Scientific Research Publishing

Journal of Applied Mathematics and Physics, 2021, 9, 2757-2764
https://www.scirp.org/journal/jamp
ISSN Online: 2327-4379
ISSN Print: 2327-4352

Constructions of *n*-Dimensional Overlap Functions Based on Bivariate Overlap Functions

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How to cite this paper: Xie, H. (2021) Constructions of *n*-Dimensional Overlap Functions Based on Bivariate Overlap Functions. *Journal of Applied Mathematics and Physics*, **9**, 2757-2764.
https://doi.org/10.4236/jamp.2021.911177

Received: September 9, 2021
Accepted: November 12, 2021
Published: November 15, 2021

Abstract

In this paper, we firstly introduce some new results on overlap functions and n-dimensional overlap functions. On the other hand, in a previous study, Gómez *et al.* presented some open problems. One of these open problems is "to search the construction of n-dimensional overlapping functions based on bi-dimensional overlapping functions". To answer this open problem, in this paper, we mainly introduce one construction method of n-dimensional overlap functions based on bivariate overlap functions. We mainly use the conjunction operator $\wedge$ to construct n-dimensional overlap functions $\mathcal{O}_n^{\wedge}$ based on bivariate overlap functions and study their basic properties.

Keywords

Overlap Functions, n-Dimensional Overlap Functions, Conjunction Operator

1. Introduction

The concepts of overlap functions and grouping functions were firstly introduced by Bustince *et al.* in [1] [2] and [3], respectively. Overlap functions and grouping functions are two particular cases of bivariate continuous aggregation functions [4] [5]. Those two concepts have been applied to some interesting problems, for example, image processing [1] [6], classification [7] [8] and decision making [3] [9]. In recent years, some extended forms of overlap functions and grouping functions were presented, for example, n-Dimensional overlap functions and grouping functions [10], general overlap functions [11]. Overlap functions and grouping functions can be constructed by using additive generator pairs [12] or multiplicative generator pairs [13]. Xie [14] proposed the concepts of multiplicative generator pairs of n-dimensional overlap functions and presented the condition under which the multiplicative generator pairs can generate an n-dimensional overlap function. In [10], some open problems were presented. One of the open problems is "to search the construction of n-dimensional overlapping functions based on bi-dimensional overlapping functions". So far, this open problem has not been solved. In this paper, we try to solve this open problem. One characteristic of the conjunction operator $\wedge$ satisfies associativity and commutativity. We construct n-dimensional overlap functions $\mathcal{O}_n^{\wedge}$ by means of $\wedge$, and study their basic properties.

The rest of this paper is organized as follows. In Section 2, we review some concepts and results about overlap functions and n-dimensional overlap functions, which will be used throughout this paper. In Section 3, we mainly introduce some new results on overlap functions and n-dimensional overlap functions. In Section 4, one construction method of n-dimensional overlap functions based on bivariate overlap functions is discussed. We provide some conclusions in Section 5.

2. Preliminaries

In this section, we recall some concepts and properties of bivariate overlap functions and n-dimensional overlap functions which shall be needed in the sequel.

Definition 2.1 (See Bustince *et al.* [1]). A bivariate function $O:[0,1]^2 \to [0,1]$ is said to be an overlap function if it satisfies the following conditions:

(O1) O is commutative;

(O2) $O(x,y)=0$ iff $xy=0$;

(O3) $O(x,y)=1$ iff $xy=1$;

(O4) O is increasing;

(O5) O is continuous.

Example 2.1 (See Qiao and Hu [15]). For any $p>0$, consider the bivariate function $O_p:[0,1]^2 \to [0,1]$ given by

$$O_p(x,y)=x^p y^p$$

for all $x,y\in[0,1]$. Then it is an overlap function and we call it p-product overlap function, here. It is obvious that 1-product overlap function is the product t-norm. Moreover, for any $p\neq 1$, the p-product overlap function is neither associative nor has 1 as neutral element. Therefore, it is not a t-norm.

Definition 2.2. (See Dimuro and Bedregal [16]) An overlap function $O:[0,1]^2 \to [0,1]$ satisfies the Property 1-section deflation if

(O6) $\forall x\in[0,1], O(x,1)\le x$, and the Property 1-section inflation if

(O7) $\forall x\in[0,1], O(x,1)\ge x$.

An overlap function $O:[0,1]^2 \to [0,1]$ satisfies the Property diagonal inflation [17] if

(O8) $O(x,x)\ge x$ for all $x\in[0,1]$.

Denote by $\mathbb{O}$ the set of all overlap functions. Then $(\mathbb{O},\le_{\mathbb{O}})$ with the ordering

$\leq_{\mathbb{O}}$ defined for $O_1, O_2 \in \mathbb{O}$ by $O_1 \leq_{\mathbb{O}} O_2$ if and only if $O_1(x, y) \leq O_2(x, y)$ for all $x, y \in [0,1]$, is a lattice [16].

Lemma 2.1 (See Wang and Liu [18]). Let $O:[0,1]^2 \to [0,1]$ be an overlap function, and $\varphi:[0,1] \to [0,1]$ be a strictly increasing automorphism. Then $O_\varphi:[0,1]^2 \to [0,1]$ is an overlap function given by

$$O_\varphi(x, y) = \varphi^{-1}\left(O\left(\varphi(x), \varphi(y)\right)\right),$$

for all $x, y \in [0,1]$.

Definition 2.3 (See Bustince *et al.* [2]). Let $G:[0,1]^2 \to [0,1]$ be a mapping and $k \in]0, \infty[$. G is homogeneous of order k if for any $\alpha \in [0, \infty[$ and for any $x, y \in [0,1]$ such that $\alpha^k x, \alpha^k y \in [0,1]$ the identity

$$G(\alpha x, \alpha y) = \alpha^k G(x, y)$$

holds.

An n-ary aggregation function $A:[0,1]^n \to [0,1]$ is said to be idempotent if $A(x, \cdots, x) = x$ for any $x \in [0,1]$.

Definition 2.4 (See Dimuro and Bedregal [19]). An overlap function $O:[0,1]^2 \to [0,1]$ is said to be Archimedean if, for each $(x, y) \in]0,1[^2$, there exists $n \in \mathbb{N} - \{0\}$ such that $x_O^{(n)} < y$, where $x_O^{(n)}$ is $x_O^{(1)} = x$ and $x_O^{(n+1)} = o\left(x, x_O^{(n)}\right)$.

Lemma 2.2 (See Dimuro and Bedregal [19]). Let $O:[0,1]^2 \to [0,1]$ be an Archimedean overlap function. Then, for all $x \in]0,1[$, it holds that $O(x, x) < x$.

Definition 2.5 (See Gómez *et al.* [10]). An n-dimensional aggregation function $\mathcal{O}:[0,1]^n \to [0,1]$ is an n-dimensional overlap function if and only if:

$\mathcal{O}1$. $\mathcal{O}$ is symmetric.

$\mathcal{O}2$. $\mathcal{O}(x_1, \cdots, x_n) = 0$ if and only if $\prod_{i=1}^n x_i = 0$.

$\mathcal{O}3$. $\mathcal{O}(x_1, \cdots, x_n) = 1$ if and only if $x_i = 1$ for all $i \in \{1, \cdots, n\}$.

$\mathcal{O}4$. $\mathcal{O}$ is increasing.

$\mathcal{O}5$. $\mathcal{O}$ is continuous.

Let us denote by $\mathcal{SO}^n$ the set of all n-dimensional overlap functions. The set $\mathcal{SO}^n$ is a lattice with the ordering $\leq_{\mathcal{SO}^n}$ defined for $\mathcal{O}_1, \mathcal{O}_2 \in \mathcal{SO}^n$ as $\mathcal{O}_1 \leq_{\mathcal{SO}^n} \mathcal{O}_2$ if and only if $\mathcal{O}_1(x) \leq \mathcal{O}_2(x)$ for all $x \in [0,1]^n$ [10].

Lemma 2.3 (See Gómez *et al.* [10]). Let $\varphi:[0,1] \to [0,1]$ be an automorphism. Then, for every overlap function O, $\varphi \circ O$ and $O\left(\varphi(x), \varphi(y)\right)$ are also overlap functions.

In this paper, the overlap function $O\left(\varphi(x), \varphi(y)\right)$ will be denoted by $O^\varphi(x, y)$, *i.e.*, $O^\varphi(x, y) = O\left(\varphi(x), \varphi(y)\right)$.

Definition 2.6 (See Gómez *et al.* [10]). Let $G:[0,1]^n \to [0,1]$ be a mapping and let $k > 0$ be a positive value. Then, the function G is homogeneous of order k if and only if for any $\alpha \in [0,1]$ and for any $x \in [0,1]$ (with $\alpha^k x_i \in [0,1]$ for all $i \in \{1, \cdots, n\}$) the identity

$$G(\alpha x_1, \cdots, \alpha x_n) = \alpha^k G(x_1, \cdots, x_n)$$

holds.

3. Some New Results on Overlap Functions and *n*-Dimensional Overlap Functions

In this section, we mainly present some new results on overlap functions and *n*-dimensional overlap functions. These new results mainly reflect three properties: 1-section deflation, 1-section inflation and diagonal inflation on overlap functions and *n*-dimensional overlap functions.

Proposition 3.1. Let $O_1, O_2 : [0,1]^2 \to [0,1]$ be two overlap functions and $O_1 \leq_{\mathbb{O}} O_2$. If O_2 satisfies the Property 1-section deflation, then O_1 also satisfies the Property 1-section deflation.

Proof. Since $O_1 \leq_{\mathbb{O}} O_2$, if O_2 satisfies the Property 1-section deflation, then for any $x \in [0,1]$, one has that $O_1(1,x) \leq O_2(1,x) \leq x$. Hence O_1 satisfies the Property 1-section deflation. □

Proposition 3.2. Let $O_1, O_2 : [0,1]^2 \to [0,1]$ be two overlap functions and $O_1 \leq_{\mathbb{O}} O_2$. If O_1 satisfies the Property 1-section inflation (or diagonal inflation), then O_2 also satisfies the Property 1-section inflation (or diagonal inflation).

Proof. It can be proven in a similar way as that of Proposition 3.1. □

Proposition 3.3. Let $O : [0,1]^2 \to [0,1]$ be an overlap function. If O satisfies the Property 1-section deflation (1-section inflation or diagonal inflation), then O_φ also satisfies the Property 1-section deflation (1-section inflation or diagonal inflation).

Proof. We only verify that the Property 1-section deflation. The other two properties can be verified in a similar way.

If O satisfies the Property 1-section deflation, then for any $x \in [0,1]$,

$$O_\varphi(x,1) = \varphi^{-1}\left(O\left(\varphi(x), \varphi(1)\right)\right) = \varphi^{-1}\left(O\left(\varphi(x), 1\right)\right) \leq \varphi^{-1}\left(\varphi(x)\right) = x .$$

Hence O_φ satisfies the Property 1-section deflation. □

Now, we extend three properties 1-section deflation, 1-section inflation and diagonal inflation to the *n*-dimensional case ($n \geq 2$).

Definition 3.1. An *n*-dimensional overlap function $\mathcal{O}_n : [0,1]^n \to [0,1]$ satisfies the Property 1-section deflation if

($\mathcal{O}6$) $\forall x \in [0,1], \mathcal{O}_n(x,1,\cdots,1) \leq x$, and the Property 1-section inflation if

($\mathcal{O}7$) $\forall x \in [0,1], \mathcal{O}_n(x,1,\cdots,1) \geq x$, and the Property diagonal inflation if

($\mathcal{O}8$) $\forall x \in [0,1], \mathcal{O}_n(x,x,\cdots,x) \geq x$.

One can extend O_φ in Lemma 2.1 to the *n*-dimensional case O_φ.

Proposition 3.4 Let $\mathcal{O} : [0,1]^n \to [0,1]$ be an *n*-dimensional overlap function, and $\varphi : [0,1] \to [0,1]$ be a strictly increasing automorphism. Then $\mathcal{O}_\varphi : [0,1]^n \to [0,1]$ is an *n*-dimensional overlap function given by

$$\mathcal{O}_\varphi(x_1, x_2, \cdots, x_n) = \varphi^{-1}\left(O\left(\varphi(x_1), \varphi(x_2), \cdots, \varphi(x_n)\right)\right),$$

for all $x_i \in [0,1] \, (i = 1, 2, \cdots, n)$.

With similar Propositions 3.1 - 3.3, we easy to get the following Propositions.

Proposition 3.5. Let $\mathcal{O}, \mathcal{O}' : [0,1]^n \to [0,1]$ be two *n*-dimensional overlap functions and $\mathcal{O} \leq_{\mathcal{SO}^n} \mathcal{O}'$. If $\mathcal{O}'$ satisfies the Property 1-section deflation, then $\mathcal{O}$

also satisfies the Property 1-section deflation.

Proposition 3.6. Let $\mathcal{O},\mathcal{O}':[0,1]^n \to [0,1]$ be two n-dimensional overlap functions and $\mathcal{O} \leq_{\mathcal{SO}^n} \mathcal{O}'$. If $\mathcal{O}$ satisfies the Property 1-section inflation (or diagonal inflation), then $\mathcal{O}'$ also satisfies the Property 1-section inflation (or diagonal inflation).

Proposition 3.7. Let $\mathcal{O}:[0,1]^n \to [0,1]$ be an n-dimensional overlap function. If $\mathcal{O}$ satisfies the Property 1-section deflation (1-section inflation or diagonal inflation), then $\mathcal{O}_\varphi$ also satisfies the Property 1-section deflation (1-section inflation or diagonal inflation).

4. Constructing *n*-Dimensional Overlap Functions Based on Bivariate Overlap Functions

In this section, we mainly introduce the construction method of n-dimensional overlap functions based on bivariate overlap functions.

Proposition 4.1. Let $O:[0,1]^2 \to [0,1]$ be a bivariate overlap function. Then the function $\mathcal{O}_n^\wedge:[0,1]^n \to [0,1]$ defined as

$$\mathcal{O}_n^\wedge(x_1,x_2,\cdots,x_n) = \bigwedge_{\substack{i,j=1\\ i<j}}^{n} O(x_i,x_j)$$

is an n-dimensional overlap function.

Proof. $\mathcal{O}1$. It is obviously that $\mathcal{O}_n^\wedge$ is symmetric, because O is symmetric.

$\mathcal{O}2$.

$$\mathcal{O}_n^\wedge(x_1,x_2,\cdots,x_n)=0 \Leftrightarrow \bigwedge_{\substack{i,j=1\\ i<j}}^{n} O(x_i,x_j)=0$$

$$\Leftrightarrow \prod_{i=1}^n x_i = 0.$$

$\mathcal{O}3$.

$$\mathcal{O}_n^\wedge(x_1,x_2,\cdots,x_n)=1 \Leftrightarrow \bigwedge_{\substack{i,j=1\\ i<j}}^{n} O(x_i,x_j)=1$$

$$\Leftrightarrow O(x_i,x_j)=1 \text{ for all } i,j\in\{1,\cdots,n\},\ i<j$$

$$\Leftrightarrow x_i = 1 \text{ for all } i\in\{1,\cdots,n\}.$$

$\mathcal{O}4$ and $\mathcal{O}5$ obviously hold. □

Example 4.1. By use of O_p in Example 2.1, we can construct an 3-dimensional overlap function $\mathcal{O}_3^\wedge(x_1,x_2,x_3)$ as follows

$$\mathcal{O}_3^\wedge(x_1,x_2,x_3) = \bigwedge_{\substack{i,j=1\\ i<j}}^{3} O_p(x_i,x_j) = x_1^p x_2^p \wedge x_1^p x_3^p \wedge x_2^p x_3^p.$$

Proposition 4.2. Let $x\in[0,1]$ be the idempotent element of bivariate overlap function O. Then x is also the idempotent element of $\mathcal{O}_n^\wedge$.

Proof. Let $x\in[0,1]$ be the idempotent element of O, then

$$\mathcal{O}_n^\wedge(x,x,\cdots,x) = \bigwedge_{\substack{i,j=1\\ i<j}}^{n} O(x,x) = \bigwedge_{\substack{i,j=1\\ i<j}}^{n} x = x.$$

Hence x is the idempotent element of $\mathcal{O}_n^{\wedge}$. □

Proposition 4.3. Let $O:[0,1]^2 \to [0,1]$ be an Archimedean overlap function. Then, for all $x \in]0,1[$, it holds that $\mathcal{O}_n^{\wedge}(x,x,\cdots,x) < x$.

Proof. Let $O:[0,1]^2 \to [0,1]$ be an Archimedean overlap function, by Lemma 2.2, for all $x \in]0,1[$, we have

$$\mathcal{O}_n^{\wedge}(x,x,\cdots,x) = \bigwedge_{\substack{i,j=1\\ i<j}}^{n} O(x,x) < \bigwedge_{\substack{i,j=1\\ i<j}}^{n} x = x. \quad \square$$

Proposition 4.4. Let $O:[0,1]^2 \to [0,1]$ be a bivariate overlap function and $\varphi:[0,1] \to [0,1]$ be an automorphism. Then

$$\left(\mathcal{O}_n^{\wedge}\right)^{\varphi}(x_1,x_2,\cdots,x_n) = \bigwedge_{\substack{i,j=1\\ i<j}}^{n} O^{\varphi}(x_i,x_j). \tag{1}$$

Proof.

$$\begin{aligned}
\left(\mathcal{O}_n^{\wedge}\right)^{\varphi}(x_1,x_2,\cdots,x_n) &= \mathcal{O}_n^{\wedge}\left(\varphi(x_1),\varphi(x_2),\cdots,\varphi(x_n)\right)\\
&= \bigwedge_{\substack{i,j=1\\ i<j}}^{n} O\left(\varphi(x_i),\varphi(x_j)\right)\\
&= \bigwedge_{\substack{i,j=1\\ i<j}}^{n} O^{\varphi}(x_i,x_j).
\end{aligned} \quad \square$$

Proposition 4.5. Let $O:[0,1]^2 \to [0,1]$ be a bivariate overlap function and O is homogeneous of order k. Then $\mathcal{O}_n^{\wedge}$ is also homogeneous of order k.

Proof. For any $\alpha \in [0,1]$ and for any $x \in [0,1]$

$$\begin{aligned}
\mathcal{O}_n^{\wedge}(\alpha x_1,\alpha x_2,\cdots,\alpha x_n) &= \bigwedge_{\substack{i,j=1\\ i<j}}^{n} O(\alpha x_i,\alpha x_j)\\
&= \bigwedge_{\substack{i,j=1\\ i<j}}^{n} \alpha^k O(x_i,x_j)\\
&= \alpha^k \bigwedge_{\substack{i,j=1\\ i<j}}^{n} O(x_i,x_j)\\
&= \alpha^k \mathcal{O}_n^{\wedge}(x_1,x_2,\cdots,x_n).
\end{aligned} \quad \square$$

Proposition 4.6. Let $O:[0,1]^2 \to [0,1]$ be a bivariate overlap function. If O satisfies the Property 1-section deflation, then $\mathcal{O}_n^{\wedge}$ also satisfies the Property 1-section deflation.

Proof. If O satisfies the Property 1-section deflation, then for any $x \in [0,1]$, we have that

$$\begin{aligned}
\mathcal{O}_n^{\wedge}(x,1,\cdots,1) &= \underbrace{O(x,1)\wedge\cdots\wedge O(x,1)}_{n-1}\wedge\underbrace{O(1,1)\wedge\cdots\wedge O(1,1)}_{C_n^2-(n-1)}\\
&\le \underbrace{x\wedge\cdots\wedge x}_{n-1}\wedge\underbrace{1\wedge\cdots\wedge 1}_{C_n^2-(n-1)}\\
&= x\wedge 1\\
&= x.
\end{aligned}$$

Therefor, $\mathcal{O}_n^{\wedge}$ satisfies the Property 1-section deflation. □

Similar to Proposition 4.6, we can get the following proposition.

Proposition 4.7. Let $O:[0,1]^2 \to [0,1]$ be a bivariate overlap function. If O satisfies the Property 1-section inflation (or diagonal inflation), then $\mathcal{O}_n^{\wedge}$ also satisfies the Property 1-section inflation (or diagonal inflation).

5. Conclusion

In this paper, we first introduce some new results on 1-section deflation, 1-section inflation and diagonal inflation. Next, three properties 1-section deflation, 1-section inflation and diagonal inflation are extended to the *n*-dimensional case ($n \geq 2$), and the corresponding results are presented. Finally, we focus on one construction method of *n*-dimensional overlap functions $\mathcal{O}_n^{\wedge}$ based on bivariate overlap functions and discuss their main properties, and well solve the open problem "to search the construction of *n*-dimensional overlapping functions based on bi-dimensional overlapping functions" in [10]. Because of the duality of *n*-dimensional overlap and grouping functions, one can also construct *n*-dimensional grouping functions based on bivariate grouping functions in a similar way.

Acknowledgements

This research was supported by National Nature Science Foundation of China (Grant Nos. 61763008, 11661028, 11661030).

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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