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A Family of the Random Attractors for a Class of Generalized Kirchhoff-Type Equations

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Abstract

In this paper, we studied the existence of a family of the random attractor for a class of generalized Kirchhoff-type equations with a strong dissipation term. Firstly, according to Ornstein-Uhlenbeck process, we transformed the equation into a stochastic equation with random variables and multiplicative white noise. Secondly, we proved the existence of a bounded random absorbing set. Finally, by using the isomorphic mapping method and the compact embedding theorem, we get the stochastic dynamical system with a family of random attractors.

Keywords

Stochastic Kirchhoff Equation, A Family of the Random Attractors, Multiplicative White Noise, Ornstein-Uhlenbeck Process

1. Introduction

In recent years, the global attractor, exponential attractor, inertial manifold, and approximate inertial manifold of the Kirchhoff equation in infinite dimensional dynamical systems have been extensively studied. With further in-depth research, people have found that many real-life problems will be interfered with by all external uncertain factors to varying degrees, and a deterministic dynamic system cannot be used to describe this type of problem. At this time, we introduce a random attractor with multiplicative white noise. The random attractor is a measurable, compact and invariant random set that attracts all solution orbits. As the smallest absorption set in the solution set of an infinite-dimensional dynamical system, the random attractor is also the largest invariant set; it can better describe the development trajectory of a disturbing object, to further predict the state of the development of things to a certain moment. In other words, the random attractor is a reasonable promotion of the global attractor of the classic deterministic dynamic system, so the random attractor has more practical and deeper properties. The random attractors can be used to study fluid mechanics, finance and other fields; they are the supplement of deterministic dynamical systems. Therefore, many scholars have done a lot of research on random attractors of nonlinear partial differential equations with white noise, and have obtained a series of research results, including stochastic parabolic equations, generalized Ginzburg-Landau equations, dissipative KdV equation, stochastic reaction-diffusion equations, stochastic Sine-Gordon equations, stochastic Boussinesq equations, stochastic Kirchhoff equations and other stochastic evolution equations have corresponding study about random attractors, more significant research can refer to [1]-[10].

Guoguang Lin, Ling Chen, Wei Wang [11] studied the stochastic strongly damped higher-order nonlinear Kirchhoff-type equation with white noise:

$$\begin{aligned} &\mathrm{d}u_t + \left[(-\Delta)^m u_t + \phi\left(\|\nabla^m u\|^2\right)(-\Delta)^m u + g(u)\right]\mathrm{d}t \\ &= f(x)\mathrm{d}t + q\mathrm{d}W(t), x\in\Omega, m>1. \end{aligned}$$

They proved the existence of a random attractor of the random dynamical system.

Guigui Xu and Libo Wang [12] studied the large-time behavior of the following initial boundary value problem for the stochastic strongly damped wave equation with white noise in a bounded domain $\mathcal{D}\subset \mathrm{R}$ with smooth boundary:

$$u_{tt} - \Delta u - \alpha\Delta u_t + \beta u_t + f(u) - g(x) = q(x)\dot{W}, (x,t)\in\mathcal{D}\times[0,+\infty);$$

$$u(x,0) = u_0(x), u_t(x,0) = u_1(x), \quad x\in\mathcal{D};$$

$$u(x,t)\big|_{\partial\Omega} = 0, \quad (x,t)\in\partial\mathcal{D}\times[0,+\infty).$$

where $(u_0,u_1)\in H_0^1(\mathcal{D})\times L^2(\mathcal{D})$, and α,β are positive constants, $u=u(x,t)$ is a real-valued function on $\mathcal{D}\times[0,+\infty)$. $\dot{W}$ is a scalar Gaussian white noise, that is, $W(t)$ is a two-sided wiener process.

The functions $f:\mathrm{R}\to\mathrm{R}$ and $g,q:\mathcal{D}\to\mathrm{R}$ satisfies the following assumptions:

1) $g\in H_0^1(\mathcal{D})$, while $q\in H^2(\mathcal{D})\times H_0^1(\mathcal{D})$ is not identically equal to zero;

2) The nonlinear term f satisfies

$$|f'(u)|\le C_0, |f(u)|\le C_1, \quad \forall u\in\mathrm{R};$$

$$|f'(u)-f'(v)|\le C_2|u-v|, \quad \forall u,v\in\mathrm{R}.$$

where C_0,C_1,C_2 are positive constants.

Guoguang Lin and Zhuoxi Li [13] studied the random attractor family of solutions to the strongly damped stochastic Kirchhoff equation with white noise:

$$u_{tt} + M\left(\|\nabla^m u\|_2\right)(-\Delta)^m u + \beta(-\Delta)^m u_t + g(x,u) = q(x)\dot{W}.$$

They get the temper random compact sets of random attractor family.

On the basis of reference [13], the stress term $\|D^m u\|^2$ is extended to $\|D^m u\|_p^p$, this paper studied the long-time dynamic behavior of a class of generalized Kirchhoff equation. According to preliminary knowledge and reasonable assumption for Kirchhoff stress term and nonlinear source term, we proved the existence of random absorbing set in stochastic dynamical system; furthermore, a family of the random attractor is obtained.

In this paper, we study the existence of a family of the random attractors for a class of generalized Kirchhoff-type equation with damping term:

$$u_{tt} + M\left(\left\|\nabla^m u\right\|_p^p\right)(-\Delta)^{2m} u + \beta(-\Delta)^{2m} u_t + g(u) = q\dot{W}, \tag{1.1}$$

$$u(x,t) = 0, \frac{\partial^i u}{\partial v^i} = 0, i = 1,2,\cdots,2m-1, x \in \partial\Omega, \tag{1.2}$$

$$u(x,0) = u_0(x), u_t(x,0) = u_1(x), x \in \Omega, t > 0. \tag{1.3}$$

where $m > 1, p \ge 2$, $M(s) \in C^2([0,+\infty); R^+)$ is a real-valued function, $\beta(-\Delta)^{2m} u_t (\beta > 0)$ denotes strong damping term, $g(u)$ is nonlinear source term, $u = u(x,t)$ is a real-valued function on $\Omega \times [0,+\infty)$, $\Omega \subset R^n (n \ge 1)$ is a bounded domain with a smooth boundary $\partial\Omega$, $q\mathrm{d}W$ denotes an additive white noise. $W(t)$ is a one-dimensional bilateral Wiener process on probability space (Ω, F, P), $\Omega = \{\omega \in C(R,R) : \omega(0) = 0\}$, F is a Borel σ-algebra generated by compact open topology on Ω, p is a probability measure, the assumption of $g(u)$ and $M(s)$ as follow:

(A1) $g(u) \in C^\infty(R)$ is Lipschitz continuous;

(A2) There existence constant $l_g > 0$, such that $\left\|\nabla^k (g(u) - g(v))\right\| \le l_g \left\|\nabla^k (u - v)\right\|$;

(A3) $J(u) = \int G(u)\mathrm{d}x$, where $G'(u) = g(u)u_t$;

(A4) $J(u) \ge -\frac{\mu}{4}\left\|\nabla^{2m+k} u\right\|^2 - c$;

(A5) $M(s) \in C^2([0,+\infty), R^+)$, $\varepsilon + 1 = \mu_0 < M(s) < \mu_1$,

$$\mu = \begin{cases} \mu_0, \frac{\mathrm{d}}{\mathrm{d}t}\left\|\nabla^{2m} u\right\|^2 \ge 0 \\ \mu_1, \frac{\mathrm{d}}{\mathrm{d}t}\left\|\nabla^{2m} u\right\|^2 < 0 \end{cases} \text{ and } 0 < \varepsilon < \min\left\{\frac{\sqrt{1 + 2\beta\lambda_1^{2m}} - 1}{2}, \frac{\mu_0 + \sqrt{\mu_0^2 - \lambda_1^{-2m}}}{\lambda_1^{-2m}}\right\}.$$

Where μ, μ_0, μ_1 are constant, λ_1 is the first eigenvalue of $-\Delta$ with homogeneous Dirichlet boundary conditions on Ω.

2. Preliminaries

For convenience, define the following spaces and notations:

$$H = L^2(\Omega), H_0^{2m}(\Omega) = H^{2m}(\Omega) \cap H_0^1(\Omega), H_0^{4m}(\Omega) = H^{4m}(\Omega) \cap H_0^1(\Omega),$$
$$H_0^{2m+k}(\Omega) = H^{2m+k}(\Omega) \cap H_0^1(\Omega), E_0 = H^{2m}(\Omega) \times L^2(\Omega),$$
$$E_k = H_0^{2m+k}(\Omega) \times H_0^k(\Omega), (k = 0,1,2,\cdots,2m), f(x) \in L^2(\Omega).$$

$(\cdot,\cdot)$ and $\|\cdot\|$ represent the inner product and norms of H respectively, *i.e.*:

$$(u,v) = \int_\Omega u(x)v(x)\mathrm{d}x, (u,u) = \|u\|^2, \|\cdot\| = \|\cdot\|_{L^2(\Omega)}, \|\cdot\|_P = \|\cdot\|_{L^P(\Omega)}, \|\cdot\|_\infty = \|\cdot\|_{L^\infty(\Omega)}.$$

$$\left(y_1,y_2\right)_{E_k}=\left(\nabla^{2m+k}u_1,\nabla^{2m+k}u_2\right)+\left(\nabla^k v_1,\nabla^k v_2\right),$$
$$\forall y_i=\left(u_i,v_i\right)\in E_k, i=1,2,\left(k=1,2,\cdots,2m\right).$$

Here are some basic knowledge of stochastic dynamic systems required:

$\left(B\left(R^+\right)\times F\times B\left(X\right),B_k\left(\omega\right)\right)\subset D\left(\omega\right)$ is a probabilistic space and define a family of measures-preserving and ergotic transformations of $\left\{\theta_t,t\in R\right\}$:

$$\theta_t w\left(\cdot\right)=w\left(\cdot+t\right)-w\left(t\right).$$

$\left(\Omega,F,P,\left(\theta_t\right)_{t\in R}\right)$ is an ergodic metric dynamical system.

Let $\left(X,\|\cdot\|\right)$ is a separable Hilbert space and $B\left(X\right)$ is a Borel σ-algebra on *X*, $\left(\Omega,F,P\right)$ be a probability space, where $\Omega=\left\{w\in C\left(\mathrm{R},\mathrm{R}\right);w\left(0\right)=0\right\}$ is endowed with compact-open topology, P is the corresponding Wiener measure, and F is the Borel σ-algebra on Ω. The space $\left(\Omega,F,P,\left(\theta_t\right)_{t\in\mathrm{R}}\right)$ is called the metric dynamical system on the probability space $\left(\Omega,F,P\right)$.

Definition 2.1. ([9]) *Let* $\left(\Omega,F,P,\left(\theta_t\right)_{t\in\mathrm{R}}\right)$ *be a metric dynamical system, if* $\left(B\left(R^+\right)\times F\times B\left(X\right),B\left(X\right)\right)$ *-measurable mapping*

$$S:R^+\times\Omega\times X\to X,\left(t,w,x\right)\mapsto S\left(t,w,x\right).$$

satisfies the following properties:

1) *For* $\forall s,t\geq 0$ *and* $w\in\Omega$, *mapping* $S\left(t,w\right):=S\left(t,w,\cdot\right)$ *satisfies*

$$S\left(0,w\right)=id,S\left(t+s,w\right)=S\left(t,\theta_s w\right)\circ S\left(s,w\right).$$

2) *For* $\forall w\in\Omega$, *mapping* $\left(t,w,x\right)\mapsto S\left(t,w,x\right)$ *is continuous. Then S is a continuous stochastic dynamical system on* $\left(\Omega,F,P,\left(\theta_t\right)_{t\in\mathrm{R}}\right)$.

Definition 2.2. ([9]) *It is said that random set* $B\left(w\right)\subset X$ *is tempered. If for* $w\in\Omega$, $\beta\geq 0$, *there is*

$$\lim_{\|s\|\to\infty}\inf \mathrm{e}^{-\beta s}d\left(B\left(\theta_{-s}w\right)\right)=0.$$

where $d\left(B\right)=\sup_{x\in B}\|x\|_X$, *for* $\forall x\in X$.

Definition 2.3. ([9]) *Let* $D\left(w\right)$ *as the set of all random sets on X, and random set* $B\left(w\right)$ *is called an absorption set on* $D\left(w\right)$. *If for any* $B\left(w\right)\in D\left(w\right)$ *and* $P_{a.e.w}\in\Omega$, *there exists* $T_{B(w)}>0$ *such that*

$$S\left(t,\theta_{-t}w\right)\left(B\left(\theta_{-t}w\right)\right)\subset B_0\left(w\right).$$

Definition 2.4. ([9]) *Random set* $A\left(w\right)$ *is called a random attractor on X for continuous stochastic dynamical system* $\left(S\left(t,w\right)\right)_{t\geq 0}$, *if random set* $A\left(w\right)$ *satisfies*

1) $A\left(w\right)$ *is a random compact set*;

2) $A\left(w\right)$ *is an invariant set, that is, for arbitrary* $t>0$, $S\left(t,w\right)A\left(w\right)=A\left(\theta_t w\right)$;

3) $A\left(w\right)$ *attracts all sets in* $D\left(w\right)$, *that is, for any* $B\left(w\right)\in D\left(w\right)$ *and* $P_{a.e.w}\in\Omega$, *we have the limit formula*

$$\lim_{t\to\infty}d\left(S\left(t,\theta_{-t}w\right)B\left(\theta_{-t}w\right),A\left(w\right)\right)=0.$$

where $d\left(A,B\right)=\sup_{x\in A}\inf_{y\in B}\|x-y\|_H$ *is the Hausdorff semi-distance.* (*There*

$A, B \subseteq H$).

Definition 2.5. ([9]) *Let random set* $B_k(w) \in D(w)$ *be the random absorption set of stochastic dynamical system* $(S(t,w))_{t\geq 0}$, *and random set* $B_k(w)$ *satisfies*

1) *Random set* $B_k(w)$ *is closed set on Hilbert space;*

2) *For* $P_{a.e.w} \in \Omega$, *random set* $B_k(w)$ *satisfies the following asymptotic compactness conditions for arbitrary sequence* $x_n \in S(t_n, \theta_{-t_n} w) B_0(\theta_{-t_n} w)$, $t \to +\infty$, *there is a convergent subsequence in space X, then the stochastic dynamical system* $(S(t,w))_{t\geq 0}$ *has a unique global attractor, i.e.,*

$$A_k(w) = \bigcap_{\tau \geq t(w)} \overline{\bigcup_{t \geq \tau} S(t, \theta_{-t} w) B_0(\theta_{-t} w)}.$$

Theorem 2.1. [9] *The Ornstein-Uhlenbeck process is given as following:*

From the above we can know that the Ornstein-Uhlenbeck process on $H_0^{2m+k}(\Omega)$ *is given by Wiener process on measurement system* $(\Omega, F, P, (\theta_t)_{t\in R})$.

Set $z(\theta_t w) = -\alpha \int_{-\infty}^{0} e^{\alpha\tau} \theta_t w(\tau) d\tau$, *where* $t \in R$. *It can be seen that for any* $t \geq 0$, *the stochastic process* $z(\theta_t w)$ *satisfies the Ito equation*

$$dz + \alpha z dt = dW(t).$$

According to the nature of the O-U process, there exists a probability measure P, θ_t*-invariant set* $\Omega_0 \subset \Omega$, *and the above stochastic process*

$$z(\theta_t w) = -\alpha \int_{-\infty}^{0} e^{\alpha\tau} \theta_t w(\tau) d\tau$$

satisfies the following properties:

1) *The mapping* $s \to z(\theta_s w)$ *is a continuous mapping, for any given* $w \in \Omega_0$;

2) *The random variable* $\|z(w)\|$ *is tempered;*

3) *There exist a slowly increasing set* $r(w) > 0$, *such that*

$$\|z(\theta_t w)\| + \|z(\theta_t w)\|^2 \leq r(\theta_t w) \leq r(w) e^{\frac{\alpha}{2}|t|};$$

4) $\lim_{t\to\infty} \frac{1}{t} \int_0^t |z(\theta_t w)|^2 d\tau = \frac{1}{2\alpha}$;

5) $\lim_{t\to\infty} \frac{1}{t} \int_0^t |z(\theta_t w)| d\tau = \frac{1}{\sqrt{\pi\alpha}}$.

3. The Existence for a Family of the Random Attractor

In this section, our objection is to prove the existence of random attractors for the initial boundary value problem (1.1)-(1.3).

At first, we define the inner product and norms on E_k as follows:

$$(y_1, y_2)_{E_k} = (\nabla^{2m+k} u_1, \nabla^{2m+k} u_2) + (\nabla^k v_1, \nabla^k v_2),$$
$$\forall y_i = (u_i, v_i) \in E_k, i = 1, 2, (k = 1, 2, \cdots, 2m);$$
$$\|y\|_{E_k}^2 = (y, y)_{E_k} = \|\nabla^{2m+k} u\|^2 + \|\nabla^k v\|^2.$$

Let $U = (u, v) \in E_k, v = u_t + \varepsilon u$, there exist $\varepsilon > 0$, such that the Equation (1.1)-

(1.3) equivalent the following evolution equation:

$$\begin{cases} \mathrm{d}u = u_t \mathrm{d}t, \\ \mathrm{d}u_t + \left[M\left(\left\|\nabla^m u\right\|_p^p\right)(-\Delta)^{2m} u + \beta(-\Delta)^{2m} u_t + g(u) \right]\mathrm{d}t = q(x)\mathrm{d}W(t), t \in [0,+\infty], \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), x \in \Omega. \end{cases} \quad (3.1)$$

Let $\varphi = (u,v)^{\mathrm{T}}, v = u_t + \varepsilon u$, the problem (3.1) can be simplified to

$$\begin{cases} \mathrm{d}\varphi + L\varphi \mathrm{d}t = F(\theta_t w, \varphi), \\ \varphi_0(w) = (u_0, u_1 + \varepsilon u_0)^{\mathrm{T}}. \end{cases} \quad (3.2)$$

where $\varphi = \begin{pmatrix} u \\ v \end{pmatrix}$, $L = \begin{pmatrix} \varepsilon I & -I \\ \left(\left(M\left(\left\|\nabla^m u\right\|_p^p\right) - \beta\varepsilon\right)(-\Delta)^{2m} + \varepsilon^2\right)I & \left(\beta(-\Delta)^{2m} - \varepsilon\right)I \end{pmatrix}$,

$F(\theta_t w, \varphi) = \begin{pmatrix} 0 \\ -g(u)\mathrm{d}t + q(x)\mathrm{d}W(t) \end{pmatrix}$. Suppose $z = v - q(x)\delta(\theta_t w)$, $\delta(\theta_t w)$ is a stochastic process, then Equation (3.1) can be written as

$$\begin{cases} \psi_t + L\psi \mathrm{d}t = \bar{F}(\theta_t w, \psi), \\ \psi_0(w) = (u_0, u_1 + \varepsilon u_0 - q(x)\delta(\theta_t w))^{\mathrm{T}}. \end{cases} \quad (3.3)$$

where $\psi = \begin{pmatrix} u \\ z \end{pmatrix}$, $L = \begin{pmatrix} \varepsilon I & -I \\ \left(\left(M\left(\left\|\nabla^m u\right\|_p^p\right) - \beta\varepsilon\right)(-\Delta)^{2m} + \varepsilon^2\right)I & \left(\beta(-\Delta)^{2m} - \varepsilon\right)I \end{pmatrix}$,

$$\bar{F}(\theta_t w, \psi) = \begin{pmatrix} q(x)\delta(\theta_t w) \\ -g(u) + \left(\varepsilon - 1 - \beta(-\Delta)^{2m}\right)q(x)\delta(\theta_t w) \end{pmatrix}.$$

Lemma 3.1. Assume that nonlinear source term $g(u)$ and Kirchhoff stress term $M(s)$ satisfy the assumption (A1), (A2), $f \in H$, $(u_0, v_0) \in E_0 = H^{2m}(\Omega) \times L^2(\Omega)$, then the initial boundary value problem (1.1)-(1.3) has smooth solution $(u,v) \in E_0$ and $v \in L^2(0,T;H^{2m}(\Omega))$ satisfy the following inequality

$$\left\|(u,v)\right\|_{E_0}^2 = \left\|\nabla^{2m} u\right\|^2 + \left\|v\right\|^2 \le \left\|Y(0)\right\| \mathrm{e}^{-b_1 t} + \frac{C_1}{b_1}\left(1 - \mathrm{e}^{-b_1 t}\right). \quad (3.4)$$

Where $v = u_t + \varepsilon u$, $b_1 = \min\left\{a_1, \frac{a_2}{\mu}, 2\varepsilon\right\}$, $Y(0) = \left\|v_0\right\|^2 + \mu\left\|\nabla^{2m} u_0\right\|^2 + \varepsilon^2\left\|u_0\right\|^2$, so there's a non-negative real number $R_0 = \sqrt{\frac{2C_1}{b_1}}$, $t_1 = \frac{1}{b_1}\ln\left(\frac{b_1\left\|Y(0)\right\|}{C_1}\right)$ and $\int_0^T \left\|\nabla^{2m} v\right\|^2 \mathrm{d}t \le C$, such that

$$\left\|(u,v)\right\|_{E_0}^2 = \left\|\nabla^{2m} u\right\|^2 + \left\|v\right\|^2 \le R_0^2 \left(t > t_1\right). \quad (3.5)$$

Proof. Taking the inner product of the second equation of (3.1) with v in $L^2(\Omega)$, we find that

$$\left(u_{tt} + M\left(\left\|\nabla^m u\right\|_p^p\right)(-\Delta)^{2m} u + \beta(-\Delta)^{2m} u_t + g(u), v\right) = \left(q(x)\dot{W}, v\right). \quad (3.6)$$

$$\left(u_{tt},v\right)=\left(v_t-\varepsilon v+\varepsilon^2 u,v\right)\geq\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v\|^2-\varepsilon\|v\|^2+\frac{\varepsilon^2}{2}\|v\|^2-\varepsilon^3\|u\|^2. \tag{3.7}$$

$$\begin{aligned}&\left(M\left(\left\|\nabla^m u\right\|_p^p\right)(-\Delta)^{2m}u,v\right)\\&=\left(M\left(\left\|\nabla^m u\right\|_p^p\right)\nabla^{2m}u,\nabla^{2m}\left(u_t+\varepsilon u\right)\right)\\&=\frac{M\left(\left\|\nabla^m u\right\|_p^p\right)}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\nabla^{2m}u\right\|^2+\varepsilon M\left(\left\|\nabla^m u\right\|_p^p\right)\left\|\nabla^{2m}u\right\|^2\\&\geq\frac{\mu}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\nabla^{2m}u\right\|^2+\varepsilon\mu_0\left\|\nabla^{2m}u\right\|^2.\end{aligned} \tag{3.8}$$

By using Poincare's inequality, we obtain

$$\begin{aligned}\left(\beta(-\Delta)^{2m}u_t,v\right)&=\beta\left\|\nabla^{2m}v\right\|^2-\left(\beta\varepsilon(-\Delta)^{2m}u,v\right)\\&\geq\frac{\beta}{2}\left\|\nabla^{2m}v\right\|^2+\frac{\beta\lambda_1^{2m}}{2}\|v\|^2-\frac{1}{2}\left\|\nabla^{2m}u\right\|^2-\frac{\beta^2\varepsilon^2}{2}\left\|\nabla^{2m}u\right\|^2.\end{aligned} \tag{3.9}$$

The following estimation can be obtained from hypothesis (A1)

$$\begin{aligned}\left(g(u),v\right)&=\left(g(u),u_t\right)+\varepsilon\left(g(u),u\right)=\frac{\mathrm{d}}{\mathrm{d}t}\int G(u)\mathrm{d}x+\varepsilon\left(g(u),u\right)\\&\geq\frac{\mathrm{d}}{\mathrm{d}t}\int G(u)\mathrm{d}x+\varepsilon^2\int G(u)\mathrm{d}x\geq\frac{\mathrm{d}}{\mathrm{d}t}J(u)+J(u).\end{aligned} \tag{3.10}$$

By using the weighted Young's inequality, we obtain

$$\left(q(x)\dot{W},v\right)\leq\left\|q(x)\dot{W}\right\|\cdot\|v\|\leq\frac{1}{2\varepsilon^2}\left\|q(x)\dot{W}\right\|^2+\frac{\varepsilon^2}{2}\|v\|^2. \tag{3.11}$$

Substitute inequality (3.6)-(3.10) into Equation (3.5), therefore

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left[\|v\|^2+\mu\left\|\nabla^{2m}u\right\|^2+2J(u)\right]+\left(\beta\lambda_1^{2m}-2\varepsilon-2\varepsilon^2\right)\|v\|^2\\&+\left(\beta-\beta^2\varepsilon^2\right)\left\|\nabla^{2m}v\right\|^2+\left(2\varepsilon\mu_0-\varepsilon^2\lambda_1^{-2m}-1\right)\left\|\nabla^{2m}u\right\|^2+2J(u)\\&\leq\frac{\left\|q(x)\dot{W}\right\|^2}{\varepsilon^2}+C_0.\end{aligned} \tag{3.12}$$

Let $a_1=\beta\lambda_1^{2m}-2\varepsilon-\varepsilon^2\geq0$, $a_2=2\varepsilon\mu_0-1\geq0$, $\beta-\beta^2\varepsilon^2-\lambda_1^{2m}\geq0$, and let $b_1=\min\left\{a_1,\frac{a_2}{\mu},2\varepsilon\right\}$, $C_1=\frac{\left\|q(x)\dot{W}\right\|^2}{\varepsilon^2}+C_0$, then the Equation (3.11) can be reduced to

$$\frac{\mathrm{d}}{\mathrm{d}t}Y(t)+b_1Y(t)+\left(\beta-\beta^2\varepsilon^2\right)\left\|\nabla^{2m}v\right\|^2\leq C_1. \tag{3.13}$$

According to hypothesis (A3)

$$\left(\|v\|^2+\left\|\nabla^{2m}u\right\|^2\right)\min\left(1,\frac{\mu}{2}\right)\leq\|v\|^2+\frac{\mu}{2}\left\|\nabla^{2m}u\right\|^2+\left(\frac{\mu}{2}\left\|\nabla^{2m}u\right\|^2+2J(u)\right)\leq C. \tag{3.14}$$

Then

$$Y(t)=\|v\|^2+\mu\left\|\nabla^{2m}u\right\|^2+2J(u)>0. \tag{3.15}$$

By using Gronwall's inequality, we get

$$\|(u,v)\|_{E_0}^2 = \|\nabla^{2m}u\|^2 + \|v\|^2 \le Y(0)e^{-b_1 t} + \frac{C_1}{b_1}\left(1-e^{-b_1 t}\right). \tag{3.16}$$

And

$$\overline{\lim_{t\to\infty}}\|(u,v)\|_{E_0}^2 \le \frac{C_1}{b_1}. \tag{3.17}$$

So, there are constants $R_0 = \sqrt{\frac{2C_1}{b_1}}$ and $t_1 = \frac{1}{b_1}\ln\left(\frac{b_1\|Y(0)\|}{C_1}\right) > 0$, we obtain

$$\|(u,v)\|_{E_0}^2 = \|\nabla^{2m}u\|^2 + \|v\|^2 \le R_0^2, (t > t_1). \tag{3.18}$$

Lemma 3.1 is proved. ■

Lemma 3.2. Let $E_k = H_0^{2m+k}(\Omega)\times H_0^k(\Omega), (k = 1,2,\cdots,2m)$, for $\forall y = (y_1, y_2)^T \in E_k$, we have

$$(Ly, y)_{E_k} \ge k_1\|y\|_{E_k}^2 + k_2\|\nabla^{2m+k}y_2\|^2. \tag{3.19}$$

where $k_1 = \min\left\{\frac{\beta\varepsilon + \varepsilon - \varepsilon^2\lambda_1^{-m}}{2\beta}, \frac{\beta\lambda_1^m - \beta\varepsilon^2 - 2\varepsilon}{2}\right\}$, $k_2 = \frac{\beta - \beta^2\varepsilon + \beta\varepsilon}{2}$.

Proof. Because of $L = \begin{pmatrix} \varepsilon I & -I \\ \left(\left(M\left(\|\nabla^m u\|_p^p\right) - \beta\varepsilon\right)(-\Delta)^{2m} + \varepsilon^2\right)I & \left(\beta(-\Delta)^{2m} - \varepsilon\right)I \end{pmatrix}$, $\forall y = (y_1, y_2)^T \in E_k$, we get

$$\begin{aligned}
&(Ly, y)_{E_k} \\
&= \left(\nabla^{2m+k}(\varepsilon y_1 - y_2), \nabla^{2m+k}y_1\right) + \Big(\nabla^k\left(M\left(\|\nabla^m u\|_p^p\right)\right)(-\Delta)^{2m}y_1 \\
&\quad - \beta\varepsilon(-\Delta)^{2m}y_1 + \varepsilon^2 y_1 + \beta(-\Delta)^{2m}y_2 - \varepsilon y_2, \nabla^k y_2\Big) \\
&= \varepsilon\|\nabla^{2m+k}y_1\|^2 - \left(\nabla^{2m+k}y_2, \nabla^{2m+k}y_1\right) + M\left(\|\nabla^m u\|_p^p\right)\left(\nabla^{2m+k}y_1, \nabla^{2m+k}y_2\right) \\
&\quad - \beta\varepsilon\left(\nabla^{2m+k}y_1, \nabla^{2m+k}y_2\right) + \varepsilon^2\left(\nabla^k y_1, \nabla^k y_2\right) + \beta\|\nabla^{2m+k}y_2\|^2 - \varepsilon\|\nabla^k y_2\|^2 \\
&\ge \varepsilon\|\nabla^{2m+k}y_1\|^2 - \left(\nabla^{2m+k}y_2, \nabla^{2m+k}y_1\right) + \left(\nabla^{2m+k}y_1, \nabla^{2m+k}y_2\right) \\
&\quad - (\beta\varepsilon - \varepsilon)\left(\nabla^{2m+k}y_1, \nabla^{2m+k}y_2\right) + \varepsilon^2\left(\nabla^k y_1, \nabla^k y_2\right) \\
&\quad + \beta\|\nabla^{2m+k}y_2\|^2 - \varepsilon\|\nabla^k y_2\|^2 \\
&\ge \varepsilon\|\nabla^{2m+k}y_1\|^2 - \frac{(\beta-1)\varepsilon}{2\beta}\|\nabla^{2m+k}y_1\|^2 - \frac{(\beta-1)\beta\varepsilon}{2} \\
&\quad - \frac{\varepsilon^2}{2\beta}\|\nabla^k y_1\|^2 - \frac{\beta\varepsilon^2}{2}\|\nabla^k y_2\|^2 + \beta\|\nabla^{2m+k}y_2\|^2 - \varepsilon\|\nabla^k y_2\|^2 \\
&\ge \frac{\beta\varepsilon + \varepsilon - \varepsilon^2\lambda_1^{-2m}}{2\beta}\|\nabla^{2m+k}y_1\|^2 + \frac{\beta - \beta^2\varepsilon + \beta\varepsilon}{2}\|\nabla^{2m+k}y_2\|^2 \\
&\quad + \frac{\beta\lambda_1^{2m} - \beta\varepsilon^2 - 2\varepsilon}{2}\|\nabla^k y_2\|^2
\end{aligned}$$

$$
\begin{aligned}
&\geq k_1\left(\left\|\nabla^{2m+k} y_1\right\|^2+\left\|\nabla^k y_2\right\|^2\right)+k_2\left\|\nabla^{2m+k} y_2\right\|^2 \\
&=k_1\|y\|_{E_k}^2+k_2\left\|\nabla^{2m+k} y_2\right\|^2 .
\end{aligned} \tag{3.20}
$$

Lemma 3.2 is proved.

Lemma 3.3. Let φ is a solution of Equation (3.2), then there exists a bounded random compact set $\tilde{B}_{0k}(w) \in D(E_k)$, such that for any random set $B_k(w) \in D(E_k)$, there exists a random variable $T_{B_k(w)} > 0$, we have

$$
\varphi(t,\theta_t w) B_k(\theta_{-t} w) \subset \tilde{B}_{0k}(w), \quad \forall t \geq T_{B_k(w)}, w \in \Omega. \tag{3.21}
$$

Proof. Let ψ is a solution of Equation (3.3), by using $\psi=(u,z)^{\mathrm{T}} \in E_k$ to taking the inner product of two sides of Equation (3.3) on E_k, we get

$$
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\psi\|_{E_k}^2+(L\psi,\psi)_{E_k}=\left(\bar{F}(\theta_t w,\psi),\psi\right). \tag{3.22}
$$

According to Lemma 3.2, we know

$$
(L\psi,\psi)_{E_k} \geq k_1\|\psi\|_{E_k}^2+k_2\left\|\nabla^{2m+k} z\right\|^2 . \tag{3.23}
$$

Furthermore, according to the inner product defined in E_k, we can get

$$
\begin{aligned}
\left(\bar{F}(\theta_t w,\psi),\psi\right)=&\left(\nabla^{2m+k} q(x)\delta(\theta_t w),\nabla^{2m+k} u\right)+\Big(\nabla^k\Big(-g(u) \\
&+\left(\varepsilon-1-\beta(-\Delta)^{2m}\right) q(x)\delta(\theta_t w)\Big),\nabla^k z\Big).
\end{aligned} \tag{3.24}
$$

By using Holder inequality, Young inequality and Poincare inequality, we get

$$
\left(\nabla^{2m+k} q(x)\delta(\theta_t w),\nabla^{2m+k} u\right) \leq \frac{\varepsilon}{2}\left\|\nabla^{2m+k} u\right\|^2+\frac{1}{2\varepsilon}\left\|\nabla^{2m+k} q(x)\left|\delta(\theta_t w)\right|\right\|^2 . \tag{3.25}
$$

$$
\left(\varepsilon\nabla^k q(x)\delta(\theta_t w),\nabla^k z\right) \leq \frac{\varepsilon\lambda_1^{-2m}}{2}\left\|\nabla^{2m+k} z\right\|^2+\frac{\varepsilon}{2}\left\|\nabla^k q(x)\right\|^2\left|\delta(\theta_t w)\right|. \tag{3.26}
$$

$$
\begin{aligned}
&\left(\nabla^k\left(1-\beta(-\Delta)^{2m}\right) q(x)\delta(\theta_t w),\nabla^k z\right) \\
&\leq \frac{\varepsilon\lambda_1^{-2m}-1}{2}\left\|\nabla^{2m+k} z\right\|^2+\frac{1}{2\varepsilon}\left\|\nabla^k q(x)\right\|^2\left|\delta(\theta_t w)\right|^2 \\
&+\frac{\beta^2}{2}\left\|\nabla^{2m+k} q(x)\right\|^2\left|\delta(\theta_t w)\right|^2 .
\end{aligned} \tag{3.27}
$$

According to assumption (A2), we get

$$
\begin{aligned}
&\left(-\nabla^k g(u),\nabla^k z\right) \leq \left\|\nabla^k\left(g(u)-g(0)\right)\right\|\left\|\nabla^k z\right\| \leq l_g\left\|\nabla^k u\right\|\left\|\nabla^k z\right\| \\
&\leq \frac{l_g}{2}\left(\left\|\nabla^k u\right\|^2+\left\|\nabla^k z\right\|^2\right) \leq \frac{l_g\lambda_1^{-2m}}{2}\left\|\nabla^{2m+k} u\right\|^2+\frac{l_g\lambda_1^{-2m}}{2}\left\|\nabla^{2m+k} z\right\|^2 .
\end{aligned} \tag{3.28}
$$

Combine (3.19)-(3.25), we have

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\|\psi\|_{E_k}^2+2k_1\|\psi\|_{E_k}^2+\left(2k_2-l_g\lambda_1^{-2m}-2\varepsilon\lambda_1^{-2m}+1\right)\left\|\nabla^{2m+k} z\right\|^2 \\
&\leq\left(\varepsilon+l_g\lambda_1^{-2m}\right)\left\|\nabla^{2m+k} u\right\|^2+\left(\beta^2+\frac{1}{\varepsilon}\right)\left\|\nabla^{2m+k} q(x)\right\|^2\left|\delta(\theta_t w)\right|^2 \\
&+\left(\varepsilon+\frac{1}{\varepsilon}\right)\left\|\nabla^k q(x)\right\|^2\left|\delta(\theta_t w)\right|^2+C_2.
\end{aligned} \tag{3.29}
$$

Let $\alpha_1 = 2k_1$, $M = \left(\beta^2 + \frac{1}{\varepsilon}\right)\left\|\nabla^{2m+k} q(x)\right\|^2 + \left(\varepsilon + \frac{1}{\varepsilon}\right)\left\|\nabla^k q(x)\right\|^2$, then

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\psi\|_{E_k}^2 + \alpha_1\|\psi\|_{E_k}^2 \le C_2 + M\left|\delta(\theta_t w)\right|^2. \tag{3.30}$$

By using Gronwall inequality $P_{a,e,w} \in \Omega$, we get

$$\left\|\psi(t,w)\right\|_{E_k}^2 \le \mathrm{e}^{-\alpha_1 t}\left\|\psi(0,w)\right\|_{E_k}^2 + \int_0^t \mathrm{e}^{-\alpha_1(t-s)}\left(C_2 + M\left|\delta(\theta_t w)\right|^2\right)\mathrm{d}s. \tag{3.31}$$

Because $\delta(\theta_t w)$ is tempered, and $\delta(\theta_t w)$ is continuous with respect to *t*, so refer to the reference [2], we can obtain a temper random variable $r_1 : \Omega \to R^+$, such that for $\forall t \in R$, $\omega \in \Omega$, there established

$$\left|\delta(\theta_t w)\right|^2 \le r_1(\theta_t w) \le \mathrm{e}^{k_1 t} r_1(w). \tag{3.32}$$

Then we use $\theta_{-t} w$ to replace the w in Equation (3.28), we can get

$$\left\|\psi(t,\theta_{-t} w)\right\|_{E_k}^2 \le \mathrm{e}^{-\alpha_1 t}\left\|\psi(0,\theta_{-t} w)\right\|_{E_k}^2 + \int_0^t \mathrm{e}^{-\alpha_1(t-s)}\left(C_2 + M\left|\delta(\theta_{s-t} w)\right|^2\right)\mathrm{d}s. \tag{3.33}$$

Let $\tau = s - t$, then

$$\begin{aligned}&\int_0^t \mathrm{e}^{-\alpha_1(t-s)}\left(C_2 + M\left|\delta(\theta_{s-t} w)\right|^2\right)\mathrm{d}s \\ &= \int_{-t}^0 \mathrm{e}^{\alpha_1 \tau}\left(C_2 + M\left|\delta(\theta_\tau w)\right|^2\right)\mathrm{d}\tau \le \frac{C_2}{\alpha_1} + \frac{2}{\alpha_1} M r_1(w).\end{aligned} \tag{3.34}$$

Because $\varphi(0,\theta_{-t} w) \in B_k(\theta_{-t} w)$ is tempered, and $\left|\delta(\theta_{-t} w)\right|$ is also tempered, so we let

$$R_0^2(w) \le \frac{C_2}{\alpha_1} + \frac{2}{\alpha_1} M r_1(w). \tag{3.35}$$

Then $R_0^2(w)$ is tempered, let $\hat{B}_{0k} = \left\{\psi \in E_k \mid \|\psi\|_{E_k} \le R_0(w)\right\}$ is a random absorption set, and because of

$$\begin{aligned}&\tilde{S}(t,\theta_{-t} w)\psi(0,\theta_{-t} w) \\ &= \varphi(t,\theta_{-t} w)\left(\psi(0,\theta_{-t} w) + \left(0, q(x)\delta(\theta_{-t} w)\right)^{\mathrm{T}}\right) - \left(0, q(x)\delta(\theta_{-t} w)\right)^{\mathrm{T}}.\end{aligned} \tag{3.36}$$

So let

$$\tilde{B}_{0k}(w) = \left\{\varphi \in E_k \mid \|\varphi\|_{E_k} \le R_0(w) + \left\|\nabla^k q(x)\delta(w)\right\| = \bar{R}_0(w)\right\}. \tag{3.37}$$

Then $\tilde{B}_{0k}(w)$ is the random absorption set of $\varphi(t,w)$, and $\tilde{B}_{0k}(w) \in D(E_k)$. Lemma 3.3 is proved.

Lemma 3.4. When $k = 1, 2, \cdots, 2m$, for $\forall B_k \in D(E_k)$, assume that $\varphi(t)$ is the solution of Equation (3.2) in initial value $\varphi_0 = (u_0, u_1 + \varepsilon u_0)^{\mathrm{T}} \in B_k$, it can be decomposed into $\varphi = \varphi_1 + \varphi_2$, where φ_1 and φ_2 satisfy

$$\begin{cases}\mathrm{d}\varphi_1 + L\varphi_1 \mathrm{d}t = 0, \\ \varphi_{10}(w) = (u_0, u_1 + \varepsilon u_0)^{\mathrm{T}}.\end{cases} \tag{3.38}$$

$$\begin{cases}\mathrm{d}\varphi_2 + L\varphi_2 \mathrm{d}t = F(w,\varphi), \\ \varphi_{20}(w) = 0.\end{cases} \tag{3.39}$$

Then $\|\varphi_1(t,\theta_{-t}w)\|_{E_k}^2 \to 0,(t\to\infty)$, For $\forall\varphi_0(\theta_{-t}w)\in B_k(\theta_{-t}w)$, there exist a tempered random radius $R_1(w)$, such that

$$\|\varphi_2(t,\theta_{-t}w)\|_{E_k}^2 \le R_1(w), \quad \forall w\in\Omega. \tag{3.40}$$

Proof. Let $\psi=\psi_1+\psi_2=(u_1,u_{1t}+\varepsilon u_1)^{\mathrm{T}}+(u_2,u_{2t}+\varepsilon u_2-q(x)\delta(\theta_t w))^{\mathrm{T}}$ is a solution of the Equation (3.3), then according to the Equation (3.35)-(3.36), we know that ψ_1,ψ_2 satisfy respectively

$$\begin{cases}\psi_{1t}+L\psi_1=0,\\ \psi_{10}=\psi_0=(u_0,u_1+\varepsilon u_0-q(x)\delta(\theta_t w))^{\mathrm{T}}.\end{cases} \tag{3.41}$$

$$\begin{cases}\psi_{2t}+L\psi_2=\bar{F}(\psi_2,\theta_t w),\\ \psi_{20}=0.\end{cases} \tag{3.42}$$

By taking the inner product of equation $\psi_1=(u_1,u_{1t}+\varepsilon u_1)^{\mathrm{T}}$ with Equation (3.38) on E_k, we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\psi_1\|_{E_k}^2+(L\psi_1,\psi_1)_{E_k}=0. \tag{3.43}$$

According to Lemma 3.2 and Gronwall inequality, we have

$$\|\psi_1(t,w)\|_{E_k}^2 \le \mathrm{e}^{-2k_1 t}\|\psi_{10}(w)\|_{E_k}^2. \tag{3.44}$$

We use $\theta_{-t}w$ to replace the w in inequality (3.41), we get

$$\|\psi_1(t,\theta_{-t}w)\|_{E_k}^2 \le \mathrm{e}^{-2k_1 t}\|\psi_0(\theta_{-t}w)\|_{E_k}^2 \to 0,(t\to\infty),\forall\psi_0(\theta_{-t}w)\in B_k. \tag{3.45}$$

Similarly, by using $\psi_2=(u_2,u_{2t}+\varepsilon u_2-q(x)\delta(\theta_t w))^{\mathrm{T}}$ to take the inner product with Equation (3.39) in E_k, and according to Lemma 3.1 Lemma 3.2 and Lemma 3.3, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\psi_2\|_{E_k}^2+\alpha_1\|\psi_2\|_{E_k}^2 \le C_2+M|\delta(\theta_t w)|^2. \tag{3.46}$$

where $\alpha_1=2k_1$, $M=\left(\beta^2+\frac{1}{\varepsilon}\right)\|\nabla^{4m}q(x)\|^2+\left(\varepsilon+\frac{1}{\varepsilon}\right)\|\nabla^{2m}q(x)\|^2$.

Then we use $\theta_{-t}w$ to replace the w in inequality (3.43) and by using Gronwall inequality, we get

$$\begin{aligned}&\|\psi_2(t,\theta_{-t}w)\|_{E_k}^2\\ &\le \mathrm{e}^{-\alpha_1 t}\|\psi_2(0,\theta_{-t}w)\|_{E_k}^2+\int_0^t \mathrm{e}^{-\alpha_1(t-s)}\left(C_2+M|\delta(\theta_{s-t}w)|^2\right)\mathrm{d}s\\ &\le \frac{C_2}{\alpha_1}+\frac{2}{\alpha_1}Mr_1(w).\end{aligned} \tag{3.47}$$

So there is a tempered random radius

$$R_1^2(w)=\frac{C_3}{\alpha_1}+\frac{2}{\alpha_1}Mr_1(w). \tag{3.48}$$

Thus, for $\forall w\in\Omega$, we have

$$\left\|\varphi_2\left(t,\theta_{-t}w\right)\right\|_{E_k} \le R_1\left(w\right). \tag{3.49}$$

Therefore, Lemma 3.4 is proved.

Lemma 3.5. The stochastic dynamical system $\{S(t,w),t\ge 0\}$, while $t=0$, $P_{a.e.w}\in\Omega$ determined by Equation (3.2) has a compact absorption set $K(w)\subset E_k$.

Proof Let $K(w)$ be a closed sphere in space E_k with a radius of $R_1(w)$. According to embedding relation $E_k\subset E_0$, $K(w)$ is a compact set in E_k. For any temper random set $B_k(w)$ in E_k, for $\forall\varphi(t,\theta_{-t}w)\in B_k$, according to Lemma 3.4, $\varphi_2=\varphi-\varphi_1\in K(w)$, so for $\forall t\ge T_{B_k(w)}>0$, we have

$$\begin{aligned}
&d_{E_k}\left(S\left(t,\theta_{-t}w\right)B_k\left(\theta_{-t}w\right),K\left(w\right)\right)\\
&=\inf_{\vartheta(t)\in K(w)}\left\|\psi\left(t,\theta_{-t}w\right)-\vartheta\left(t\right)\right\|_{E_k}^2 \le \left\|\psi_1\left(t,\theta_{-t}w\right)\right\|_{E_k}^2\\
&\le \mathrm{e}^{-2k_1t}\left\|\psi_{10}\left(t,\theta_{-t}w\right)\right\|_{E_k}^2 \to 0,\left(t\to\infty\right).
\end{aligned} \tag{3.50}$$

Lemma 3.5 is proved. According to Lemma 3.1-Lemma 3.5, we have the following theorem.

Theorem 3.1. The stochastic dynamical system $\{S(t,w),t\ge 0\}$ has a family of random attractors $A_k(w)\subset K(w)\subset E_k, w\in\Omega$ and there exists a tempered random set $K(w)$, so that $P_{a.e.w}\in\Omega$

$$A_k\left(w\right)=\bigcap_{t\ge 0,\tau\ge t}\overline{\bigcup S\left(t,\theta_{-\tau}w,K\left(\theta-\tau w\right)\right)}. \tag{3.51}$$

and $S(t,w)A_k(w)=A_k(\theta_t w)$.

In conclusion, according to Ornstein-Uhlenbeck process, we transformed the equation into a stochastic equation with random variables and multiplicative white noise; then we proved the existence of a bounded random absorbing set; through the isomorphic mapping method and the compact embedding theorem, we get the stochastic dynamical system with a family of the random attractors.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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