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The Family of Exponential Attractors and Random Attractors for a Class of Kirchhoff Equations

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Abstract

To prove the existence of the family of exponential attractors, we first define a family of compact, invariant absorbing sets B_k. Then we prove that the solution semigroup has Lipschitz property and discrete squeezing property. Finally, we obtain a family of exponential attractors and its estimation of dimension by combining them with previous theories. Next, we obtain Kirchhoff-type random equation by adding product white noise to the right-hand side of the equation. To study the existence of random attractors, firstly we transform the equation by using Ornstein-Uhlenbeck process. Then we obtain a family of bounded random absorbing sets via estimating the solution of the random differential equation. Finally, we prove the asymptotic compactness of semigroup of the stochastic dynamic system; thereby we obtain a family of random attractors.

Keywords

Family of Exponential Attractors, Lipschitz Continuous, Squeezing Property, Stochastic Dynamic System, Family Random Attractors

1. Introduction

In 1991, Eden A. *et al.* [1] proposed the concept of inertial fractal set and how inertial fractal set is constructed. Meanwhile, they provided some applications for people to study how to prove the existence of exponential attractors. The authors' relevant research results can be referred to [2] [3] [4] [5].

With the advent of Kirchhoff [6] equation and the existence of its solution, scholars began to study the existence of exponential attractors of Kirchhoff equation. Recently, Jia Lan, Ma Qiaozhen [7] studied the Kirchhoff-type suspension bridge equations:

$$\begin{cases} u_{tt}+\Delta^2 u+\alpha u_t+\left(p-\beta\left\|\nabla^m u\right\|^2\right)\Delta u+k^2u^+ + f(u)=g(x),(x,t)\in\Omega\times R^+, \\ \Delta u(x,t)=u(x,t)=0, x\in\partial\Omega, t\geq 0. \end{cases}$$

They proved the asymptotic compactness of the semigroup and showed the existence of exponential attractors by a new method of enhanced flattening property under a weaker condition of nonlinearity.

Lin Guoguang, Wang Wei [8] discussed a class of higher-order Kirchhoff-type equation with nonlinear damped term:

$$\begin{cases} u_{tt}+(-\Delta)^m u+\phi\left(\left\|\nabla^2 u\right\|^2\right)(-\Delta)^m u+h(u_t)=f(x), \\ u(x,t)=0, \dfrac{\partial^j u}{\partial v^j}=0, \quad j=1,2,\cdots,m-1, \quad x\in\partial\Omega, t>0. \end{cases}$$

They obtained the exponential attractors via proving the Lipschitz continuity and discrete squeezing property of dynamical system.

In paper [9], we studied Kirchhoff equation:

$$\begin{cases} u_{tt}+M\left(\left\|\nabla^m u\right\|_p^p\right)(-\Delta)^{2m} u+\beta(-\Delta)^{2m} u_t+|u|^\rho u=f(x), \\ u(x,t)=\dfrac{\partial^j u}{\partial v^j}=0, \quad j=1,2,\cdots,2m-1, \quad x\in\partial\Omega, \\ u(x,0)=u_0(x), u_t(x,0)=u_1(x), \quad x\in\partial\Omega, \quad t>0, \end{cases} \tag{1.1}$$

where $m>1$, $p\geq 2$, $\Omega\subseteq R^n\,(n\geq 1)$ is a bounded region with a smooth boundary $\partial\Omega$, $Q=\Omega\times[0,\infty)$ stands for the cylinder in $R_x^n\times R_t$, the rigid term $M(s)\in C^1[0,\infty)$ is a general function, $\beta(-\Delta)^{2m}u_t\,(\beta>0)$ is the strong dissipative term, $|u|^\rho u$ is the nonlinear term and $\rho\geq -1$, $f(x)$ denotes the external force. We have proved the existence and uniqueness of solution, a family of global attractors and its dimension estimation. In this paper, we will discuss the family of the exponential attractors and its dimension estimation. Meanwhile, we will discuss the family of random attractors of stochastic Kirchhoff equation with product white noise:

$$\begin{cases} u_{tt}+M\left(\left\|\nabla^m u\right\|_p^p\right)(-\Delta)^{2m} u+\beta(-\Delta)^{2m} u_t+|u|^\rho u=q(x)\dfrac{\mathrm{d}W(t)}{\mathrm{d}t}, \\ u(x,t)=\dfrac{\partial^j u}{\partial v^j}=0, \quad j=1,2,\cdots,2m-1, \quad x\in\partial\Omega, \\ u(x,0)=u_0(x), u_t(x,0)=u_1(x), \quad x\in\partial\Omega, \quad t>0, \end{cases} \tag{1.2}$$

where $W(t)$ is independent of time denotes a two-side process in probability space $(\Omega,\mathcal{F},P)$, $\Omega=\{\omega\in C(R,R):\omega(0)=0\}$, $\mathcal{F}$ denotes a Borel σ-algebra generated by compact-open topology on Ω, P denotes a probability measure.

Random attractor plays an important role in stochastic dynamic systems because of its property. Lu D. proposed the concept of stochastic process and its application in the literature [10]. Then Guo Boling, Pu Xueke introduced the knowledge of random infinite dimensional dynamical system in the literature

[11]. Lin G.G., Qin C.L. [12] discussed the existence of the random attractors of weekly damped Kirchhoff equation:

$$\begin{cases} du_t + \alpha du + \left[-\left(1+\left(\int_D |\nabla u|^2 dx\right)^\rho\right)\Delta u + g(u) \right] dt = f(x)dt + q(x)dW(t), t \in [\tau, +\infty), \\ u(x,t)|_{x\in\partial D} = 0, t > \tau, \end{cases}$$

Following, Lin, G.G., *et al.* [13] proved the exponential attractor of Kirchhoff-type equations with strongly damped terms and source terms:

$$du_t + \left[(-\Delta)^m u_t + \phi\left(\|\nabla^m u\|^2\right)(-\Delta)^m u + g(u) \right] dt = f(x)dt + q(x)dW(t).$$

More relevant results can be referred to [14] [15] [16] [17].

2. Preliminaries

Combine paper [9] with some new definitions and assumptions, we have:

$$H = L^2(\Omega),\ H_0^m(\Omega) = H^m(\Omega) \cap H_0^1(\Omega),\ E_k = H_0^{2m+k}(\Omega) \times H_0^k(\Omega),$$

where $k = 1, 2, \cdots, 2m$, $U = (u, v)^{\mathrm{T}}$, $v = u_t + \varepsilon u$, $C_i > 0 (i = 0, 1, 2, \cdots)$ are constants. λ_j denotes the *j*th eigenvalue of $-\Delta$ with the homogeneous Dirichlet boundary on Ω. Define the inner of E_k as following:

$$\left((\xi, \eta), (\bar{\xi}, \bar{\eta})\right) = \left(\nabla^{2m+k}\xi, \nabla^{2m+k}\bar{\xi}\right) + \left(\nabla^k \eta, \nabla^k \bar{\eta}\right).$$

$M(s), \rho$ and p satisfy the following conditions:

(A1) For $\forall s \ge 0$, we have $\varepsilon + 1 \le \mu_0 \le M(s) \le \mu_1$, where μ_0, μ_1 are constants, and

$$\mu = \begin{cases} \mu_0, & \frac{d}{dt}\|\Delta^m u\|^2 \ge 0 \\ \mu_1, & \frac{d}{dt}\|\Delta^m u\|^2 \le 0 \end{cases};$$

(A2) $\begin{cases} 1 \le \rho \le \frac{8m}{n-4m}, & n > 4m \\ \rho \ge -1, & n \le 4m \end{cases};$

(A3) $\begin{cases} 2 \le p \le \frac{2n}{n-2m}, & n > 2m \\ p \ge 2, & n \le 2m \end{cases};$

(A4) $0 \le \varepsilon \le \min\left\{ \sqrt{1 + \frac{\beta\lambda_1^{2m}}{4}} - 1, \frac{2\mu}{\beta}, \frac{2\mu\lambda_1^{2m}}{\beta\lambda_1^{2m}+1}, \left(2\beta - 2 - \frac{1}{\lambda_1^{2m-k}}\right)\left(1+\mu_1^2\right) \right\}.$

3. Exponential Attractors

Definition 3.1. Compact set M_k is called a family of exponential attractors of $\left(\{S(t)\}_{t\ge 0}, E_k\right)$, if a family of compact attractors $A_k \subseteq M_k \subseteq E_k$ satisfies:

1) $S(t)M_k \subseteq M_k, \forall t \ge 0$;

2) M_k has finite fractal dimension d_{M_k};

3) There exist constants $a_0, a_1 > 0$ such that

$$\forall t \geq 0, dist\left(S(t)E_k, M_k\right) \leq a_0 e^{-a_1 t}.$$

Definition 3.2. [1] Solution semigroup $S(t)$ is Lipschitz continuity, if there is a bounded function $l(t)$, such that

$$\left\|S(t)U - S(t)V\right\|_{E_k} \leq l(t)\left\|U - V\right\|_{E_k}, \forall U, V \in E_k.$$

Definition 3.3. [1] Assume that solution semigroup $S(t): E_k \to E_k$ is a map satisfies Lipschitz continuity, then we say $S_* = S(t_*)$ is squeezing in E_k, if $\forall \delta \in \left(0, \frac{1}{8}\right)$, $\exists t_* > 0$, $N = N(\delta)$, there is

$$\left\|Q_N\left(S_*U - S_*V\right)\right\|_{E_k} \geq \left\|P_N\left(S_*U - S_*V\right)\right\|_{E_k}, \forall U, V \in E_k,$$

or

$$\left\|S_*U - S_*V\right\|_{E_k} < \delta\left\|U - V\right\|_{E_k}, \forall U, V \in E_k,$$

where P_N is an orthogonal projection in $E_k, Q_N = I - P_N$.

Theorem 3.1. [15] Assuming that

1) $S(t)$ possesses a family of (E_k, E_0)-compact attractors A_k;

2) $S(t)$ exists a family of positive, invariant compact sets in E_k;

3) $S(t)$ is Lipschitz continuous and squeezing on E_k;

then $S(t)$ exists a family of (E_k, E_0)-type exponential attractors M_k and

$$M_k = \bigcup_{0 \leq t \leq t_*} S(t) M_*, \; M_* = A_k \cup \left(\bigcup_{j=1}^{\infty} \bigcup_{k=1}^{\infty} S(t_*)^j \left(E^{(l)}\right)\right),$$

moreover, the fractal dimension of M_k satisfies $d_{M_k} \leq cN_0 + 1$, where N_0 is the least of N which makes squeezing found.

Theorem 3.2. [9] Suppose that (A1)-(A4) are valid. Let $(u_0, u_1) \in E_k$, $k = 1, 2, \cdots, 2m$, $f(x) \in L^2(\Omega)$, then the initial boundary value problem (1.1) has a global solution (u, v) that satisfies $u \in L^\infty\left(0, +\infty; H_0^{2m+k}(\Omega)\right)$, $v \in L^\infty\left(0, +\infty; H_0^k(\Omega)\right) \cap L^2\left(0, T; H_0^{2m+k}(\Omega)\right)$, and there exists a nonnegative real number R_k and $T = t(\Omega) > 0$ so that

$$\left\|(u, v)\right\|_{E_k}^2 = \left\|\nabla^{2m+k} u\right\|^2 + \left\|\nabla^k v\right\|^2 \leq R_k^2, \quad (t > T).$$

According to paper [9], the solution semigroup $S(t)$ of the initial boundary value problem (1.1) exists a family of (E_k, E_0)-compact attractors A_k, and we can define a family of positive, invariant compact sets $B_k = \overline{\bigcup_{0 \leq t \leq T} S(t) B_{0k}}$, where $B_{0k} = \left\{U \in E_k : \left\|U\right\|_{E_k} \leq R_k^2\right\}$.

Next, we prove the problem (1.1) exists a family of exponential attractors.

Let Equation (1.1) transform into a first-order evolution equation:

$$U_t + F(U) + R(U) = 0, U \in E_k, \tag{3.1}$$

where

$$F(U) = \left(\varepsilon u - v, \varepsilon\beta(-\Delta)^{2m} u + \beta(-\Delta)^{2m} v + \varepsilon^2 u - \varepsilon v\right)^{\mathrm{T}},$$

$$R(U) = \left(0, M\left(\left\|\nabla^m u\right\|_p^p\right)(-\Delta)^{2m} u + |u|^\rho u - f(x)\right)^{\mathrm{T}}.$$

Lemma 3.1. (Lipschitz property) $\forall U_0, V_0 \in B_k$, there is

$$\left\|S(t)U_0 - S(t)V_0\right\|_{E_k} \le \mathrm{e}^{\alpha_2 t}\left\|U_0 - V_0\right\|_{E_k},$$

where $\alpha_2 = \max\left\{\varepsilon^2\beta^2 + \frac{C_1^2 + C_2^2\lambda_1^m}{\lambda_1^{2m+k}}, \varepsilon\right\}$.

Proof. Let $\bar{U}(t) = U(t) - V(t) = \left(\bar{u}(t), \bar{v}(t)\right)^{\mathrm{T}}$, where $U(t) = S(t)U_0 = \left(u_1(t), v_1(t)\right)^{\mathrm{T}}$, $V(t) = S(t)V_0 = \left(u_2(t), v_2(t)\right)^{\mathrm{T}}$, $\bar{u}(t) = u_1(t) - u_2(t)$, then we have

$$\bar{U}_t + F(\bar{U}) + R(U) - R(V) = 0, \tag{3.2}$$

Taking the inner product of Equation (3.2) with W in E_k, and we get that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\bar{U}_t\right\|_{E_k}^2 + \left(F(\bar{U}), \bar{U}\right)_{E_k} + \left(R(U) - R(V), W\right)_{E_k} = 0, \tag{3.3}$$

By using Young's inequality, Holder's inequality, Poincare's inequality and differential mean value theorem, we obtain

$$\begin{aligned}\left(F(\bar{U}), \bar{U}\right)_{E_k} &= \varepsilon\left\|\nabla^{2m+k}\bar{u}\right\|^2 - \left(\nabla^{2m+k}\bar{u}, \nabla^{2m+k}\bar{v}\right) + \beta\left\|\nabla^{2m+k}\bar{v}\right\|^2 \\ &\quad - \varepsilon\beta\left(\nabla^{2m+k}\bar{u}, \nabla^{2m+k}\bar{v}\right) + \varepsilon^2\left(\nabla^k\bar{u}, \nabla^k\bar{v}\right) - \varepsilon\left\|\nabla^k\bar{v}\right\|^2 \\ &\ge \left(\frac{\varepsilon - \varepsilon^2\beta^2}{2}\right)\left\|\nabla^{2m+k}\bar{u}\right\|^2 + \left(\beta - \frac{1}{2\varepsilon} - \frac{1}{2}\right)\left\|\nabla^{2m+k}\bar{v}\right\|^2 - \varepsilon\left\|\nabla^k\bar{v}\right\|^2.\end{aligned} \tag{3.4}$$

$$\begin{aligned}&\left|\left(R(U) - R(V), \bar{U}\right)_{E_k}\right| \\ &= \left|\left(\nabla^k\left(|u_1|^\rho u_1 - |u_2|^\rho u_2\right), \nabla^k\bar{v}\right)\right| \\ &\quad + \left|\left(M\left(\left\|\nabla^m u_1\right\|_p^p\right)\nabla^{2m+k}u_1 - M\left(\left\|\nabla^m u_2\right\|_p^p\right)\nabla^{2m+k}u_2, \nabla^{2m+k}\bar{v}\right)\right| \\ &\le \left(\frac{C_1^2 + C_2^2\lambda_1^m}{2\lambda_1^{2m+k}} + \frac{\varepsilon}{2}\right)\left\|\nabla^{2m+k}\bar{u}\right\|^2 + \left(\frac{1}{2\lambda_1^{2m-k}} + \frac{\varepsilon + \mu_1^2}{2\varepsilon}\right)\left\|\nabla^{2m+k}\bar{v}\right\|^2.\end{aligned} \tag{3.5}$$

Substitute (3.4) and (3.5) into (3.3), we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left\|\bar{U}\right\|_{E_k}^2 + \left(2\beta - 2 - \frac{1 + \mu_1^2}{\varepsilon} - \frac{1}{\lambda_1^{2m-k}}\right)\left\|\nabla^{2m+k}\bar{v}\right\|^2 \\ &\le \left(\varepsilon^2\beta^2 + \frac{C_1^2 + C_2^2\lambda_1^m}{\lambda_1^{2m+k}}\right)\left\|\nabla^{2m+k}\bar{u}\right\|^2 + \varepsilon\left\|\nabla^k\bar{v}\right\|^2.\end{aligned}$$

Let $\alpha_1 = 2\beta - 2 - \frac{1 + \mu_1^2}{\varepsilon} - \frac{1}{\lambda_1^{2m-k}}$, $\alpha_2 = \max\left\{\varepsilon^2\beta^2 + \frac{C_1^2 + C_2^2\lambda_1^m}{\lambda_1^{2m+k}}, \varepsilon\right\}$, then we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\|\bar{U}\right\|_{E_k}^2 + \alpha_1\left\|\nabla^{2m+k}\bar{v}\right\|^2 \le \alpha_2\left\|\bar{U}\right\|_{E_k}^2. \tag{3.6}$$

By using Gronwall's inequality, we get

$$\left\|\bar{U}(t)\right\|_{E_k}^2 \le \mathrm{e}^{\alpha_2 t}\left\|\bar{U}(0)\right\|_{E_k}^2. \tag{3.7}$$

Meanwhile, we obtain $Lip_{B_k}\left(S(t)\right)\le e^{\frac{\alpha_2 t}{2}}$.

Lemma 3.1 is proved.

Lemma 3.2. (discrete squeezing) $\forall U_0, V_0 \in B_k$, if

$$\left\|Q_N\left(S_*U_0 - S_*V_0\right)\right\|_{E_k} \ge \left\|P_N\left(S_*U_0 - S_*V_0\right)\right\|_{E_k},$$

then we have

$$\left\|S_*U_0 - S_*V_0\right\|_{E_k} < \frac{1}{8}\left\|U_0 - V_0\right\|_{E_k}.$$

Proof. Applying Q_N to Equation (3.2), we get

$$Q_N\bar{U}_t(t) + Q_N F(\bar{U}) + Q_N\left(R(U) - R(V)\right) = 0. \quad (3.8)$$

Taking the inner product of Equation (3.8) with $Q_N\bar{U}$ in E_k, we obtain

$$\frac{1}{2}\frac{d}{dt}\left\|Q_N\bar{U}_t\right\|_{E_k}^2 + \left(Q_N F(\bar{U}), Q_N\bar{U}\right)_{E_k} + \left(Q_N\left(R(U) - R(V)\right), Q_N\bar{U}\right)_{E_k} = 0, \quad (3.9)$$

Similar to the process of Lemma 3.1, we have

$$\frac{d}{dt}\left\|Q_N\bar{U}\right\|_{E_k}^2 + \alpha_3\left\|\nabla^{2m+k}Q_N\bar{v}\right\|^2 \le \alpha_4\left\|Q_N\bar{U}\right\|_{E_k}^2. \quad (3.10)$$

where $\alpha_3 = 2\beta - 2 - \frac{1+\mu_1^2}{\varepsilon} - \frac{1}{\lambda_{N+1}^{2m-k}}$, $\alpha_4 = \max\left\{\varepsilon^2\beta^2 + \frac{C_1^2 + C_2^2\lambda_{N+1}^m}{\lambda_{N+1}^{2m+k}}, \varepsilon\right\}$.

By using Gronwall's inequality, we get

$$\left\|Q_N\bar{U}(t)\right\|_{E_k}^2 \le e^{\alpha_4 t}\left\|Q_N\bar{U}(0)\right\|_{E_k}^2.$$

Suppose $\left\|Q_N\left(S_*U_0 - S_*V_0\right)\right\|_{E_k} \ge \left\|P_N\left(S_*U_0 - S_*V_0\right)\right\|_{E_k}$, then we have

$$\begin{aligned}\left\|S_*U_0 - S_*V_0\right\|_{E_k}^2 &\le \left\|P_N\left(S_*U_0 - S_*V_0\right)\right\|_{E_k}^2 + \left\|Q_N\left(S_*U_0 - S_*V_0\right)\right\|_{E_k}^2\\ &\le 2\left\|Q_N\left(S_*U_0 - S_*V_0\right)\right\|_{E_k}^2 \le 2e^{\alpha_4 t_*}\left\|Q_N\bar{U}(0)\right\|_{E_k}^2\\ &\le 2e^{\alpha_4 t_*}\left\|U_0 - V_0\right\|_{E_k}^2.\end{aligned}$$

Therefore, when $t_* \le -\frac{7\ln 2}{\alpha_4}$, we have

$$\left\|S_*U_0 - S_*V_0\right\|_{E_k} \le \frac{1}{8}\left\|U_0 - V_0\right\|_{E_k}.$$

Lemma 3.2 is proved.

In fact, Theorem 3.1 has provided the theoretical basis to prove the existence of random attractors. Because of the value of $k(k = 1, \cdots, 2m)$, furthermore, we can obtain the existence theorem of the family of exponential attractors via Lemma 3.1 and Lemma 3.2 as following:

Theorem 3.3. Assume (A1)-(A4) are valid, $f \in H$, then $\forall U \in E_k$, the solution semigroup $S(t)$ of the initial boundary value problem (1.1) exists a family of (E_k, E_0) -type exponential attractors M_k, and

$$M_k = \bigcup_{0 \le t \le t_*} S(t)\left(A_k \cup \left(\bigcup_{j=1}^{\infty} \bigcup_{k=1}^{\infty} S(t_*)^j \left(E^{(l)}\right)\right)\right),$$

$$d_{M_k} \le cN_0 + 1, k = 1, 2, \cdots, 2m.$$

4. Random Attractors

Define the time-translation operator on Ω: $\theta_s \omega(t) = \omega(t+s) - \omega(s)$, then $\left(\Omega, \mathcal{F}, P, \{\theta_t\}_{t \in R}\right)$ constitutes an ergodic, metric dynamical system. According to paper [8], we have some definitions and theorems as following:

Definition 4.1. $\left(\Omega, \mathcal{F}, P, \{\theta_t\}_{t \in R}\right)$ is a metric dynamical system, $\left(B(R^+) \times \mathcal{F} \times B(E_k), B(E_k)\right)$ is measurable. ϕ is called a continuous stochastic dynamical system, if map $\phi : R^+ \times \Omega \times E_k \to E_k$ satisfies

1) $\phi(0, \omega)x = x, x \in E_k, \omega \in \Omega$;

2) $\phi(t+s, \omega) = \phi(t, \theta_s \omega) \circ \phi(s, \omega), \forall t, s \in R, x \in E_k, \omega \in \Omega$;

3) $(t, \omega, x) \to \phi(t, \omega, x)$ is continuous.

Definition 4.2. [8] $B_k \subset E_k$ is called a family of tempered random sets, if

$$\lim_{|s| \to \infty} \inf \mathrm{e}^{-\beta s} d\left(B_k(\theta_{-s}\omega)\right) = 0, \omega \in \Omega,$$

where $\beta > 0$, $d(B_k) = \sup_{x \in B_k} \|x\|_{E_k}$.

Definition 4.3. [8] $D(\omega)$ is the set of all the random sets on E_k. Random set $B_k(\omega)$ is called a family of absorption sets on $D(\omega)$, if $\forall B_k(\omega) \in D(\omega)$, $\exists T > 0$, such that

$$\phi(t, \theta_{-t}\omega) B_k(\theta_{-t}\omega) \subset B_{0k}(\omega), \quad P - a.e. \omega \in \Omega.$$

Definition 4.4. [8] Random set $\mathcal{A}_k(\omega)$ is called a family of random attractors of continuous random dynamic system $\{S(t, s; \omega)\}_{t \ge s}$ on E_k, if $\mathcal{A}_k(\omega)$ satisfies:

1) $\mathcal{A}_k(\omega)$ is a family of random compact sets;

2) $\mathcal{A}_k(\omega)$ is a family of invariant sets, which means $\forall t > 0, S(t, \omega)\mathcal{A}_k(\omega) = \mathcal{A}_k(\theta_t \omega)$;

3) $\mathcal{A}_k(\omega)$ attracts all sets on $D(\omega)$, which means $\forall B_k(\omega) \in D(\omega)$, there is

$$\lim_{t \to \infty} d\left(S(t, \theta_{-t}\omega) B_k(\Theta_{-t}\omega), \mathcal{A}_k(\omega)\right) = 0,$$

where $d(A, B) = \sup_{x \in A} \inf_{y \in B} \|x - y\|$ denotes the Hausdorff half-distance.

Theorem 4.1. Assume the family of random sets $B_k(\omega) \in D(\omega)$ is a family of random absorbing sets of the stochastic dynamic system $\{S(t, s; \omega)\}_{t \ge s}$, and it satisfies:

1) random set $B_k(\omega)$ is closed set on E_k;

2) $\forall P - a.e. \omega \in \Omega$, $B_k(\omega)$ is asymptotically compact. Namely, when $t_n \to +\infty$, there exists a convergent subsequence in E_k for $\forall x_n \in S(t_n, \theta_{-t_n}\omega)$.

Then there exists a unique family of global attractors $\mathcal{A}_k(\omega)$ of the stochastic dynamic system $\{S(t, \omega)\}_{t \ge 0}$, and

$$\mathcal{A}_k(\omega) = \bigcap_{\tau \ge t(\omega)} \overline{\bigcup_{t \ge \tau} S(t, \theta_{-t}\omega) B_k(\theta_{-t}\omega)}.$$

Rewrite Equation (1.2) into a stochastic differential equation:

$$\begin{cases} \mathrm{d}u_t + \left(M\left(\left\|\nabla^m u\right\|_p^p\right)(-\Delta)^{2m} u + \beta(-\Delta)^{2m} u_t + |u|^\rho u\right)\mathrm{d}t = q(x)\mathrm{d}W(t), \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), \quad x \in \partial\Omega, \quad t > 0, \end{cases} \tag{4.1}$$

Equation (4.1) can be reduced to

$$\begin{cases} \mathrm{d}\varphi + L\varphi \mathrm{d}t = F(\theta_t\omega, \varphi), \\ \varphi_0(\omega) = (u_0, u_1 + \varepsilon u_0)^{\mathrm{T}}, \end{cases} \tag{4.2}$$

where $\varphi = (u,v)^{\mathrm{T}}$, $v = u_t + \varepsilon u$, $F(\theta_t\omega,\varphi) = \left(0, -|u|^\rho u\mathrm{d}t + q(x)\mathrm{d}W(t)\right)^{\mathrm{T}}$,

$$L = \begin{pmatrix} \varepsilon I & -I \\ \left(\left(M\left(\left\|\nabla^m u\right\|_p^p\right) - \varepsilon\beta\right)\Delta^{2m} + \varepsilon^2\right)I & \left(\beta\Delta^{2m} - \varepsilon\right)I \end{pmatrix}.$$

Let $\delta = \delta(\theta_t\omega) = -\int_{-\infty}^0 \mathrm{e}^s \theta_t\omega(s)\mathrm{d}s$, where $\delta(\theta_t\omega)$ denotes Ornstein-Uhlenbeck process and is the solution of Ito equation:

$$\mathrm{d}\delta + \delta \mathrm{d}t = \mathrm{d}W.$$

Let $z = v - q(x)\delta(\theta_t\omega)$, furthermore, rewrite Equation (4.1)

$$\begin{cases} \psi_t + L\psi = \bar{F}(\theta_t\omega, \psi), \\ \psi_0(\omega) = \left(u_0, u_1 + \varepsilon u_0 - q(x)\delta(\theta_t\omega)\right)^{\mathrm{T}}, \end{cases} \tag{4.3}$$

where $\varphi = (u,z)^{\mathrm{T}}$, $\bar{F}(\theta_t\omega,\varphi) = \left(\theta_t\omega, -|u|^\rho u + \left(\varepsilon + 1 - \beta\Delta^{2m}\right)q(x)\delta(\theta_t\omega)\right)^{\mathrm{T}}$.

Lemma 4.1. $\forall x = (x_1, x_2)^{\mathrm{T}} \in E_k$, there is

$$(Lx, x)_{E_k} \ge \alpha_5 \|x\|_{E_k}^2 + \frac{\alpha_6}{2}\left\|\nabla^{2m+k} x_2\right\|^2,$$

where $\alpha_5 = \min\left\{\frac{\beta\varepsilon + \varepsilon}{2\beta} - \frac{\varepsilon^2}{2\beta\lambda^{2m}}, \frac{\beta\lambda^{2m} - \beta\varepsilon^2}{2} - \varepsilon\right\}$, $\alpha_6 = \beta - \beta\varepsilon^2 + \beta\varepsilon$.

Proof.

$$\begin{aligned} (Lx,x)_{E_k} &= \left(\nabla^{2m+k}(\varepsilon x_1 - x_2), \nabla^{2m+k} x_1\right) + \left(M\left(\left\|D^m u\right\|_p^p\right)\nabla^{2m+k} x_1, \nabla^{2m+k} x_2\right) \\ &\quad - \beta\varepsilon\left(\nabla^{2m+k} x_1, \nabla^{2m+k} x_2\right) + \varepsilon^2\left(\nabla^k x_1, \nabla^k x_2\right) + \beta\left\|\nabla^{2m+k} x_2\right\|^2 - \varepsilon\left\|\nabla^k x_2\right\|^2 \\ &\ge \varepsilon\left\|\nabla^{2m+k} x_1\right\|^2 - (\beta\varepsilon - \varepsilon)\left\|\nabla^{2m+k} x_2\right\|^2 - \frac{\varepsilon^2}{2}\left\|\nabla^k x_1\right\|^2 - \frac{\varepsilon^2}{2}\left\|\nabla^k x_2\right\|^2 \\ &\quad + \frac{\beta\lambda^{2m}}{2}\left\|\nabla^k x_2\right\|^2 + \frac{\beta}{2}\left\|\nabla^{2m+k} x_2\right\|^2 - \varepsilon\left\|\nabla^k x_2\right\|^2 \\ &\ge \left(\frac{\beta\varepsilon + \varepsilon}{2\beta} - \frac{\varepsilon^2}{2\beta\lambda^{2m}}\right)\left\|\nabla^{2m+k} x_1\right\|^2 + \left(\frac{\beta\lambda^{2m} - \beta\varepsilon^2}{2} - \varepsilon\right)\left\|\nabla^k x_2\right\|^2 \\ &\quad + \left(\frac{\beta - \beta\varepsilon^2 + \beta\varepsilon}{2}\right)\left\|\nabla^{2m+k} x_2\right\|^2. \end{aligned}$$

Let $\alpha_5 = \min\left\{\frac{\beta\varepsilon + \varepsilon}{2\beta} - \frac{\varepsilon^2}{2\beta\lambda^{2m}}, \frac{\beta\lambda^{2m} - \beta\varepsilon^2}{2} - \varepsilon\right\}$, $\alpha_6 = \beta - \beta\varepsilon^2 + \beta\varepsilon$, we obtain

$$(Lx,x)_{E_k} \ge \alpha_5 \|x\|_{E_k}^2 + \frac{\alpha_6}{2}\|\nabla^{2m+k}x_2\|^2.$$

Lemma 4.2. Assume φ is the solution of problem (4.3), then there exists a bounded random compact set $\tilde{B}_{0k}(\omega)\in D(E_k)$, such that for arbitrarily random set $B_k(\omega)\in D(E_k)$, there is a random variable $T_k>0$, we have

$$\varphi(t,\theta_t\omega)B_k(\theta_{-t}\omega)\subset \tilde{B}_{0k}(\omega), \forall t>T_k, \omega\in\Omega.$$

Proof. Suppose ψ is the solution of problem (4.3). Taking the inner product of Equation (4.3) with ψ in E_k, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\psi\|_{E_k}^2 + (L\psi,\psi)_{E_k} = \left(\bar{F}(\theta_t\omega,\psi),\psi\right)_{E_k}. \tag{4.4}$$

According to Lemma 4.1, we get

$$(L\psi,\psi)_{E_k} \ge \alpha_5\|\psi\|_{E_k}^2 + \frac{\alpha_6}{2}\|\nabla^{2m+k}z\|^2. \tag{4.5}$$

$$\begin{aligned}\left(\bar{F}(\theta_t\omega,\psi),\psi\right)_{E_k} &= \left(\nabla^{2m+k}q(x)\delta,\nabla^{2m+k}u\right)-\left(\nabla^k\left(|u|^\rho u\right),\nabla^k z\right)\\ &\quad-\beta\left(\nabla^{2m+k}q(x)\delta,\nabla^k z\right)+(\varepsilon+1)\left(\nabla^k q(x)\delta,\nabla^k z\right)\\ &\le \frac{\varepsilon}{2}\|\psi\|_{E_k}^2+\left(\frac{1+\lambda_1^k}{2\lambda_1^{2m}}+\frac{\beta}{2}\right)\|\lambda^{2m+k}z\|^2+C_3\\ &\quad-\left(\frac{1}{2\varepsilon}+\frac{\beta}{2}+\frac{\varepsilon+1}{2\lambda_1^{2m}}\right)\|\nabla^{2m+k}q(x)\|^2|\delta(\theta_t\omega)|^2.\end{aligned} \tag{4.6}$$

Substitute (4.5)-(4.6) into (4.4), we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\|\psi\|_{E_k}^2+(2\alpha_5-\varepsilon)\|\psi\|_{E_k}^2+\left(\alpha_6-\beta-\frac{1+\lambda_1^k}{2\lambda_1^{2m}}\right)\|\nabla^{2m+k}z\|^2\\ &\le 2C_3+\left(\frac{1}{\varepsilon}+\beta+\frac{\varepsilon+1}{\lambda_1^{2m}}\right)\|\nabla^{2m+k}q(x)\|^2|\delta(\theta_t\omega)|^2.\end{aligned}$$

Let $C_4=2\alpha_5-\varepsilon$, $C_5=2C_3$, $C_6=\frac{1}{\varepsilon}+\beta+\frac{\varepsilon+1}{\lambda_1^{2m}}\|\nabla^{2m+k}q(x)\|^2$, then we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\psi\|_{E_k}^2+C_4\|\psi\|_{E_k}^2 \le C_5+C_6|\delta(\theta_t\omega)|^2. \tag{4.7}$$

By using Gronwall's inequality, we get

$$\|\psi(t,\omega)\|_{E_k}^2 \le \mathrm{e}^{-C_4t}\|\psi_0(\omega)\|_{E_k}^2+\int_0^t \mathrm{e}^{-C_4(t-r)}\left(C_5+C_6|\delta(\theta_t\omega)|^2\right)\mathrm{d}r. \tag{4.8}$$

Because $\delta(\theta_t\omega)$ is tempered, $\delta(\theta_t\omega)$ is continuous about *t*. According to paper [11], we can obtain a temper random variable $r_1:\Omega\to R^+$, so that $\forall t\in R,\omega\in\Omega$, we have

$$|\delta(\theta_t\omega)|^2 \le r_1(\theta_t\omega)\le \mathrm{e}^{\frac{C_4}{2}|t|}r_1(\omega). \tag{4.9}$$

Substitute $\theta_{-t}\omega$ for ω in inequality (4.8), and let $\tau=r-t$, then we have

$$\begin{aligned}\|\psi(t,\theta_{-t}\omega)\|_{E_k}^2 &\le e^{-C_4 t}\|\psi_0(\theta_{-t}\omega)\|_{E_k}^2+\int_{-t}^{0}e^{C_4\tau}\left(C_5+C_6|\delta(\theta_\tau\omega)|^2\right)d\tau \\ &\le e^{-C_4 t}\|\psi_0(\theta_{-t}\omega)\|_{E_k}^2+\frac{C_5}{C_6}+\frac{2C_6}{C_4}r_1(\omega).\end{aligned} \tag{4.10}$$

Because $\psi_0(\theta_{-t}\omega)\in B_k(\theta_{-t}\omega)$ is tempered and $|\delta(\theta_{-t}\omega)|$ is tempered, $R_0^2(\omega)=\frac{C_5}{C_6}+\frac{2C_6}{C_4}r_1(\omega)$ is also tempered. Then

$\tilde{B}_{0k}=\left\{\psi\in E_k:\|\psi\|_{E_k}\le R_0(\omega)\right\}$ is a random attractor set. Because of

$$\begin{aligned}&\tilde{S}(t,\theta_{-t}\omega)\psi_0(\theta_{-t}\omega)\\ &=\varphi(t,\theta_{-t}\omega)(\psi_0(\theta_{-t}\omega))+\left(0,q(x)\delta(\theta_{-t}\omega)^{\mathrm{T}}\right)-\left(0,q(x)\delta(\theta_{-t}\omega)\right)^{\mathrm{T}},\end{aligned}$$

$\tilde{B}_{0k}=\left\{\varphi\in E_k:\|\varphi\|_{E_k}\le R_0(\omega)+\|\nabla^k q(x)\delta(\theta_{-t}\omega)\|=\bar{R}_0(\omega)\right\}$ is a random attractor set of $\varphi(t,\omega)$.

Lemma 4.3. When $k=1,2,\cdots,2m$, $\forall B_k(\omega)\in D(E_k)$, assume $\varphi(t)$ is the solution of problem (4.2), and we decompose $\varphi=\varphi_1+\varphi_2$, where

$$\begin{cases}d\varphi_1+L\varphi_1 dt=0,\\ \varphi_{01}(\omega)=(u_0,u_1+\varepsilon u_0)^{\mathrm{T}},\end{cases} \tag{4.11}$$

$$\begin{cases}d\varphi_2+L\varphi_2 dt=F(\theta_t\omega,\varphi),\\ \varphi_{02}(\omega)=0,\end{cases} \tag{4.12}$$

then $\|\varphi_1(t,\theta_{-t}\omega)\|_k^2\to 0(t\to\infty)$, $\forall\varphi_0(\theta_{-t}\omega)\in B_k(\theta_{-t}\omega)$.

Also, there exists a temper random radius $R_1(\omega)$, such that $\forall\omega\in\Omega$, $R_1(\omega)$ satisfies $\|\varphi_2(t,\Theta_{-t}\omega)\|_k^2\le R_1(\omega)$.

Proof. Suppose $\psi=\psi_1+\psi_2$ is the solution of Equation (4.4), then we can know from Equation (4.11) and Equation (4.12) that ψ_1,ψ_2 satisfy

$$\begin{cases}\psi_{1t}+L\psi_1=0,\\ \psi_{01}=\left(u_0,u_1+\varepsilon u_0-q(x)\delta(\theta_t\omega)\right)^{\mathrm{T}},\end{cases} \tag{4.13}$$

$$\begin{cases}\psi_{2t}+L\psi_2=\bar{F}(\psi_2,\theta_t\omega),\\ \psi_{02}(\omega)=0,\end{cases} \tag{4.14}$$

Taking the inner product of Equation (4.13) with ψ_1 in E_k, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\psi_1\|_{E_k}^2+(L\psi_1,\psi_1)=0. \tag{4.15}$$

By using Gronwall's inequality and Lemma 4.1, we have

$$\|\psi_1(t,\omega)\|_{E_k}^2\le e^{-2\alpha_5 t}\|\psi_{01}(\omega)\|_{E_k}^2. \tag{4.16}$$

Substitute $\theta_{-t}\omega$ for ω in inequality (4.16), and $\delta(\theta_t\omega)\in B_k$ is tempered, thus

$$\|\psi_1(t,\theta_{-t}\omega)\|_{E_k}^2\le e^{-2\alpha_5 t}\|\psi_{01}(\omega)\|_{E_k}^2\to 0(t\to\infty),\forall\psi_{01}(\theta_{-t}\omega)\in B_k(\theta_{-t}\omega)$$

Taking the inner product of Equation (4.14) with ψ_2 in E_k, with Lemma 4.1

and Lemma 4.2 we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\psi_2\|_{E_{2m}}^2 + C_4\|\psi_2\|_{E_k}^2 \le C_7 + C_6\left|\delta(\theta_t\omega)\right|^2. \tag{4.17}$$

Substitute $\theta_{-t}\omega$ for ω in inequality (4.17), and by using Gronwall's inequality, we obtain

$$\begin{aligned}\left\|\psi_2(t,\theta_{-t}\omega)\right\|_{E_k}^2 &\le \mathrm{e}^{-C_4 t}\left\|\psi_{02}(t,\theta_{-t}\omega)\right\|_{E_k}^2 + \int_0^t \mathrm{e}^{C_4(\tau)}\left(C_7 + C_6\left|\delta(\theta_\tau\omega)\right|^2\right)\mathrm{d}r \\ &\le \frac{C_7}{C_4} + \frac{2C_6}{C_4} r_1(\omega),\end{aligned} \tag{4.18}$$

Thus, there exists a temper random radius $R_1^2(\omega) = \frac{C_7}{C_4} + \frac{2C_6}{C_4} r_1(\omega)$, such that $\forall\omega\in\Omega$, $\left\|\varphi_2(t,\Theta_{-t}\omega)\right\|_k^2 \le R_1^2(\omega)$.

Lemma 4.4. The stochastic dynamic system $\{S(t,\omega), t\ge 0\}$ determined by problem (4.3) has a family of compact absorbing sets $D_k(\omega)$, while $t=0$, $P-a.e.\omega\in\Omega$.

Proof. Suppose $D_k(\omega)$ be a closed ball with radius $R_1^2(\omega)$ in E_k. According to Rellich-Kondrachov Compact Embedding Theorem, $E_k \to E_0$. Then $D_k(\omega)$ is the compact set in E_k. For any tempered random set $\forall B_k(\omega), \varphi(t,\theta_{-t}\omega)\in B_k$, according to Lemma 4.3, we have $\varphi_2 = \varphi - \varphi_1 \in D_k(\omega)$, then for all $t \ge T_k > 0$, when $t\to\infty$, via Lemma 4.2 we have

$$\begin{aligned}d_{E_k}\left(S(t,\theta_{-t}\omega)B_k(\theta_{-t}\omega), D_k(\omega)\right) &= \inf_{v(t)\in D_k(\omega)}\left\|\varphi(t,\theta_{-t}\omega) - v(t)\right\|_{E_k}^2 \\ &\le \left\|\varphi_1(t,\theta_{-t}\omega)\right\|_{E_k}^2 \\ &\le \mathrm{e}^{-2C_4 t}\left\|\varphi_{01}(t,\theta_{-t}\omega)\right\|_{E_k}^2 \to 0\end{aligned}$$

Lemma 4.4 is proved.

According to the lemma mentioned above, we verified two conditions in Theorem 4.1. Similarly, we can obtain the theorem as following:

Theorem 4.2. The stochastic dynamic system $\{S(t,\omega), t\ge 0\}$ has a family of random attractors $\mathcal{A}_k(\omega)\subset D_k(\omega)\subset E_k, \omega\in\Omega$, and there exists a slowly increasing family of random sets $D_k(\omega)$, such that

$$\mathcal{A}_k(\omega) = \bigcap_{\tau\ge t(\omega)}\overline{\bigcup_{t\ge\tau} S(t,\Theta_{-t}\omega)D_k(\Theta_{-t}\omega)}, P-a.e.\omega\in\Omega.$$

and

$$S(t,\omega)\mathcal{A}_k(\omega) = \mathcal{A}_k(\theta_t\omega), k = 1,2,\cdots,2m.$$

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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