such that

$$E = \begin{cases} (V, v) \bullet_H (V, v) = \left\|(V, v)\right\|_H^2, \\ \left(\sqrt{M}V, \sqrt{m}v\right) \bullet_J \left(\sqrt{M}V, \sqrt{m}v\right) = \left\|\left(\sqrt{M}V, \sqrt{m}v\right)\right\|_J^2, \end{cases}$$

and velocities (V, v) having the same h-kinetic energy are on a hyperbola.

The direction $\left(\sqrt{M}, \sqrt{m}\right)$ will play a special role in the sequel. Let us note α the hyperbolic angle of this direction with a horizontal axis OV, so we have

$$\begin{cases} \cosh(\alpha) = \sqrt{\dfrac{M}{M-m}}, \\ \sinh(\alpha) = \sqrt{\dfrac{m}{M-m}}. \end{cases}$$

We will consider two kinds of interactions. The first kind will be between the two points of masses M and m. It will be called h-elastic interactions which means that both h-momentum and h-kinetic energy remain constant. The second kind will be between the point of mass m and the wall. It will produce sign changes of the velocity of the point of mass m, so the h-momentum of the system changes while keeping constant its h-kinetic energy.

By following rules that ensure the alternation of the two kinds of interactions, we will consider the dynamics of the two point masses. We will look at the total number of interactions, counting interactions between the two point masses and interactions between a point mass and the wall. We will see that under certain conditions, the number of interactions corresponds to the first digits of $\ln(2)$.

3. Direct Analysis: The Natural Coordinate System

In this section, we analyze the system with respect to its natural coordinate system *VOv*.

3.1. Observations

Let the velocity be given by

$$(V,v)=\left(\frac{E}{M-m}\right)^{1/2}\left(\frac{\cosh(\phi)}{\cosh(\alpha)},\frac{\sinh(\phi)}{\sinh(\alpha)}\right)$$

with $\phi\in\mathbb{R}$, such that $MV^2-mv^2=E$. Let us point out that $V>0$. The rule of the process is such that if

1) $\phi\in(-\infty,0)$, so $v<0$, there will be a point$_m$-wall interaction;
2) $\phi\in[0,\alpha]$, so $0\le v\le V$, there will be no interaction;
3) $\phi\in(\alpha,+\infty)$, so $0<V<v$, there will be a point$_M$-point$_m$ interaction.

Let us observe that for $\phi=\alpha$

$$(V,v)=\left(\frac{E}{M-m}\right)^{1/2}(1,1).$$

3.2. Two-Point Mass System

For the point$_M$-point$_m$ interaction, the velocity (V,v) of constant h-kinetic energy and constant h-momentum lends itself well to a decomposition. The possible velocities in this case are given by the intersection points of a hyperbola (h-kinetic energy) and a line (h-momentum). There are no more than two intersection points. This decomposition will be helpful to explain the transformation of the velocity during an interaction between the two point masses.

3.2.1. Decomposition of the Velocity

We will break down the velocity (V,v) using a h-orthogonal basis.

Theorem 1 *The set* $\{(1,1),(m,M)\}$ *is a h-orthogonal basis with respect to the quadratic form used to define the hyperbola of constant h-kinetic energy.*

Proof. Indeed we have

$$(m,M)\bullet_H(1,1)=0.$$

Moreover

$$\|(1,1)\|_H^2=(1,1)\bullet_H(1,1)=M-m,$$

and

$$\|(m,M)\|_H^2=(m,M)\bullet_H(m,M)=-Mm(M-m).$$

We can now decompose the velocity (V,v) as follows.

Theorem 2 *The velocity* (V,v) *can be written as*

$$\begin{pmatrix}V\\v\end{pmatrix}=R\begin{pmatrix}1\\1\end{pmatrix}+S\begin{pmatrix}m\\M\end{pmatrix}=\begin{pmatrix}1&m\\1&M\end{pmatrix}\begin{pmatrix}R\\S\end{pmatrix}$$

where

$$R=\frac{(1,1)\bullet_H(V,v)}{(1,1)\bullet_H(1,1)}=\frac{MV-mv}{M-m}=\frac{Q}{M-m},$$

and

$$S=\frac{(m,M)\bullet_H(V,v)}{(m,M)\bullet_H(m,M)}=-\frac{V-v}{M-m}.$$

Corollary 1 *We have*

$$V-v=\frac{1}{Mm}(m,M)\bullet_H(V,v),$$

so

$$V\begin{Bmatrix}>\\=\\<\end{Bmatrix}v\quad\text{if and only if}\quad(m,M)\bullet_H(V,v)\begin{Bmatrix}>\\=\\<\end{Bmatrix}0.$$

Thanks to this decomposition of (V,v), we will see that a point$_M$-point$_m$ interaction consists simply in making a change of sign of the coefficient S in this decomposition.

3.2.2. Compatibility Condition

Using the hyperbolic Cauchy-Bunyakovski-Schwarz inequality [4], we get

$$\begin{aligned}Q^2&=\left[(V,v)\bullet_H(1,1)\right]^2\\&\geq\|(V,v)\|_H^2\,\|(1,1)\|_H^2\\&=E(M-m).\end{aligned}$$

A more precise expression is given in the next theorem.

Theorem 3 *The h-kinetic energy and the h-momentum of the point$_M$-point$_m$ system are related by the relation*

$$E(M-m)=Q^2-mM(V-v)^2.$$

Proof. Using the decomposition of Theorem 2, to be on the hyperbola (V,v) must satisfy

$$E=(V,v)\bullet_H(V,v)=\frac{(MV-mv)^2}{M-m}-\frac{mM(V-v)^2}{M-m},$$

so the result follows.

For given compatible E and Q, possible values of the velocity are given in the next theorem.

Theorem 4 *Under the compatibility condition*

$$E(M-m)=Q^2-mM(V-v)^2,$$

if

1) $E(M-m)<Q^2$, so $V\neq v$, we have two possible velocities

$$\begin{pmatrix}V\\v\end{pmatrix}=\frac{Q}{M-m}\begin{pmatrix}1\\1\end{pmatrix}\pm\frac{1}{M-m}\left(\frac{Q^2-E(M-m)}{mM}\right)^{1/2}\begin{pmatrix}m\\M\end{pmatrix},$$

2) $E(M-m)=Q^2$, so $V=v$, we have only one possible velocity

$$\begin{pmatrix} V \\ v \end{pmatrix} = \frac{Q}{M-m}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

We can also obtain a decomposition of the h-kinetic energy and the velocity of the system. Let us introduce the following two average velocities

$$\bar{V} = \frac{MV-mv}{M-m} = \frac{Q}{M-m},$$

and

$$\overline{V^2} = \frac{MV^2-mv^2}{M-m} = \frac{E}{M-m}.$$

Theorem 5 *The h-kinetic energy and the velocity of the system are decomposable as follows*

$$E = (M-m)\bar{V}^2 + M\left(V-\bar{V}\right)^2 - m\left(v-\bar{V}\right)^2,$$

and

$$\overline{V^2} = \bar{V}^2 + \frac{M}{M-m}\left(V-\bar{V}\right)^2 - \frac{m}{M-m}\left(v-\bar{V}\right)^2.$$

3.2.3. Elastic Interaction

For a h-elastic interaction, h-momentum and h-kinetic energy remain constant. The velocity (V,v) is therefore one of the two points on the hyperbola described above.

Theorem 6 *Let* (V_-,v_-) *be the velocity before the interaction, and* (V_+,v_+) *be the velocity after the interaction. Then*

$$\begin{pmatrix} V_+ \\ v_+ \end{pmatrix} = T\begin{pmatrix} V_- \\ v_- \end{pmatrix} \quad \text{where } T = \begin{pmatrix} \dfrac{M+m}{M-m} & -\dfrac{2m}{M-m} \\ \dfrac{2M}{M-m} & -\dfrac{M+m}{M-m} \end{pmatrix},$$

and

$$\begin{pmatrix} R_+ \\ S_+ \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} R_- \\ S_- \end{pmatrix}.$$

Moreover

$$V_- \begin{Bmatrix} < \\ = \\ > \end{Bmatrix} v_- \quad \text{if and only if} \quad V_+ \begin{Bmatrix} > \\ = \\ < \end{Bmatrix} v_+.$$

Proof. From Theorem 4, (V_+,v_+) is the second point on the hyperbola obtained by changing the sign of the coefficient *S*, so

$$\begin{pmatrix} R_+ \\ S_+ \end{pmatrix} = \begin{pmatrix} R_- \\ -S_- \end{pmatrix}.$$

From Theorem 2, we get

$$\begin{pmatrix} V_+ \\ v_+ \end{pmatrix} = \begin{pmatrix} 1 & m \\ 1 & M \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & m \\ 1 & M \end{pmatrix}^{-1}\begin{pmatrix} V_- \\ v_- \end{pmatrix} = T\begin{pmatrix} V_- \\ v_- \end{pmatrix}.$$

Moreover, we have

$$\begin{aligned}V_+ - v_+ &= \frac{1}{Mm}(m,M)\bullet_H (V_+,v_+)\\ &= \frac{V_- - v_-}{Mm(M-m)}(m,M)\bullet_H (m,M)\\ &= -(V_- - v_-),\end{aligned}$$

so the last result follows.

Corollary 2 $T^2 = I$

On the hyperbola of **Figure 1**, the velocity moves down left along the direction opposite to (m,M) from points above the line of direction $(1,1)$ to points below the line of direction $(1,1)$, for example from P_0 to P_1, P_2 to P_3, and so on.

3.3. Point Mass and Wall System

The point$_m$-wall interaction is easier to analyze.

Theorem 7 *Let* (V_-,v_-) *be the velocity before the interaction of mass*$_m$ *with the wall and* (V_+,v_+) *be the velocity after the interaction. Let* $v_- < 0$ *to have a interaction with the wall, so we have*

$$\begin{pmatrix}V_+\\ v_+\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}V_-\\ v_-\end{pmatrix}.$$

and

$$\begin{pmatrix}R_+\\ S_+\end{pmatrix} = \tilde{T}\begin{pmatrix}R_-\\ S_-\end{pmatrix} \quad \text{where } \tilde{T} = \begin{pmatrix}\dfrac{M+m}{M-m} & \dfrac{2Mm}{M-m}\\ -\dfrac{2}{M-m} & -\dfrac{M+m}{M-m}\end{pmatrix}.$$

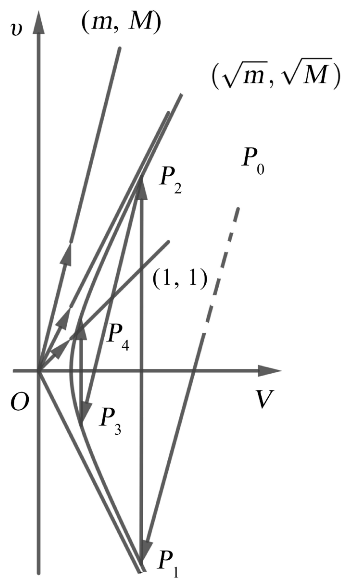

Figure 1. (V,v) displacements on the hyperbola $MV^2 - mv^2 = E$.

The h-momentum decreases at each interaction with the wall, and we have

$$Q_+ = Q_- + 2mv_- < Q_-.$$

Proof. When the point$_m$ interacts with the wall, it bounces with opposite velocity of the same magnitude, *i.e.* $v_+ = -v_-$. Since the point$_M$ doesn't interact with the wall, $V_+ = V_-$. For the coefficients R and S, using Theorem 2 we have

$$\begin{pmatrix} R_+ \\ S_+ \end{pmatrix} = \begin{pmatrix} 1 & m \\ 1 & M \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & m \\ 1 & M \end{pmatrix} \begin{pmatrix} R_- \\ S_- \end{pmatrix} = \tilde{T} \begin{pmatrix} R_- \\ S_- \end{pmatrix}.$$

For the h-momentum we have

$$Q_+ = MV_+ - mv_+ = MV_- + mv_- = MV_- - mv_- + 2mv_- = Q_- + 2mv_-,$$

and since $v_- < 0$, the h-momentum decreases at each interaction of the point$_m$ with the wall.

Corollary 3 $\tilde{T}^2 = I$

On the hyperbola of **Figure 1**, the velocity moves upward from points below the OV axis to points above the OV axis, for example from P_1 to P_2, P_3 to P_4, and so on.

3.4. Stopping Criterion and Trajectory on the Hyperbola

Suppose the velocity (V, v) is

$$(V, v) = \left(\frac{E}{M-m}\right)^{1/2} \left(\frac{\cosh(\phi)}{\cosh(\alpha)}, \frac{\sinh(\phi)}{\sinh(\alpha)}\right)$$

with $\phi \in \mathbb{R}$, such that $MV^2 - mv^2 = E$. There will be no new interaction if (V, v)

1) is the initial velocity and $\phi \in [0, \alpha]$;

2) is the velocity after a point$_m$-wall interaction (after moving up vertically) and $\phi \in (0, \alpha]$;

3) is the velocity after a point$_M$-point$_m$ interaction (after moving down left) and $\phi \in [0, \alpha)$.

On **Figure 1**, the trajectory of the velocity (V, v) on the hyperbola is given for the process with at least one interaction. We see that it moves successively from P_0 to P_1, to P_2, to P_3, and eventually up to the final point. There is alternation of moving down left and moving up vertically.

4. A Useful Transformation

The standard parametrization of the hyperbola suggests a way to transform the graph of the h-kinetic energy. Let us consider $(W, w) = (\sqrt{M}V, \sqrt{m}v)$. So we can write

$$\begin{pmatrix} W \\ w \end{pmatrix} = \Sigma \begin{pmatrix} V \\ v \end{pmatrix}$$

where

$$\Sigma = \begin{pmatrix} \sqrt{M} & 0 \\ 0 & \sqrt{m} \end{pmatrix} = \sqrt{M-m} \begin{pmatrix} \cosh(\alpha) & 0 \\ 0 & \sinh(\alpha) \end{pmatrix}.$$

We will suit to call (W,w) the velocity.

The expression for the h-kinetic energy becomes

$$E = W^2 - w^2 = (W,w) \bullet_J (W,w),$$

which is such that this quadratic form coincides now with the hyperbolic standard inner product in $\mathbb{R}^2$, say the matrix H is now the matrix J. The h-momentum is now

$$Q = \sqrt{M}W - \sqrt{m}w = \sqrt{M-m}\left(\cosh(\alpha), \sinh(\alpha)\right) \bullet_J (W,w),$$

and the lines of constant h-momentum are of direction $\left(\sinh(\alpha), \cosh(\alpha)\right)$.

We also have

$$V \begin{Bmatrix} > \\ = \\ < \end{Bmatrix} v \quad \text{if and only if} \quad \left(\sinh(\alpha), \cosh(\alpha)\right) \bullet_J (W,w) \begin{Bmatrix} > \\ = \\ < \end{Bmatrix} 0.$$

5. On Hyperbolic Reflection and Rotation

Let us recall that

$$\cosh(\beta) = \frac{\exp(\beta) + \exp(-\beta)}{2} \quad \text{and} \quad \sinh(\beta) = \frac{\exp(\beta) - \exp(-\beta)}{2}$$

so we have

$$\begin{cases} \cosh(\beta+\gamma) = \cosh(\beta)\cosh(\gamma) + \sinh(\beta)\sinh(\gamma), \\ \sinh(\beta+\gamma) = \sinh(\beta)\cosh(\gamma) + \cosh(\beta)\sinh(\gamma). \end{cases}$$

Moreover, for $\gamma = -\beta$, we have

$$\cosh^2(\beta) - \sinh^2(\beta) = 1.$$

Some useful results about hyperbolic rotations and reflections are now given. Let us consider any angle δ. For the rotation matrix $\operatorname{roth}(\beta)$ of an angle β we have

$$\operatorname{roth}(\beta) = \begin{pmatrix} \cosh(\beta) & \sinh(\beta) \\ \sinh(\beta) & \cosh(\beta) \end{pmatrix},$$

and

$$\operatorname{roth}(\beta) \begin{pmatrix} \cosh(\delta) \\ \sinh(\delta) \end{pmatrix} = \begin{pmatrix} \cosh(\delta+\beta) \\ \sinh(\delta+\beta) \end{pmatrix}.$$

For the reflection matrix $\operatorname{refh}(\gamma)$ which represents a reflection with respect to a line which makes an angle γ with the OW axis, we have

$$\operatorname{refh}(\gamma) = \begin{pmatrix} \cosh(2\gamma) & -\sinh(2\gamma) \\ \sinh(2\gamma) & -\cosh(2\gamma) \end{pmatrix}$$

and

$$\operatorname{refh}(\gamma) \begin{pmatrix} \cosh(\gamma+\delta) \\ \sinh(\gamma+\delta) \end{pmatrix} = \begin{pmatrix} \cosh(\gamma-\delta) \\ \sinh(\gamma-\delta) \end{pmatrix}.$$

To complete this section let us present some identities whose proofs are simple and omitted.

Lemma 8 *For any angle* β, *we have*

$$\operatorname{refh}(\beta)=\begin{cases}\operatorname{roth}(\beta)\operatorname{refh}(0)\operatorname{roth}(-\beta),\\ \operatorname{roth}(2\beta)\operatorname{refh}(0),\\ \operatorname{refh}(0)\operatorname{roth}(-2\beta).\end{cases}$$

Lemma 9 *For two angles* β *and* γ, *we have*

1) $\operatorname{roth}(\beta)\operatorname{roth}(\gamma)=\operatorname{roth}(\beta+\gamma)$;
2) $\operatorname{roth}(\beta)\operatorname{refh}(\gamma)=\operatorname{roth}(2\gamma+\beta)\operatorname{refh}(0)$;
3) $\operatorname{refh}(\gamma)\operatorname{roth}(\beta)=\operatorname{roth}(2\gamma-\beta)\operatorname{refh}(0)$;
4) $\operatorname{refh}(\beta)\operatorname{refh}(\gamma)=\operatorname{roth}(2(\beta-\gamma))$.

6. Indirect Analysis: The Transformed Coordinate System

In this section we analyze the system with respect to the transformed coordinate system WOw.

6.1. Observations

Let the velocity be given by

$$(W,w)=E^{1/2}(\cosh(\phi),\sinh(\phi))$$

such that $W^2-w^2=E$, and $\phi\in\mathbb{R}$. Now, the rule of the process is such that if

1) $\phi\in(-\infty,0)$, so $w<0$, there will be a point$_m$-wall interaction;
2) $\phi\in[0,\alpha]$, so $0\le w/\sqrt{m}\le W/\sqrt{M}$, there will be no more interaction;
3) $\phi\in(\alpha,+\infty)$, so $0<W/\sqrt{M}<w/\sqrt{m}$, there will be a point$_M$-point$_m$ interaction.

6.2. Two-Point Mass System

The velocity (W,w) can also be decomposed, and the possible velocities are given by the intersection points of a hyperbola (h-kinetic energy) and a line (h-momentum).

6.2.1. Decomposition of the Velocity

Let us express the velocity in terms of the new variables and an appropriate orthonormal basis.

Theorem 10 *The set* $\{(\cosh(\alpha),\sinh(\alpha)),(\sinh(\alpha),\cosh(\alpha))\}$ *is an h-orthogonal basis with respect to the quadratic form used to define the hyperbola of constant h-kinetic energy.*

Proof. Indeed we have

$$(\cosh(\alpha),\sinh(\alpha))\bullet_J(\sinh(\alpha),\cosh(\alpha))=0.$$

Moreover

$$\left\|(\cosh(\alpha),\sinh(\alpha))\right\|_J^2=\cosh^2(\alpha)-\sinh^2(\alpha)=1,$$

and

$$\left\|\left(\sinh(\alpha),\cosh(\alpha)\right)\right\|_J^2=\sinh^2(\alpha)-\cosh^2(\alpha)=-1.$$

Theorem 11 *The expression of the velocity* (W,w) *is*

$$\begin{pmatrix}W\\w\end{pmatrix}=r\begin{pmatrix}\cosh(\alpha)\\\sinh(\alpha)\end{pmatrix}+s\begin{pmatrix}\sinh(\alpha)\\\cosh(\alpha)\end{pmatrix}=\operatorname{roth}(\alpha)\begin{pmatrix}r\\s\end{pmatrix},$$

where

$$r=\left(\cosh(\alpha),\sinh(\alpha)\right)\bullet_J(W,w)=\frac{Q}{\sqrt{M-m}},$$

and

$$s=-\left(\sinh(\alpha),\cosh(\alpha)\right)\bullet_J(W,w)=\frac{\sqrt{M}w-\sqrt{m}W}{\sqrt{M-m}}=\sqrt{\frac{Mm}{M-m}}(v-V).$$

Moreover

$$E=W^2-w^2=r^2-s^2.$$

6.2.2. Compatibility Condition

The condition remains the same, but we can rewrite the expressions for the velocity.

Theorem 12 *Under the condition*

$$E(M-m)=Q^2-mM(V-v)^2,$$

1) if $E(M-m)<Q^2$, so $V\neq v$, there are two possible velocities

$$\begin{pmatrix}W\\w\end{pmatrix}=\frac{Q}{\sqrt{M-m}}\begin{pmatrix}\cosh(\alpha)\\\sinh(\alpha)\end{pmatrix}\pm\left(\frac{Q^2-E(M-m)}{M-m}\right)^{1/2}\begin{pmatrix}\sinh(\alpha)\\\cosh(\alpha)\end{pmatrix};$$

2) if $E(M-m)=Q^2$, so $V=v$, there is only one possible velocity

$$\begin{pmatrix}W\\w\end{pmatrix}=\frac{Q}{\sqrt{M-m}}\begin{pmatrix}\cosh(\alpha)\\\sinh(\alpha)\end{pmatrix}.$$

6.2.3. Elastic Interaction

For a h-elastic interaction, h-momentum and h-kinetic energy remain constant. The velocity (W,w) is therefore one of the two points on the hyperbola as described above.

Theorem 13 *Let* (W_-,w_-) *be the velocity before the interaction, and* (W_+,w_+) *be the velocity after the interaction. The velocities are related by the relation*

$$\begin{pmatrix}W_+\\w_+\end{pmatrix}=\operatorname{refh}(\alpha)\begin{pmatrix}W_-\\w_-\end{pmatrix},$$

and the coefficients by the relation

$$\begin{pmatrix}r_+\\s_+\end{pmatrix}=\operatorname{refh}(0)\begin{pmatrix}r_-\\s_-\end{pmatrix}.$$

Moreover

$$\left(\sinh(\alpha),\cosh(\alpha)\right)\bullet_J\left(W_-,w_-\right)\begin{Bmatrix}>\\=\\<\end{Bmatrix}0$$

if and only if

$$\left(\sinh(\alpha),\cosh(\alpha)\right)\bullet_J\left(W_+,w_+\right)\begin{Bmatrix}<\\=\\>\end{Bmatrix}0.$$

Proof. From Theorem 12, we have

$$\begin{pmatrix}r_+\\s_+\end{pmatrix}=\begin{pmatrix}r_-\\-s_-\end{pmatrix}=\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}r_-\\s_-\end{pmatrix}=\operatorname{refh}(0)\begin{pmatrix}r_-\\s_-\end{pmatrix}.$$

From the decomposition of Theorem 11, we get

$$\begin{pmatrix}W_+\\w_+\end{pmatrix}=\operatorname{roth}(\alpha)\operatorname{refh}(0)\operatorname{roth}(-\alpha)\begin{pmatrix}W_-\\w_-\end{pmatrix}=\operatorname{refh}(\alpha)\begin{pmatrix}W_-\\w_-\end{pmatrix}.$$

Moreover a direct computation leads to

$$\left(\sinh(\alpha),\cosh(\alpha)\right)\bullet_J\left(W_+,w_+\right)=s_-=-\left(\sinh(\alpha),\cosh(\alpha)\right)\bullet_J\left(W_-,w_-\right),$$

and the last result follows.

Remark *We have the following decomposition for the matrix T of the linear system of Theorem 6*

$$T=\begin{pmatrix}\dfrac{M+m}{M-m}&-\dfrac{2m}{M-m}\\\dfrac{2M}{M-m}&-\dfrac{M+m}{M-m}\end{pmatrix}=\Sigma^{-1}\operatorname{refh}(\alpha)\Sigma.$$

On the unit hyperbola of **Figure 2**, the velocity moves down left along the direction opposite to $\left(\sqrt{m},\sqrt{M}\right)$ from points above the line of direction $\left(\sqrt{M},\sqrt{m}\right)$ to points P_+ below the line of direction $\left(\sqrt{M},\sqrt{m}\right)$, for example from P_0 to P_1, P_2 to P_3, and so on.

6.3. Point Mass and Wall System

In the new coordinate system, for a point$_m$-wall interaction we have $w_+=-w_-$ and $W_+=W_-$.

Theorem 14 *If* $\left(W_-,w_-\right)$, *is the velocity before the interaction of the point$_m$ with the wall with* $w_-<0$, *we have*

$$\begin{pmatrix}W_+\\w_+\end{pmatrix}=\operatorname{refh}(0)\begin{pmatrix}W_-\\w_-\end{pmatrix},$$

which is a reflection with respect to the OW axis, the line $w=0$. For the coefficients we have

$$\begin{pmatrix}r_+\\s_+\end{pmatrix}=\operatorname{refh}(-\alpha)\begin{pmatrix}r_-\\s_-\end{pmatrix}.$$

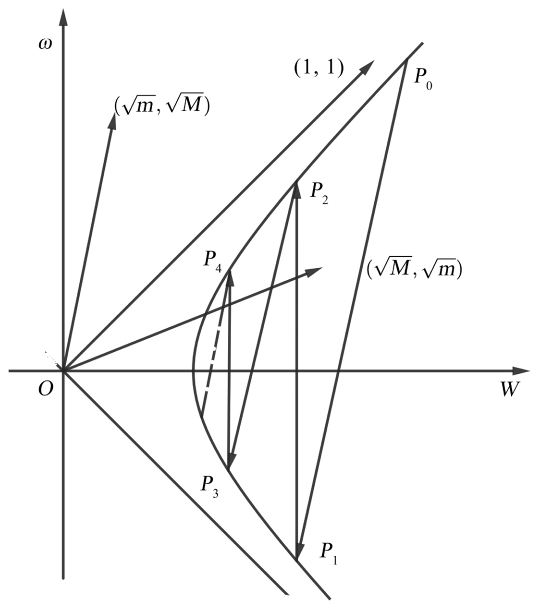

Figure 2. (W, w) displacements on the hyperbola $W^2 - w^2 = E$.

The momentum decreases at each interaction with the wall, and we have

$$Q_+ = Q_- + 2\sqrt{M-m}\sinh(\alpha) w_- < Q_-.$$

Proof. The first result is obvious. For the coefficients, we use Theorem 11 to get

$$\begin{pmatrix} r_+ \\ s_+ \end{pmatrix} = \operatorname{roth}(-\alpha)\operatorname{refh}(0)\operatorname{roth}(\alpha)\begin{pmatrix} r_- \\ s_- \end{pmatrix} = \operatorname{refh}(-\alpha)\begin{pmatrix} r_- \\ s_- \end{pmatrix}.$$

For the momentum

$$Q_+ = \sqrt{M}W_+ - \sqrt{m}w_+ = \sqrt{M}W_- + \sqrt{m}w_- = Q_- + 2\sqrt{m}w_-,$$

with $w_- < 0$.

On the hyperbola of **Figure 2**, the velocity moves upward from points below the OW axis to points above the OW axis, for example from P_1 to P_2, P_3 to P_4, and so on.

Remark *Directly, or from Theorem 7 and Theorem 14, we have*

$$J = \operatorname{refh}(0) \text{ and } \operatorname{refh}(0) = \Sigma^{-1}\operatorname{refh}(0)\Sigma.$$

6.4. Stopping Criterion and Trajectory on the Hyperbola

Suppose the velocity (W, w) is

$$(W, w) = E^{1/2}(\cosh(\phi), \sinh(\phi))$$

with $\phi \in \mathbb{R}$, such that $W^2 - w^2 = E$. There will be no new interaction if (W, w)

1) is the initial velocity and $\phi \in [0, \alpha]$;

2) is the velocity after a point$_m$-wall interaction (after moving up vertically)

and $\phi \in (0, \alpha]$;

3) is the velocity after a point$_M$-point$_m$ interaction (after moving down left) and $\phi \in [0, \alpha)$.

On **Figure 2**, the trajectory of the velocity (W, w) on the hyperbola is given after at least one interaction. We see that it moves successively from P_0 to P_1, to P_2, to P_3, to P_4, and eventually up to the final point. There is alternation of moving down left and moving up vertically.

7. Sequence of Interactions

7.1. The Problem

To any point on the hyperbola $W^2 - w^2 = E$ there exist an unknown angle ϕ such that

$$(W, w) = E^{1/2}\left(\cosh(\phi), \sinh(\phi)\right).$$

Using this initial condition, two sequences of alternating interactions are analyzed. We will see that we get an approximation of ϕ which depends on the choice of M and m ($M > m$). We will use the notation (W_k, w_k) for the velocity after the k-th interaction.

7.2. Two-Point Mass Interaction First

For the sequence of interactions starting with a point$_M$-point$_m$ interaction followed by a point$_m$-wall interaction, we must have $V < v$, or

$$\frac{1}{\sqrt{M-m}}\left(\sqrt{m}, \sqrt{M}\right) \bullet_J (W, w) = \left(\sinh(\alpha), \cosh(\alpha)\right) \bullet_J (W, w) < 0,$$

which also means that $\phi \in (\alpha, +\infty)$.

Theorem 15 *A sequence of two interactions, a point$_M$-point$_m$ interaction followed by a point$_m$-wall interaction, is a rotation of angle* -2α.

Proof. Using part (d) of Lemma 9, we have

$$\begin{pmatrix} W_2 \\ w_2 \end{pmatrix} = \operatorname{refh}(0)\begin{pmatrix} W_1 \\ w_1 \end{pmatrix} = \operatorname{refh}(0)\operatorname{refh}(\alpha)\begin{pmatrix} W \\ w \end{pmatrix} = \operatorname{roth}(-2\alpha)\begin{pmatrix} W \\ w \end{pmatrix},$$

so the result follows.

Thereafter there is alternation of interactions: point$_M$-point$_m$, point$_m$-wall, etc.

Theorem 16 *The velocity after*

1) $2n$ interactions (with a last point$_m$-wall interaction) is

$$\begin{pmatrix} W_{2n} \\ w_{2n} \end{pmatrix} = E^{1/2}\begin{pmatrix} \cosh(\phi - 2n\alpha) \\ \sinh(\phi - 2n\alpha) \end{pmatrix};$$

2) $2n+1$ interactions (with a last point$_M$-point$_m$ interaction) is

$$\begin{pmatrix} W_{2n+1} \\ w_{2n+1} \end{pmatrix} = E^{1/2}\begin{pmatrix} \cosh(2(n+1)\alpha - \phi) \\ \sinh(2(n+1)\alpha - \phi) \end{pmatrix}.$$

Proof. The process ends after $2n$ or $2n+1$ interactions.

1) For $2n$ interactions, we use part (*a*) of Lemma 9 to get

$$\begin{pmatrix} W_{2n} \\ w_{2n} \end{pmatrix} = \underbrace{\mathrm{roth}(-2\alpha)\cdots\mathrm{roth}(-2\alpha)}_{n\text{-times}} \begin{pmatrix} W \\ w \end{pmatrix} = \mathrm{roth}(-2n\alpha) \begin{pmatrix} W \\ w \end{pmatrix}.$$

2) For $2n+1$ interactions, one more point$_M$-point$_m$ interaction is needed, so

$$\begin{pmatrix} W_{2n+1} \\ w_{2n+1} \end{pmatrix} = \mathrm{refh}(\alpha) \begin{pmatrix} W_{2n} \\ w_{2n} \end{pmatrix}.$$

Then from part (c) of Lemma 9, we get

$$\begin{pmatrix} W_{2n+1} \\ w_{2n+1} \end{pmatrix} = \mathrm{refh}(\alpha)\,\mathrm{roth}(-2n\alpha) \begin{pmatrix} W \\ w \end{pmatrix} = \mathrm{roth}(2(n+1)\alpha)\,\mathrm{refh}(0) \begin{pmatrix} W \\ w \end{pmatrix}.$$

Starting with a point$_M$-point$_m$ interaction, considering the preceding expressions for the velocity, and applying the stopping criterion, we conclude that the process will end after

1) $2n$ interactions if the last interaction is a point$_m$-wall interaction, so (W_{2n}, w_{2n}), of angle $\phi - 2n\alpha$, is on the hyperbolic arc of angle in $(0,\alpha]$;

2) $2n+1$ interactions if the last interaction is a point$_M$-point$_m$ interaction, so (W_{2n+1}, w_{2n+1}), of angle $2(n+1)\alpha - \phi$, is on the hyperbolic arc of angle in $[0,\alpha)$.

In both cases, if K is the number of interactions we obtain

$$\frac{\phi}{\alpha} - 1 \le K < \frac{\phi}{\alpha},$$

or

$$K = \begin{cases} \dfrac{\phi}{\alpha} - 1 & \text{if } \dfrac{\phi}{\alpha} \text{ is an integer,} \\ \left\lfloor \dfrac{\phi}{\alpha} \right\rfloor & \text{if } \dfrac{\phi}{\alpha} \text{ is not an integer.} \end{cases}$$

Note that this result also holds for $\phi \in [0,\alpha]$ because $K = 0$.

7.3. Point Mass and Wall Interaction First

For the sequence of interactions starting with a point$_m$-wall interaction followed by a point$_M$-point$_m$ interaction, we must have (V,v) with $v<0$, so (W,w) with $w<0$. It also means that $\phi \in (-\infty, 0)$.

Theorem 17 *A sequence of two interactions, a point$_m$-wall interaction followed by a point$_M$-point$_m$ interaction, is a rotation of angle* 2α.

Proof. Using part (d) of Lemma 9, we get

$$\begin{pmatrix} W_2 \\ w_2 \end{pmatrix} = \mathrm{refh}(\alpha) \begin{pmatrix} W_1 \\ w_1 \end{pmatrix} = \mathrm{refh}(\alpha)\,\mathrm{refh}(0) \begin{pmatrix} W \\ w \end{pmatrix} = \mathrm{roth}(2\alpha) \begin{pmatrix} W \\ w \end{pmatrix},$$

so the result follows.

Thereafter there is alternation of interactions: point$_m$-wall, point$_M$-point$_m$, etc.

Theorem 18 *The velocity after*

1) $2n$ interactions (with a last point$_M$-point$_m$ interaction) is

$$\begin{pmatrix} W_{2n} \\ w_{2n} \end{pmatrix} = E^{1/2} \begin{pmatrix} \cosh(\phi + 2n\alpha) \\ \sinh(\phi + 2n\alpha) \end{pmatrix}.$$

2) $2n + 1$ interactions (with a last point$_m$-wall interaction) is

$$\begin{pmatrix} W_{2n+1} \\ w_{2n+1} \end{pmatrix} = E^{1/2} \begin{pmatrix} \cosh(-2n\alpha - \phi) \\ \sinh(-2n\alpha - \phi) \end{pmatrix}.$$

Proof. The process ends after $2n$ or $2n + 1$ interactions.

1) For $2n$ interactions, we use part (a) of Lemma 9 to get

$$\begin{pmatrix} W_{2n} \\ w_{2n} \end{pmatrix} = \underbrace{\mathrm{roth}(2\alpha)\cdots\mathrm{roth}(2\alpha)}_{n\text{-times}} \begin{pmatrix} W \\ w \end{pmatrix} = \mathrm{roth}(2n\alpha) \begin{pmatrix} W \\ w \end{pmatrix}.$$

2) For $2n + 1$ interactions, one more point$_m$-wall interaction is needed, so

$$\begin{pmatrix} W_{2n+1} \\ w_{2n+1} \end{pmatrix} = \mathrm{refh}(0) \begin{pmatrix} W_{2n} \\ w_{2n} \end{pmatrix}.$$

Then from part (c) of Lemma 9, we get

$$\begin{pmatrix} W_{2n+1} \\ w_{2n+1} \end{pmatrix} = \mathrm{refh}(0)\mathrm{roth}(2n\alpha) \begin{pmatrix} W \\ w \end{pmatrix} = \mathrm{roth}(-2n\alpha)\mathrm{refh}(0) \begin{pmatrix} W \\ w \end{pmatrix}.$$

Starting with a point$_m$-wall interaction, considering the preceding expressions for the velocity, and applying the stopping criterion, we conclude that the process will end after

1) $2n + 1$ interactions if the last interaction is a point$_m$-wall interaction, so (W_{2n+1}, w_{2n+1}), of angle $-2n\alpha - \phi$, is on the hyperbolic arc of angle in $(0, \alpha]$;

2) $2n$ interactions if the last interaction is a point$_M$-point$_m$ interaction, so (W_{2n}, w_{2n}), of angle $2n\alpha + \phi$, is on the hyperbolic arc of angle in $[0, \alpha)$.

In both case, if K is the number of interactions, we have

$$-\frac{\phi}{\alpha} \le K < -\frac{\phi}{\alpha} + 1,$$

or

$$K = \begin{cases} -\dfrac{\phi}{\alpha} & \text{if } -\dfrac{\phi}{\alpha} \text{ is an integer,} \\ \left\lfloor -\dfrac{\phi}{\alpha} \right\rfloor + 1 & \text{if } -\dfrac{\phi}{\alpha} \text{ is not an integer.} \end{cases}$$

7.4. Summary of Results

Table 1 presents a summary of our results on the values of K depending on the values of ϕ. Let us use the notation K_ϕ for K associated to ϕ. So let us observe that for $\phi > 0$ we have

$$K_{-\phi} = K_\phi + 1 = K_{\phi+\alpha}.$$

8. Digits of the Logarithmic Constant ln(2)

Let p be a prime number and consider the logarithmic constant $\ln(p)$. Since

Table 1. Values of K.

K			
$\phi<0$	$\phi\in[0,\alpha]$	$\phi>\alpha$	condition
$-\frac{\phi}{\alpha}$	0	$\frac{\phi}{\alpha}-1$	$\frac{\phi}{\alpha}$ is an integer
$\left\lfloor -\frac{\phi}{\alpha}\right\rfloor+1$	0	$\left\lfloor \frac{\phi}{\alpha}\right\rfloor$	$\frac{\phi}{\alpha}$ is not an integer

$$\begin{cases}\cosh(\ln(p))=\dfrac{p^2+1}{2p}\\ \sinh(\ln(p))=\dfrac{p^2-1}{2p},\end{cases}$$

we can apply the preceding result with $(W,w)=\left(\frac{p^2+1}{2p},\frac{p^2-1}{2p}\right)$ to find the first digits of $\ln(p)$. We will consider $p=2$ to illustrate the process, so our initial point on the hyperbola will be $(W,w)=(5/4,3/4)$, with $E=1$, and $\ln(2)=0.693\cdots$ [3].

8.1. Observations

In any integer base $b\geq 2$ of a number system, the integer part of $\ln(2)\cdot b^N$, noted $\lfloor \ln(2)\cdot b^N\rfloor_b$ in base b, add, to the integer part of $\ln(2)$, the first N digits of the fractional part of $\ln(2)$ in base b. So let us consider an angle $\alpha\approx b^{-N}$ and look at the value of the number K of interactions.

To get $\alpha\approx b^{-N}$ we can consider the following two cases:

A) $\tanh(\alpha)=b^{-N}$, so $\alpha=\operatorname{arctanh}(b^{-N})\approx b^{-N}$, which means that $M=b^{2N}m$;

B) $\sinh(\alpha)=b^{-N}$, so $\alpha=\operatorname{arcsinh}(b^{-N})\approx b^{-N}$, which means that $M=(b^{2N}+1)m$. It remains to verify that, if K is the number of interactions, the following conjecture is true.

Conjecture *For* $M=b^{2N}m$ (*i.e.* $\alpha=\operatorname{arctanh}(b^{-N})$), *or* $M=(b^{2N}+1)m$ (*i.e.* $\alpha=\operatorname{arcsinh}(b^{-N})$), *the total number K of interactions which is given by its representation in base b by*

$$K=\begin{cases}\left\lfloor\dfrac{\ln(2)}{\alpha}\right\rfloor_b-1 & \text{if } \dfrac{\ln(2)}{\alpha} \text{ is an integer,}\\ \left\lfloor\dfrac{\ln(2)}{\alpha}\right\rfloor_b & \text{if } \dfrac{\ln(2)}{\alpha} \text{ is not an integer.}\end{cases}$$

consists of the digits of the integer part $\ln(2)$ and the first N digits of the fractional part of $\ln(2)$ in base b, so $K=\lfloor\ln(2)\cdot b^N\rfloor_b$.

In the sequel, we will use the following representations in base b

$$\begin{cases} \left[\ln(2)\right]_b = 0.a_1a_2\cdots a_N a_{N+1}\cdots a_{2N}a_{2N+1}\cdots \\ \left[\dfrac{\ln(2)}{2}\right]_b = 0.\tilde{a}_1\tilde{a}_2\cdots\tilde{a}_N\tilde{a}_{N+1}\cdots\tilde{a}_{2N}\tilde{a}_{2N+1}\cdots \end{cases}$$

and

$$\begin{cases} \left[\ln(2)\cdot b^N\right]_b = a_1a_2\cdots a_N.a_{N+1}\cdots a_{2N}a_{2N+1}a_{2N+2}\cdots \\ \left[\dfrac{\ln(2)}{2}\cdot b^{-N}\right]_b = 0.\underbrace{0\cdots0}_{N\text{-times}}\tilde{a}_1\tilde{a}_2\cdots \end{cases}$$

8.1.1. Case (A)

The Taylor expansion of $\operatorname{arctanh}(x)$ is

$$\operatorname{arctanh}(x) = \sum_{i=0}^{n}\frac{x^{2i+1}}{2i+1} + B_n(x)$$

for $|x|<1$. Also $0 < B_n(x) \le \dfrac{x^{2i+3}}{2i+3}$ for $x>0$. It can be shown that

$$\frac{1}{x}-\frac{x}{2} < \frac{1}{\operatorname{arctanh}(x)} < \frac{1}{x},$$

for $0 < x \le 1/2$. Multiplying by $\ln(2)$ and take $x = b^{-N}$, we have

$$\ln(2)\cdot b^N - \frac{\ln(2)}{2}\cdot b^{-N} < \frac{\ln(2)}{\operatorname{arctanh}\left(b^{-N}\right)} < \ln(2)\cdot b^N.$$

Since $\ln(2)\cdot b^N$ is not an integer

$$\left[\frac{\ln(2)}{\operatorname{arctanh}\left(b^{-N}\right)}\right]_b < \left[\ln(2)\cdot b^N\right]_b < \left\lfloor \ln(2)\cdot b^N \right\rfloor_b + 1.$$

Using the representations in base *b*, since

$$\left[\ln(2)\cdot b^N\right]_b - \left[\frac{\ln(2)}{2}\cdot b^{-N}\right]_b = \left[\ln(2)\cdot b^N - \frac{\ln(2)}{2}\cdot b^{-N}\right]_b,$$

under the condition that there exists $n\in[N+1,2N]$ such that $a_n>0$, we get

$$\left\lfloor \ln(2)\cdot b^N \right\rfloor_b < \left[\ln(2)\cdot b^N - \frac{\ln(2)}{2}\cdot b^{-N}\right]_b < \left[\frac{\ln(2)}{\operatorname{arctanh}\left(b^{-N}\right)}\right]_b.$$

So

$$\left\lfloor \ln(2)\cdot b^N \right\rfloor_b < \left[\frac{\ln(2)}{\operatorname{arctanh}\left(b^{-N}\right)}\right]_b < \left\lfloor \ln(2)\cdot b^N \right\rfloor_b + 1,$$

consequently $\left[\dfrac{\ln(2)}{\operatorname{arctanh}\left(b^{-N}\right)}\right]_b$ is not an integer and

$$\left\lfloor \ln(2)\cdot b^N \right\rfloor_b = \left\lfloor \frac{\ln(2)}{\operatorname{arctanh}\left(b^{-N}\right)} \right\rfloor_b.$$

8.1.2. Case (B)

The Taylor expansion of $\operatorname{arcsinh}(x)$,

$$\operatorname{arcsinh}(x)=\sum_{i=0}^{n}(-1)^i\frac{(2i)!}{4^i(i!)^2}\frac{x^{2i+1}}{2i+1}+B_n(x)$$

for $|x|<1$. Also $0<B_n(x)\le\frac{x^{2n+3}}{2n+3}$ for $x>0$. It can be shown that

$$\frac{1}{x}<\frac{1}{\operatorname{arcsinh}(x)}<\frac{1}{x}+\frac{x}{2}$$

for $0<x<1$. Multiplying by $\ln(2)$ and take $x=b^{-N}$, then

$$\ln(2)\cdot b^N<\frac{\ln(2)}{\operatorname{arcsinh}(b^{-N})}<\ln(2)\cdot b^N+\frac{\ln(2)}{2}\cdot b^{-N}.$$

Since $\ln(2)\cdot b^N$ is never an integer, so

$$\lfloor\ln(2)\cdot b^N\rfloor_b<\lceil\ln(2)\cdot b^N\rceil_b<\left\lceil\frac{\ln(2)}{\operatorname{arcsinh}(b^{-N})}\right\rceil_b.$$

Using the representations in base *b*, since

$$\lceil\ln(2)\cdot b^N\rceil_b+\left\lceil\frac{\ln(2)}{2}\cdot b^{-N}\right\rceil_b=\left\lceil\ln(2)\cdot b^N+\frac{\ln(2)}{2}\cdot b^{-N}\right\rceil_b,$$

under the condition that there exists $n\in[N+1,2N]$ such that $a_n<b-1$, we get

$$\left\lceil\frac{\ln(2)}{\operatorname{arcsinh}(b^{-N})}\right\rceil_b<\left\lceil\ln(2)\cdot b^N+\frac{\ln(2)}{2}\cdot b^{-N}\right\rceil_b<\lfloor\ln(2)\cdot b^N\rfloor_b+1.$$

Hence

$$\lfloor\ln(2)\cdot b^N\rfloor_b<\left\lceil\frac{\ln(2)}{\operatorname{arcsinh}(b^{-N})}\right\rceil_b<\lfloor\ln(2)\cdot b^N\rfloor_b+1,$$

consequently $\left\lceil\frac{\ln(2)}{\operatorname{arcsinh}(b^{-N})}\right\rceil_b$ is not an integer and

$$\lfloor\ln(2)\cdot b^N\rfloor_b=\left\lfloor\frac{\ln(2)}{\operatorname{arcsinh}(b^{-N})}\right\rfloor_b.$$

8.2. Consequences

There are consequences of the preceding results. For

Case (A): if $a_{N+1}>0$, then

$$\left\lfloor\ln(2)\cdot b^{N-k}-\frac{\ln(2)}{2}\cdot b^{-(N-k)}\right\rfloor_b=\lfloor\ln(2)\cdot b^{N-k}\rfloor_b$$

Case (B): if $a_{N+1}<b-1$, then

$$\left\lfloor \ln(2)\cdot b^{N-k} \right\rfloor_b = \left\lfloor \ln(2)\cdot b^{N-k} + \frac{\ln(2)}{2}\cdot b^{-(N-k)} \right\rfloor_b$$

for $k = 0,\cdots,\left\lfloor \frac{N-1}{2} \right\rfloor$. So the result holds for the powers of b from $N - \left\lfloor \frac{N-1}{2} \right\rfloor$ up to N.

This last observation suggests a Cauchy induction like method [5]. With an algorithm which can find a digit of $\ln(2)$ at a precise position N without calculating all digits in positions less than N, see for example [3] [6], we could deduce the result for a number of lower positions. We proceed in the following way. Suppose the property true for $n = 1,\cdots,N$. Then look for the smallest $\ell \in \{0,1,\cdots,N\}$ such that in case (A) $a_{2N-\ell+1} > 0$ or in case (B) $a_{2N-\ell+1} < b-1$, then the result holds for $n = 1,\cdots,2N-\ell$.

8.3. Conjecture Almost Proved

It should be verified, with modern computational facilities, that up to very large values of N, no sequences such that

Case (A): $a_n = 0$ for $n \in [N+1, 2N]$,

Case (B): $a_n = b-1$ for $n \in [N+1, 2N]$,

are present in the expansion of $\ln 2$. So the conjecture would be verified for up to very large values of N.

8.4. Final Remark

There exists in fact infinitely many angles α for which we get the result $K_b = \left\lfloor \pi \cdot b^N \right\rfloor_b$. Indeed, if we use α_λ with

$$\tanh^2(\alpha_\lambda) = \frac{m}{M_\lambda} = \frac{b^{-2N}}{1+\lambda b^{-2N}}$$

for $\lambda \in [0,1]$, we also get the result. We observe that

$$M_\lambda = (1-\lambda)M_A + \lambda M_B = \left(b^{2N} + \lambda\right)m,$$

where $M_0 = M_A = b^{2N}m$ and $M_1 = M_B = \left(b^{2N}+1\right)m$ are the masses for case (A) and case (B). Also $\alpha_A = \alpha_0 \ge \alpha_\lambda \ge \alpha_1 = \alpha_B$, and

$$\frac{\pi}{\alpha_B} \le \frac{\pi}{\alpha_\lambda} \le \frac{\pi}{\alpha_A}.$$

Since the result holds for $\frac{\pi}{\alpha_A}$ and $\frac{\pi}{\alpha_B}$, it also holds for $\frac{\pi}{\alpha_\lambda}$ for any $\lambda \in [0,1]$.

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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