Journal of Applied Mathematics and Physics, 2021, 9, 2511-2526
https://www.scirp.org/journal/jamp
ISSN Online: 2327-4379
ISSN Print: 2327-4352

Neighbor Sum Distinguishing Index of Graphs with Maximum Average Degree

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How to cite this paper: Sun, X.Z. (2021) Neighbor Sum Distinguishing Index of Graphs with Maximum Average Degree. *Journal of Applied Mathematics and Physics*, **9**, 2511-2526.
https://doi.org/10.4236/jamp.2021.910161

Received: September 27, 2021
Accepted: October 24, 2021
Published: October 27, 2021

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Abstract

A proper k-edge coloring of a graph $G=(V(G),E(G))$ is an assignment $c:E(G)\to\{1,2,\cdots,k\}$ such that no two adjacent edges receive the same color. A neighbor sum distinguishing k-edge coloring of G is a proper k-edge coloring of G such that $\sum_{u\in e}c(e)\neq\sum_{v\in e}c(e)$ for each edge $uv\in E(G)$. The neighbor sum distinguishing index of a graph G is the least integer k such that G has such a coloring, denoted by $\chi'_{\Sigma}(G)$. Let $mad(G)=\max\left\{\frac{2|E(H)|}{|V(H)|}\mid H\subseteq G\right\}$ be the maximum average degree of G. In this paper, we prove $\chi'_{\Sigma}(G)\le\max\{9,\Delta(G)+1\}$ for any normal graph G with $mad(G)<\frac{37}{12}$. Our approach is based on the discharging method and Combinatorial Nullstellensatz.

Keywords

Proper Edge Coloring, Neighbor Sum Distinguishing Edge Coloring, Maximum Average Degree, Combinatorial Nullstellensatz

1. Introduction

All graphs mentioned in this paper are undirected, finite and simple. For a graph G, we denote its vertex set, edge set, maximum degree, minimum degree by $V(G)$, $E(G)$, $\Delta(G)$, $\delta(G)$, respectively. Let $N_G(v)$ be the set of neighbors of the vertex v in G, and $d_G(v)=|N_G(v)|$ be the degree of v in G. The *average degree* of a graph G is defined as $\frac{2|E(G)|}{|V(G)|}$. The *maximum average de-*

gree $mad(G)$ of G is the maximum of the average degrees of its subgraphs.

A *proper k-edge coloring* of a graph $G=(V(G),E(G))$ is an assignment $c:E(G)\to\{1,2,\cdots,k\}$ such that $c(e_1)\neq c(e_2)$ for any two adjacent edges e_1 and e_2. Let c be a proper k-edge coloring of G. We use $f(v)$ to denote the sum of colors of the edges incident to v. If $f(u)\neq f(v)$ for each edge $uv\in E(G)$, then c is called as a *neighbor sum distinguishing k-edge coloring* or an nsd-k-coloring of G for short. The *neighbor sum distinguishing index* of a graph G is the least integer k such that G has an nsd-k-coloring, denoted by $\chi'_{\Sigma}(G)$. By $S(v)$, we denote the set of colors taken on the edges incident to v, *i.e.* $S(v)=\{c(uv)\mid uv\in E(G)\}$. The proper k-edge coloring c is a *neighbor set distinguishing k-edge coloring*, if $S(u)\neq S(v)$ for each edge $uv\in E(G)$. Let $\chi'_a(G)$ be the smallest value k such that G has a neighbor set distinguishing k-edge coloring. It is easy to observe that G has a neighbor sum(or set) distinguishing edge coloring if and only if G does not contain isolated edges. A graph with no isolated edges is called as *a normal* graph. Then $\chi'_{\Sigma}(G)\geq\chi'_a(G)$ for any normal graph by definition.

In 2002, Zhang *et al.* [1] introduced the concept of the neighbor set distinguishing edge coloring and posed the following conjecture.

Conjecture 1.1 ([1]) *If G is a connected normal graph with at least* 6 *vertices, then* $\chi'_a(G)\leq\Delta(G)+2$.

Hatami [2] proved $\chi'_a(G)\leq\Delta(G)+300$ by probabilistic method for normal graph G with $\Delta(G)>10^{20}$. Akbari *et al.* [3] showed that $\chi'_a(G)\leq 3\Delta(G)$ for any normal graph. Wang *et al.* [4] improved this bound to that $\chi'_a(G)\leq 2.5\Delta(G)$ for any normal graph.

The neighbor sum distinguishing edge coloring was introduced by Flandrin *et al.* [5]. They determined the neighbor sum distinguishing index of graph classes including paths, trees, cycles, complete graphs and complete bipartite graphs, and posed the following conjecture.

Conjecture 1.2. ([5]) *If G is a connected normal graph with at least* 3 *vertices and* $G\neq C_5$, *then* $\Delta(G)\leq\chi'_{\Sigma}(G)\leq\Delta(G)+2$.

Flandrin *et al.* [5] proved that $\chi'_{\Sigma}(G)\leq\left\lceil\frac{7\Delta(G)-4}{2}\right\rceil$ for each connected normal graph G with maximum degree $\Delta(G)\geq 2$. Wang and Yan [6] improved this bound to $\left\lceil\frac{10\Delta(G)+2}{3}\right\rceil$. Bonamy and Przybylo [7] showed that $\chi'_{\Sigma}(G)\leq\Delta(G)+1$ for planar graph with $\Delta(G)\geq 28$. Dong *et al.* [8] studied the connections between neighbor sum distinguishing index and maximum average degree, and proved that if G is a normal graph with $mad(G)<\frac{5}{2}$ and $\Delta(G)\geq 5$, then $\chi'_{\Sigma}(G)\leq\Delta(G)+1$. Later, Gao *et al.* [9] showed that if G is a normal graph with $mad(G)<\frac{8}{3}$ and $\Delta(G)\geq 5$, then $\chi'_{\Sigma}(G)\leq\Delta(G)+1$. Hocquard and Przybylo [10] proved that $\chi'_{\Sigma}(G)\leq\Delta(G)+1$ for any normal graph G with $mad(G)<3$

and $\Delta(G)\ge 6$. Wang *et al.* [11] proved that if G is a normal graph with $mad(G)<\frac{37}{12}$ and $\Delta(G)\ge 7$, then $\chi'_\Sigma(G)\le\Delta(G)+2$. Recently, Wang *et al.* [12] proved that if G is a normal graph with $mad(G)<\frac{10}{3}$ and $\Delta(G)\ge 8$, then $\chi'_\Sigma(G)\le\Delta(G)+2$.

In this paper, we improve the result given by Wang *et al.* [11] and obtain the following result:

Theorem 1.1. *Let G be a normal graph. If* $mad(G)<\frac{37}{12}$, *then* $\chi'_\Sigma(G)\le\max\{9,\Delta(G)+1\}$.

Corollary 1.2. *Let G be a normal graph. If* $mad(G)<\frac{37}{12}$ *and* $\Delta(G)\ge 8$, *then* $\chi'_\Sigma(G)\le\Delta(G)+1$.

2. Preliminaries

To prove our main result, we need to introduce some notations. A vertex v is called a *k*-vertex (a k^+-vertex, or a k^--vertex, respectively) if $d(v)=k$ ($d(v)\ge k$, or $d(v)\le k$, respectively). A vertex v is called a *leaf* if $d(v)=1$.

At first, we introduce several lemmas.

Lemma 2.1. ([11]) *Suppose that k and n are positive integers with* $k\le n$, S_i *is a set of integers with* $|S_i|=l_i\ge n$, $i=1,2,\cdots,k$. *Let* $T_k=\left\{\sum_{i=1}^{k}x_i \mid x_i\in S_i, x_i\ne x_j\,(i\ne j)\right\}$, *then* $|T_k|\ge\sum_{i=1}^{k}l_i-k^2+1$.

Lemma 2.2. ([13]) *Let F be an arbitrary field, and let* $P=P(x_1,\cdots,x_n)$ *be a polynomial in* $F[x_1,\cdots,x_n]$. *Suppose the degree* $deg(P)$ *of P equals* $\sum_{i=1}^{n}k_i$, *where each* k_i *is a non-negative integer, and suppose the coefficient of* $x_1^{k_1}\cdots x_n^{k_n}$ *in P is non-zero. Then if* $S_1,\cdots,S_n$ *are subsets of F with* $|S_i|>k_i$, *there are* $s_1\in S_1,\cdots,s_n\in S_n$ *so that* $P(s_1,\cdots,s_n)\ne 0$.

Lemma 2.3. ([14]) *Let*

$$P(x_1,x_2,\cdots,x_n)=\prod_{1\le i<j\le n}(x_i-x_j)\left(\sum_{k=1}^{n}x_k\right)^2$$

be a polynomial in n variables, where $n\ge 2$. *Let* $C_P\left(x_1^{n_1}x_2^{n_2}\cdots x_n^{n_n}\right)$ *denote the coefficient of* $x_1^{n_1}x_2^{n_2}\cdots x_n^{n_n}$ *in P, then*

$$C_P\left(x_1^{n}x_2^{n-1}x_3^{n-3}x_4^{n-4}\cdots x_{n-1}\right)=1.$$

3. Proof of Theorem 1.1

3.1. Unavoidable Configuration

In this paper, we will prove Theorem 1.1 by contradiction. Let $K=\max\{9,\Delta(G)+1\}$. Let G be a counterexample of theorem 1.1 such that $|V(G)|+|E(G)|$ is the smallest. Obviously, G is connected. Let H be a normal subgraph of *G*. By the minimality of *G*, H has an nsd-*K*-coloring c using the color set $\{1,2,\cdots,K\}$.

Remark 1. Let $u \in V(G)$. Suppose that u is adjacent to a 2-vertex v with $N_G(v) = \{u, w\}$ and $d_G(w) \le 4$. If $G' = G - vw$ admits an nsd-*K*-coloring c such that $c(uv) \ne f(w)$, then there are at least $K - 3 - 3 - 1 - 1 \ge 1$ colors available for *vw*. Hence we can get an nsd-*K*-coloring of *G*. Hence, in the following discussion, we will omit the proof of recoloring or coloring of *vw*, and just show that $G' = G - vw$ has an nsd-*K*-coloring c with $c(uv) \ne f(w)$.

Let H be the graph which is obtained by removing all the leaves of *G*. Let $d_k(v)$ ($d_{k^+}(v)$, $d_{k^-}(v)$) be the number of neighbors of v with degree k (at most k, at least k) in H.

Claim 3.1. The graph H has the following properties:

(1) $\delta(H) \ge 2$.

(2) If $d_H(u) \le 4$, then $d_G(u) = d_H(u)$.

(3) If uvw is a path in H such that $d_H(v) = 2$, $2 \le d_H(w) \le 4$, then $d_G(u) = d_H(u)$.

Proof: (1) This statement follows from [8].

(2) Assume to the contrary that $d_G(u) > d_H(u) = d$. Let $d_G(u) - d_H(u) = l \ge 1$. Then u in G is adjacent to d 2^+-vertices $u_1, u_2, \cdots, u_d$ and l leaves $v_1, v_2, \cdots, v_l$.

Suppose that $l = 1$. Let $G' = G - uv_1$. Then $\chi'_\Sigma(G') \le K$ by the minimality of *G*. The colors in $\{c(uu_i) \mid 1 \le i \le d\} \cup \{f(u_i) - f(u) \mid 1 \le i \le d\}$ are forbidden for uv_1. So we have at least $K - 2d \ge K - 8 \ge 1$ available colors for uv_1. Therefore, we can get an nsd-*K*-coloring of *G*, a contradiction.

Suppose that $2 \le l \le \Delta(G) - d$. Let $G' = G - \{uv_i \mid 1 \le i \le l\}$ and S_i denote the feasible color set which uv_i can use for each $1 \le i \le l$. Then $|S_i| \ge K - d$ colors available for uv_i. By Lemma 2.1, $|T_l| \ge l(K - d) - l^2 + 1$. Let $f(l) = l(K - d) - l^2 + 1 = l(K - d - l) + 1$. Note that $K - d - l \ge \Delta(G) + 1 - d - l \ge 1$. If $l \ge 4$, then $f(l) \ge l + 1 > 4$. If $2 \le l \le 3$, then $K - d - l \ge 9 - 4 - 3 \ge 2$ and $f(l) \ge 2 \times 2 + 1 > 4$. Thus, we can show that $f(l) > 4$. Hence we can get an nsd-*K*-coloring of *G*, a contradiction.

(3) According to Claim 3.1 (2), we have $d_G(v) = d_H(v) = 2$ and $d_G(w) = d_H(w)$. Assume to the contrary that $d_G(u) > d_H(u) = d$, this means that there exist at least one leave v_1 adjacent to u in *G*. Let $G' = G - vw$. Then G' has an nsd-*K*-coloring c by the minimality of *G*. If $c(uv) \ne f(w)$, we can get an nsd-*K*-coloring of G by Remark 1, a contradiction. If $c(uv) = f(w)$, then we exchange the colors of uv_1 and uv, and get an nsd-*K*-coloring of G by Remark 1, a contradiction. □

A 2-vertex is called *bad* if it is adjacent to a 2-vertex; *weak* if it is adjacent to a 3-vertex or a 4-vertex; *good* if it is adjacent to two 5^+-vertices; *deficient* if it is bad or weak. Let $d_{b2}(u)$ ($d_{def}(v)$, $d_{g2}(u)$) be the number of bad 2-vertices (deficient vertices, good 2-vertices) adjacent to u in *H*.

Claim 3.2. Suppose that u is a weak 2-vertex in *H*. Let $N_H(u) = \{u_1, u_2\}$, where $d_H(u_1) = 3$ or 4.

(1) If $d_H(u_1) = 3$, then $d_{5^+}(u_1) = 2$.

(2) If $d_H(u_1)=4$, then $d_{4^+}(u_1)=3$.

Proof: (1) By Claim 3.1 (2), $d_G(u_1)=d_H(u_1)=3$ and $d_G(u)=d_H(u)=2$. Let $N_G(u_1)=\{u,x,y\}$. Assume to the contrary that $d_H(x)\le 4$. According to Claim 3.1 (2), $d_G(x)=d_H(x)\le 4$. Let $G'=G-\{uu_1,u_1x\}$. Then G' has an nsd-*K*-coloring by the minimality of *G*. It is easy to see that there are at least $K-3-3-1-1\ge 1$ available colors for u_1x. So we can extend the coloring to G by Remark 1, a contradiction.

(2) By Claim 3.1 (2), $d_G(u_1)=d_H(u_1)=4$ and $d_G(u)=d_H(u)=2$. Let $N_G(u_1)=\{u,x,y,z\}$. Assume to the contrary that $d_H(x)\le 3$. By Claim 3.1 (2), $d_G(x)=d_H(x)\le 3$. Let $G'=G-\{uu_1,u_1x\}$. Then G' has an nsd-*K*-coloring by the minimality of *G*. There are at least $K-2-2-2-1\ge 2$ available colors for u_1x. So we can obtain an nsd-*K*-coloring of *G* by Remark 1, a contradiction. □

If uxy is a path of *H* such that $d_H(y)=2$ and $d_H(x)=3$, then *u* is called the *source* vertex of *y*, *y* is called *sink* vertex of *u*. We use $s(u)$ to denote the number of sink vertices of *u*. By Claim 3.2 (1), we know that $s(u)\le d_3(u)$.

Claim 3.3 Let $u\in V(H)$ with $d_H(u)=d$. Let $u_1,\cdots,u_d$ be the neighbors of *u* in *H*.

(1) If $d=2$, then $d_{4^-}(u)\le 1$.

(2) If $d=3$, then $d_{3^-}(u)\le 1$.

(3) If $d\ge 5$, then $d_{b2}(u)\le 1$.

(4) If $d=5$, then $d_2(u)\le 2$.

(4.1) If $d_{def}(u)=1$, then $s(u)=0$.

(4.2) If $d_2(u)=2$, then $d_{def}(u)\le 1$.

(4.3) If $d_2(u)=2$ and $d_{def}(u)=1$, then $d_3(u)=0$.

(5) If $d=6$, then $d_{def}(u)\le 1$.

(5.1) If $d_{def}(u)=0$ and $d_{g2}(u)\ge 5$, then $d_{3^-}(u)=5$.

(5.2) If $d_{def}(u)=1$, then $d_{g2}(u)\le 1$.

Proof: (1) Assume to the contrary that *u* is adjacent to two 4^--vertices u_1 and u_2. By Claim 3.1 (2), $d_G(u)=d_H(u)=2$ and $d_G(u_i)=d_H(u_i)$ for $i=1,2$. Suppose that $u_1u_2\notin E(G)$ (If $u_1u_2\in E(G)$, we can prove it similarly). Let $G'=G-u$. Then G' has an nsd-*K*-coloring *c* by the minimality of *G*. It is easy to show that the colors in $\{c(u_1w)\mid w\in N_{G'}(u_1)\}\cup\{f(w)-f(u_1)\mid w\in N_{G'}(u_1)\}\cup\{f(u_2)\}$ are forbidden for uu_1. So we have at least $K-3-3-1\ge 2$ available colors for uv_1. Then we can get an nsd-*K*-coloring of *G* by Remark 1, a contradiction.

(2) Assume to the contrary that *u* is adjacent to two 3^--vertices u_1 and u_2. By Claim 3.1 (2), $d_G(u)=d_H(u)=3$ and $d_G(u_i)=d_H(u_i)$ for $i=1,2$. Let $G'=G-\{uu_1,uu_2\}$. Then G' has an nsd-*K*-coloring *c* by the minimality of *G*. The colors in $\{c(u_1w),f(w)-f(u_1)\mid w\in N_{G'}(u_1)\}\cup\{c(uu_3),f(u_2)-c(uu_3)\}$ are forbidden for uu_1. So we have at least $K-2-2-1-1\ge 3$ available colors for uu_1. Next, there are at least $K-2-2-2-2\ge 1$ colors available for uu_2. Hence we have $\chi'_\Sigma(G)\le K$, a contradiction.

(3) Assume to the contrary that u is adjacent to two bad 2-vertices u_1 and u_2. By Claim 3.1 (2), we have $d_G(u_i)=d_H(u_i)=2$ for $i=1,2$.

Case 1: $u_1u_2 \in E(G)$.

Let $G'=G-u_1u_2$. The colors in $\{f(u)-c(uu_1), f(u)-c(uu_2), c(uu_1), c(uu_2)\}$ are forbidden for u_1u_2. So there are at least $K-4\ge 5$ available colors for u_1u_2. Hence, we can extend this coloring to *G*, a contradiction.

Case 2: $u_1u_2 \notin E(G)$.

Let $N_G(u_i)=\{u,w_i\}$ for $i=1,2$. By Claim 3.1 (2), $d_G(w_i)=d_H(w_i)=2$ for $i=1,2$. By Claim 3.2 (1), $w_1\ne w_2$ and $w_1w_2\notin E(G)$. Let $N_G(w_i)=\{u_i,v_i\}$ for $i=1,2$.

Case 2.1: $v_1=v_2=v$.

Let $G'=G-\{u_1w_1,u_2w_2\}$. Then G' has an nsd-*K*-coloring c by the minimality of *G*. Note that $c(uu_1)\ne c(uu_2)$ and $c(vw_1)\ne c(vw_2)$. If $c(uu_1)=c(vw_1)$ (similarly for $c(uu_2)=c(vw_2)$), then we switch the colors of uu_1 and uu_2. It is worth noting that $f(u_1)\ne f(w_1)$ and $f(u_2)\ne f(w_2)$ for this new nsd-*K*-coloring c' of G'. The colors in $\{c'(uu_1), f(u)-c'(uu_1), c'(vw_1), f(v)-c'(vw_1)\}$ are forbidden for u_1w_1. So we have at least $K-1-1-1-1\ge 5$ available colors for u_1w_1. Similarly, there are at least 5 colors available for u_2w_2. Hence we have $\chi'_\Sigma(G)\le K$, a contradiction. If $c(uu_1)\ne c(vw_1)$ and $c(uu_2)\ne c(vw_2)$, then $f(u_i)\ne f(w_i)$ ($i=1,2$). Now we can extend this coloring to G with the similar discussion as above, a contradiction.

Case 2.2: $v_1\ne v_2$.

Let $G'=G-\{u_1w_1,u_2w_2\}+\{w_1w_2\}$. Now we will show that $mad(G')<\frac{37}{12}$. In fact, let H' be the subgraph of G'. If $w_1w_2\notin E(H')$, then $H'\subseteq G$ and $mad(H')<\frac{37}{12}$. So suppose that $w_1w_2\in E(H')$. If at most one of w_1v_1 and w_2v_2 belongs to $E(H')$, say $w_1v_1\in E(H')$ if it exists, then $H=H'-w_2$, is the subgraph of *G*. Note that $\frac{2|E(H)|}{|V(H)|}<\frac{37}{12}$. Therefore,

$$\frac{2|E(H')|}{|V(H')|}=\frac{2(|E(H)|+1)}{|V(H)|+1}=\frac{2|E(H)|+2}{|V(H)|+1}<\frac{\frac{37}{12}|V(H)|+2}{|V(H)|+1}<\frac{37}{12}.$$

If $w_1v_1\in E(H')$ and $w_2v_2\in E(H')$, then $H=H'-\{w_1,w_2\}$ is the subgraph of *G*. Note that $\frac{2|E(H)|}{|V(H)|}<\frac{37}{12}$. Therefore,

$$\frac{2|E(H')|}{|V(H')|}=\frac{2(|E(H)|+3)}{|V(H)|+2}=\frac{2|E(H)|+6}{|V(H)|+2}<\frac{\frac{37}{12}|V(H)|+6}{|V(H)|+2}<\frac{37}{12}.$$

Thus, we obtain that G' is a graph with $mad(G')<\frac{37}{12}$ and

$|E(G')|+|V(G')|<|E(G)|+|V(G)|$. Then G' has an nsd-K-coloring c by the minimality of G. Note that $c(w_1v_1)\neq c(w_2v_2)$ by $f(w_1)\neq f(w_2)$. Then we can achieve an nsd-K-coloring of G with the similar discussion as Case 2.1.

(4) Assume to the contrary that u in H is adjacent to three 2-vertices u_1,u_2 and u_3. By Claim 3.1 (2), it holds that $d_G(u_i)=d_H(u_i)=2$ for $i\in\{1,2,3\}$. Let $N_G(u_i)=\{u,w_i\}$ for each $i\in\{1,2,3\}$. Assume that u in G is adjacent to l leaves $v_1,v_2,\cdots,v_l$ with $l\geq 0$.

Case 1: $l=0$.

Then $d_G(u)=d_H(u)=5$. Let $G'=G-\{uu_i \mid i=1,2,3\}$. Then G' has an nsd-K-coloring by the minimality of G. It is easy to see that there are at least $K-2-1-1\geq 5$ colors available for uu_i ($i\in\{1,2,3\}$). By Lemma 2.1, $5\times 3-3^2+1=7>5$. So we can color uu_1, uu_2 and uu_3 properly such that $f(u)\neq f(u_i)$ for each $1\leq i\leq 5$, a contradiction.

Case 2: $l=1$.

Let $G'=G-\{uv_1,uu_1,uu_2,uu_3\}$. Then G' has an nsd-K-coloring by the minimality of G. It is evident that there are at least $K-2-1-1\geq 5$ colors available for uu_i ($i\in\{1,2,3\}$) and at least $K-2\geq 7$ colors available for uv_1. By Lemma 2.1, $7+5\times 3-4^2+1=7>5$. So we can color uv_1, uu_1, uu_2 and uu_3 properly and obtain an nsd-K-coloring of G, which is a contradiction.

Case 3: $l\geq 2$.

Let $G'=G-(\{uv_i \mid i=1,2,\cdots,m\}\cup\{uu_1\})$. Then G' has an nsd-K-coloring by the minimality of G. We use colors $x_1,\cdots,x_m,x_{m+1}$ to color $uv_1,\cdots,uv_m,uu_1$. Let S_i and S_{m+1} devote the available color set for uv_i ($1\leq i\leq m$) and uu_1, respectively. Now we consider the following polynomial:

$$P(x_1,\cdots,x_{m+1})=\prod_{1\leq i<j\leq m+1}(x_i-x_j)\prod_{n=2}^{5}\left(\sum_{i=1}^{m+1}x_i+\sum_{t=s+1}^{l}c(uv_t)+\sum_{k=2}^{5}c(uu_k)-f(u_n)\right)$$
$$\cdot\left(\sum_{i=1}^{m+1}x_i+\sum_{t=m+1}^{l}c(uv_t)+\sum_{k=2}^{5}c(uu_k)-x_{m+1}-f(u_1)\right).$$

Let

$$Q(x_1,\cdots,x_{m+1})=\prod_{1\leq i<j\leq m+1}(x_i-x_j)\left(\sum_{i=1}^{m+1}x_i\right)^4\left(\sum_{i=1}^{m}x_i\right).$$

Suppose that $l=2$. Let $m=2$. Notice that $|S_1|=|S_2|\geq K-(7-3)\geq 5$ and $|S_3|\geq K-(7-3)-1-1\geq 3$. By computations, we obtain that $C_P(x_1^4x_2^3x_3)=C_Q(x_1^4x_2^3x_3)=3\neq 0$. According to Lemma 2.2, we can choose $x_i\in S_i$ ($1\leq i\leq m+1$) such that $P(x_1,\cdots,x_{m+1})\neq 0$. That is, we can get an nsd-K-coloring of G, a contradiction.

Suppose that $l\geq 3$. Let $m=3$. It is evident that for each $1\leq i\leq m$, we have $|S_i|\geq K-(d_G(u)-4)\geq 5$ and $|S_{m+1}|\geq K-(d_G(u)-4)-1-1\geq 3$. By computations, we obtain that $C_P(x_1^4x_2^3x_3^2x_4^2)=C_Q(x_1^4x_2^3x_3^2x_4^2)=-1\neq 0$. According to Lemma 2.2, we can choose $x_i\in S_i$ ($1\leq i\leq m+1$) such that $P(x_1,\cdots,x_{m+1})\neq 0$. That is, we can get an nsd-K-coloring of G, a contradiction.

(4.1) Assume to the contrary that u in H is adjacent to one deficient vertex u_1

and has one sink vertex x. Let $u_2 \in N_G(x) \cap N_G(u)$. Then by the definition of sink vertex, we have $d_H(u_2)=3$, $d_H(x)=2$. By Claim 3.1, it holds that $d_G(u)=d_H(u)=5$, $d_G(u_1)=d_H(u_1)=2$, $d_G(u_2)=d_H(u_2)=3$ and $d_G(x)=d_H(x)=2$. Suppose that $N_G(u_1)=\{u,w_1\}$. Let $G'=G-uu_1$. Then G' has an nsd-K-coloring by the minimality of G. We erase the colors on edges u_1w_1 and u_2x. There are at least $K-4-1 \geq 4$ available colors for uu_1. Thus, we can color properly the edge uu_1 such that $f(u) \neq f(u_i)$ for $i \in \{3,4,5\}$. Then we can get an nsd-K-coloring of G by Remark 1, a contradiction.

(4.2) Assume to the contrary that u in H is adjacent to two deficient vertices u_1 and u_2. By Claim 3.1, it holds that $d_G(u)=d_H(u)=5$, $d_G(u_i)=d_H(u_i)=2$ and $d_G(w_i)=d_H(w_i) \leq 4$ ($i=1,2$). Suppose that $N_G(u_i)=\{u,w_i\}$ for each $i=1,2$. Let $G'=G-\{uu_1,uu_2\}$. Then G' has an nsd-K-coloring by the minimality of G. We erase the colors on edges u_iw_i for $i=1,2$. There are at least $K-3-1 \geq 5$ available colors for uu_1. So we can color uu_1. Next, we have at least $K-4-1 \geq 4$ colors available for uu_2. Thus, we can color properly the edge uu_2 such that $f(u) \neq f(u_i)$ for $i \in \{3,4,5\}$. Then we can get an nsd-K-coloring of G by Remark 1, a contradiction.

(4.3) Assume that u in H is adjacent to one deficient vertex u_1, one good 2-vertex u_2 and one 3-vertex u_3. By Claim 3.1, it holds that $d_G(u)=d_H(u)=5$, $d_G(u_i)=d_H(u_i)=2$ ($i=1,2$) and $d_G(w_1)=d_H(w_1) \leq 4$. Suppose that $N_G(u_i)=\{u,w_i\}$ for each $i=1,2$. Let $G'=G-\{uu_i \mid i=1,2,3\}$. Then G' has an nsd-K-coloring by the minimality of G. Firstly, we erase the colors on edge u_1w_1. Then there are at least $K-2-1 \geq 6$ available colors for uu_1 and at least $K-2-1-1 \geq 5$ available colors for uu_2, and at least $K-2-2-2 \geq 3$ colors available for uu_3. By Lemma 2.1, $6+5+3-3^2+1=6>5$. So we can color uu_1, uu_2 and uu_3 properly and obtain an nsd-K-coloring of G by Remark 1, which is a contradiction.

(5) Assume that u of H is adjacent to two deficient vertices u_1 and u_2. By Claim 3.1, it holds that $d_G(u)=d_H(u)=6$, $d_G(u_i)=d_H(u_i)=2$ and $d_G(w_i)=d_H(w_i) \leq 4$ for $i=1,2$. Suppose that $N_G(u_i)=\{u,w_i\}$ for $i=1,2$. Let $G'=G-\{uu_1,uu_2\}$. Then G' has an nsd-K-coloring by the minimality of G. We erase the colors on edges u_iw_i ($i=1,2$). Then there are at least $K-4-1 \geq 4$ available colors for uu_i. By Lemma 2.1, $4\times 2-2^2+1=5>4$. So we can color uu_1 and uu_2 properly such that $f(u) \neq f(v)$ for $v \in N_G(u)-\{u_1,u_2\}$. Hence, we obtain an nsd-K-coloring of G by Remark 1, a contradiction.

(5.1) Assume to the contrary that u in H is adjacent to five good 2-vertices u_1, u_2, u_3, u_4, u_5 and one 3^--vertex u_6. By Claim 3.1 (2), it holds that $d_G(u_i)=d_H(u_i)$ for each $1 \leq i \leq 6$. Assume that u in G is adjacent to l leaves $v_1, v_2, \cdots, v_l$ with $l \geq 0$.

Suppose that $l=0$. Then $d_G(u)=d_H(u)=6$. Let $G'=G-\{uu_1,uu_2\}$. Then G' has an nsd-K-coloring by the minimality of G. When $6=d_G(u) \leq \left\lceil \frac{\Delta(G)}{2} \right\rceil$, *i.e.* $\Delta(G)\ge 11$, we know that there are at least $K-4-1-1=\Delta(G)-5\ge 6$ colors available for uu_i ($i=1,2$). By Lemma 2.1, $6\times 2-2^2+1=9>6$. So we can color uu_1 and uu_2 properly such that $f(u)\ne f(u_i)$ for each $1\le i\le 6$, a contradiction. When $6=d_G(u)\ge\left\lceil\frac{\Delta(G)}{2}\right\rceil+1$, then $1+2+\cdots+5>\Delta(G)+1$. So we have $f(u)\ne f(u_i)$ for each $1\le i\le 5$. There are at least $K-4-1-1\ge 3$ available colors for uu_i ($i=1,2$). By Lemma 2.1, $3\times 2-2^2+1=3>1$. Thus, we can color properly the edges uu_1 and uu_2 such that $f(u)\ne f(u_6)$, a contradiction.

Suppose that $l\ge 1$. Let $G'=G-uv_1$. Then G' has an nsd-*K*-coloring by the minimality of *G*. When $6+l=d_G(u)\le\left\lceil\frac{\Delta(G)}{2}\right\rceil$, *i.e.* $\Delta(G)\ge 2l+11$, it is evident that there are at least $K-(6+l-1)-6\ge l+1$ colors available for uv_1. So we get an nsd-*K*-coloring of *G*, a contradiction. When $6+l=d_G(u)\ge\left\lceil\frac{\Delta(G)}{2}\right\rceil+1$, then $1+2+\cdots+(5+l)>\Delta(G)+1$. So we have $f(u)\ne f(u_i)$ for each $1\le i\le 5$. There are at least $K-(d_G(u)-1)-1\ge 1$ available colors for uv_1. Then we get an nsd-*K*-coloring of *G*, a contradiction.

(5.2) Assume that u of H is adjacent to a deficient vertex u_1 and two good 2-vertices u_2 and u_3. By Claim 3.1, it holds that $d_G(u)=d_H(u)=6$, $d_G(w_1)=d_H(w_1)\le 4$ and $d_G(u_i)=d_H(u_i)=2$ for each $1\le i\le 3$. Let $N_G(u_i)=\{u,w_i\}$ for $1\le i\le 3$. Let $G'=G-\{uu_i\mid i=1,2,3\}$. Then G' has an nsd-*K*-coloring by the minimality of *G*. We erase the colors on edges u_1w_1. For each $1\le i\le 3$, we use x_i to color uu_i. Let S_i be the available color set for uu_i. Then $|S_1|\ge K-3-1\ge 5$ and $|S_2|=|S_3|\ge K-3-1-1\ge 4$. Now we consider the following polynomial:

$$P(x_1,x_2,x_3)=\prod_{1\le i<j\le 3}(x_i-x_j)\prod_{n=1}^{3}\left(\sum_{i=1}^{3}x_i+\sum_{k=4}^{6}c(uu_k)-x_n-f(u_n)\right)\cdot\prod_{n=4}^{6}\left(\sum_{i=1}^{3}x_i+\sum_{k=4}^{6}c(uu_k)-f(u_n)\right).$$

Let

$$Q(x_1,x_2,x_3)=\prod_{1\le i<j\le 3}(x_i-x_j)\prod_{n=1}^{3}\left(\sum_{i=1}^{3}x_i-x_n\right)\left(\sum_{i=1}^{3}x_i\right)^3.$$

By matlab, we obtain that $C_P\left(x_1^4x_2^3x_3^2\right)=2\ne 0$. According to Lemma 2.2, we can choose $x_i\in S_i$ ($1\le i\le 3$) such that $P(x_1,x_2,x_3)\ne 0$. Thus, we get an nsd-*K*-coloring of *G* by Remark 1, a contradiction. □

Remark 2. Note that if $d_G(u)\ge\left\lceil\frac{\Delta(G)}{2}\right\rceil+1$, then $1+2+\cdots+(d_G(u)-1)\ge 1+2+\cdots+\left\lceil\frac{\Delta(G)}{2}\right\rceil>\Delta(G)+1$. So $f(u)\ne f(u_i)$ when

$d_G(u_i)=2$ for any proper edge coloring of *G*.

Claim 3.4. Let $u \in V(H)$ with $d_H(u)=d$. Suppose that $7 \le d \le \Delta(G)-1$.

(1) $d_2(u) \le d-1$.

(2) If $d \le \left\lceil \frac{\Delta(G)}{2} \right\rceil$, then $d_{def}(u)=0$.

(3) If $d \ge \left\lceil \frac{\Delta(G)}{2} \right\rceil + 1$, then $d_{def}(u) \le \left\lceil \frac{d}{2} \right\rceil - 1$.

(3.1) If $d_{def}(u)=1$, then $d_2(u) \le d-3$.

(3.2) If $d_{def}(u)=\left\lceil \frac{d}{2} \right\rceil - 1$, then $d_{g2}(u) \le 1$.

Proof: (1) Assume to the contrary that *u* in *H* is adjacent to *d* 2-vertices $u_1,\cdots,u_d$. By Claim 3.1 (2), we have $d_G(u_i)=d_H(u_i)=2$ for each $1 \le i \le d$. Assume that *u* in *G* is adjacent to *l* leaves $v_1,v_2,\cdots,v_l$.

Suppose that $l=0$. Then $d_G(u)=d_H(u)=d$. Let $G'=G-u$. Then G' has an nsd-*K*-coloring by the minimality of *G*. If $7 \le d \le \left\lceil \frac{\Delta(G)}{2} \right\rceil$. Then $\Delta(G) \ge 2d-1$. There are at least $K-1-1 \ge \Delta(G)-1$ colors available for uu_i ($1 \le i \le d$). By Lemma 2.1, $(\Delta(G)-1)d-d^2+1 \ge (d-2)d+1 > d$. So we can color uu_i properly such that $f(u) \ne f(u_i)$ for each $1 \le i \le d$. So we get an nsd-*K*-coloring of *G*, a contradiction. So suppose that $d \ge \left\lceil \frac{\Delta(G)}{2} \right\rceil + 1$. By Remark 2, $f(u) \ne f(u_i)$ ($1 \le i \le d$) for any proper edge coloring of *G*. Note that there are at least $K-1-1=\Delta(G)-1 \ge d$ available colors for uu_i ($1 \le i \le d$). Hence, we get an nsd-*K*-coloring of *G*, a contradiction.

Suppose that $l \ge 1$. Let $G'=G-uv_1$. Then G' has an nsd-*K*-coloring by the minimality of *G*. If $d+l \le \left\lceil \frac{\Delta(G)}{2} \right\rceil$, *i.e.* $\Delta(G) \ge 2d+2l-1$, then there are at least $K-(d+l-1)-d \ge l+1$ colors available for uv_1. So we get an nsd-*K*-coloring of *G*, a contradiction. If $d+l \ge \left\lceil \frac{\Delta(G)}{2} \right\rceil + 1$, then $f(u) \ne f(u_i)$ for each $1 \le i \le d$ for any proper coloring of *G* by Remark 2. Note that there are at least $K-(d_G(u)-1) \ge 2$ available colors for uv_1. Hence, we get an nsd-*K*-coloring of *G*, a contradiction.

(2) Assume that *u* of *H* is adjacent to a deficient vertex u_1. Let $N_G(u_1)=\{u,w_1\}$. By Claim 3.1, it holds that $d_G(u)=d_H(u)=d \le \left\lceil \frac{\Delta(G)}{2} \right\rceil$, $d_G(u_1)=d_H(u_1)=2$ and $d_G(w_1)=d_H(w_1) \le 4$. Let $G'=G-u_1$. Then G' has an nsd-*K*-coloring by the minimality of *G*. There are at least $K-2(d-1)-1 \ge \Delta(G)-2\left\lceil \frac{\Delta(G)}{2} \right\rceil + 2 \ge 1$ available colors for uu_1. Hence, we

can get an nsd-K-coloring of G by Remark 1, a contradiction.

(3) Assume to the contrary that u in H is adjacent to $\left\lceil \frac{d}{2} \right\rceil$ deficient vertices $u_1, u_2, \cdots, u_{\lceil \frac{d}{2} \rceil}$. Let $N_G(u_i) = \{u, w_i\}$ for each $1 \le i \le \left\lceil \frac{d}{2} \right\rceil$. By Claim 3.1, it holds that $d_G(u) = d_H(u) = d$, $d_G(u_i) = d_H(u_i) = 2$ and $d_G(w_i) = d_H(w_i) \le 4$ for $1 \le i \le \left\lceil \frac{d}{2} \right\rceil$. Let $G' = G - \left\{ uu_i \mid i = 1, 2, \cdots, \left\lceil \frac{d}{2} \right\rceil \right\}$. Then G' has an nsd-K-coloring by the minimality of G. We erase the colors on edges $u_i w_i$ for each $1 \le i \le \left\lceil \frac{d}{2} \right\rceil$. There are at least $K - \left(d - \left\lceil \frac{d}{2} \right\rceil \right) - 1 \ge \Delta(G) - d + \left\lceil \frac{d}{2} \right\rceil \ge \left\lceil \frac{d}{2} \right\rceil + 1$ available colors for uu_i ($1 \le i \le \left\lceil \frac{d}{2} \right\rceil$). By Lemma 2.1,

$\left(\left\lceil \frac{d}{2} \right\rceil + 1 \right) \times \left\lceil \frac{d}{2} \right\rceil - \left\lceil \frac{d}{2} \right\rceil^2 + 1 = \left\lceil \frac{d}{2} \right\rceil + 1 > d - \left\lceil \frac{d}{2} \right\rceil$. So we can color uu_i properly such that $f(u) \ne f(u_i)$ for each $1 \le i \le d$. Hence, we get an nsd-K-coloring of G by Remark 1, a contradiction.

(3.1) Assume that u of H is adjacent to $(d-2)$ 2-vertices $u_1, \cdots, u_{d-2}$ such that u_1 is a deficient vertex. Let $N_G(u_i) = \{u, w_i\}$ for $1 \le i \le d-2$. By Claim 3.1, it holds that $d_G(u) = d_H(u) = d$, $d_G(w_1) = d_H(w_1) \le 4$ and $d_G(u_i) = d_H(u_i) = 2$ for each $1 \le i \le d-2$. Let $G' = G - \{uu_i \mid i = 1, 2, \cdots, d-2\}$. Then G' has an nsd-K-coloring by the minimality of G. Firstly, we erase the colors on edge $u_1 w_1$. For each $1 \le i \le d-2$, we use x_i to color uu_i. Let S_i be the available color set for uu_i. Then $|S_1| \ge K - 2 - 1 = \Delta(G) - 2 \ge d - 1$ and $|S_i| \ge K - 2 - 1 - 1 \ge d - 2$ ($2 \le i \le d-2$). Now we consider the following polynomial:

$$P(x_1, x_2, \cdots, x_{d-2}) = \prod_{1 \le i < j \le d-2} (x_i - x_j) \prod_{n=d-1}^{d} \left(\sum_{i=1}^{d-2} x_i + \sum_{k=d-1}^{d} c(uu_k) - f(u_n) \right).$$

Let

$$Q(x_1, x_2, \cdots, x_{d-2}) = \prod_{1 \le i < j \le d-2} (x_i - x_j) \left(\sum_{i=1}^{d-2} x_i \right)^2.$$

By Lemma 2.3, we obtain that $C_Q\left(x_1^{d-2} x_2^{d-3} x_3^{d-5} x_4^{d-6} \cdots x_{d-3} \right) = 1 \ne 0$. Thus, we get an nsd-$K$-coloring of G by Remark 1, a contradiction.

(3.2) Assume to the contrary that, u in H is adjacent to ($\left\lceil \frac{d}{2} \right\rceil + 1$) 2-vertices $u_1, \cdots, u_{\lceil \frac{d}{2} \rceil + 1}$ such that $u_1, \cdots, u_{\lceil \frac{d}{2} \rceil - 1}$ are deficient vertices. Then $u_{\lceil \frac{d}{2} \rceil}$ and $u_{\lceil \frac{d}{2} \rceil + 1}$ are good 2-vertices by Claim 3.4 (3). Let $N_G(u_i) = \{u, w_i\}$ for each $1 \le i \le \left\lceil \frac{d}{2} \right\rceil + 1$. By Claim 3.1, it holds that $d_G(u) = d_H(u) = d$,

$d_G(u_i) = d_H(u_i) = 2$ for $1 \le i \le \left\lceil \frac{d}{2} \right\rceil + 1$. Consider that $G' = G - \left\{ uu_i \mid i = 1, 2, \cdots, \left\lceil \frac{d}{2} \right\rceil + 1 \right\}$. Then G' has an nsd-K-coloring by the minimality of G. We erase the color on edge $u_i w_i$ for each $1 \le i \le \left\lceil \frac{d}{2} \right\rceil - 1$. There are at least $K - \left(d - \left\lceil \frac{d}{2} \right\rceil - 1 \right) - 1 \ge \Delta(G) - d + \left\lceil \frac{d}{2} \right\rceil + 1 \ge \left\lceil \frac{d}{2} \right\rceil + 2$ available colors for uu_i ($1 \le i \le \left\lceil \frac{d}{2} \right\rceil - 1$), at least $K - \left(d - \left\lceil \frac{d}{2} \right\rceil - 1 \right) - 1 - 1 \ge \left\lceil \frac{d}{2} \right\rceil + 1$ available colors for uu_i ($i = \left\lceil \frac{d}{2} \right\rceil, \left\lceil \frac{d}{2} \right\rceil + 1$). By Lemma 2.1, $\left(\left\lceil \frac{d}{2} \right\rceil + 2 \right) \left(\left\lceil \frac{d}{2} \right\rceil - 1 \right) + 2 \left(\left\lceil \frac{d}{2} \right\rceil + 1 \right) - \left(\left\lceil \frac{d}{2} \right\rceil + 1 \right)^2 + 1 = \left\lceil \frac{d}{2} \right\rceil > d - \left\lceil \frac{d}{2} \right\rceil - 1$. So we can color uu_i properly such that $f(u) \ne f(u_i)$ for each $1 \le i \le d$. Thus, we get an nsd-K-coloring of G by Remark 1, a contradiction. □

Claim 3.5. Let $u \in V(H)$ with $d_H(u) = \Delta(G) \ge 7$. If $d_{def}(u) = 1$ and $d_2(u) = \Delta(G) - 1$, then $d_{3^-}(u) = \Delta(G) - 1$.

Proof: Assume that u of H is adjacent to $\Delta(G)$ 3^--vertices $u_1, \cdots, u_{\Delta(G)}$ such that u_1 is a deficient vertex and $u_2, \cdots, u_{\Delta(G)-1}$ are 2-vertices. By Claim 3.1, it holds that $d_G(u_i) = d_H(u_i)$. Suppose that $N_G(u_i) = \{u, w_i\}$ for each $1 \le i \le d - 1$. Let $G' = G - uu_1$. Then G' has an nsd-K-coloring by the minimality of G. Since $1 + 2 + \cdots + (\Delta(G) - 1) = \frac{\Delta(G)(\Delta(G) - 1)}{2} > 2\Delta(G) + 1$, then we can get $f(u) \ne f(u_i)$ for each $1 \le i \le \Delta(G)$. Firstly, we erase the color on edge $u_1 w_1$. There are at least $K - (\Delta(G) - 1) - 1 \ge 1$ available colors for uu_1. Thus, we obtain an nsd-K-coloring of G by Remark 1, a contradiction. □

3.2. Discharging Process

In order to complete the proof of Theorem 1.1, we use the discharging method. For this aim we first define the original charge function $w(u) = d_H(u) - \frac{37}{12}$ for each $u \in V(H)$, then

$$\sum_{u \in V(H)} w(u) = \sum_{u \in V(H)} \left(d_H(u) - \frac{37}{12} \right) \le |V(H)| \left(mad(G) - \frac{37}{12} \right) < 0.$$

We will design several discharging rules and reassign the charges according to the rules below. Let w^* be the final charge. We will show that for each $u \in V(H)$, $w^*(u) \ge 0$. This leads to the following contradiction:

$$0 \le \sum_{u \in V(H)} w^*(u) = \sum_{u \in V(H)} w(u) < 0$$

and it shows the nonexistence of G.

We define the discharging rules as follows:

(R1) Every 5^{+}-vertex gives $\frac{13}{12}$ to each adjacent bad 2-vertex; $\frac{15}{24}$ to each adjacent weak 2-vertex; $\frac{13}{24}$ to each adjacent good 2-vertex.

(R2) Every source vertex gives $\frac{11}{48}$ to its sink vertex.

(R3) Every 4-vertex gives $\frac{11}{24}$ to each adjacent 2-vertex.

(R4) Every 4^{+}-vertex gives $\frac{1}{24}$ to each adjacent 3-vertex.

Now we are going to show $w^{*}(v)\geq 0$ for each $u\in V(H)$. Let $u\in V(H)$ with $d_H(u)=d$. By Claim 3.1 (1), $d\geq 2$.

(1) $d=2$. According to Claim 3.3 (1), u is adjacent to at least one 5^{+}-vertex. First, suppose that u is a bad 2-vertex. Then by (R1), $w^{*}(u)=2-\frac{37}{12}+\frac{13}{12}=0$. Next, suppose that u is a weak 2-vertex. If u is adjacent to a 3-vertex, then u has two source vertices. By (R1) and (R2), $w^{*}(u)=2-\frac{37}{12}+\frac{15}{24}+2\times\frac{11}{48}=0$. If u is adjacent to a 4-vertex, then by (R1) and (R3), $w^{*}(u)=2-\frac{37}{12}+\frac{15}{24}+\frac{11}{24}=0$. Finally, suppose that u is a good 2-vertex, by (R1), $w^{*}(u)=2-\frac{37}{12}+2\times\frac{13}{24}=0$.

(2) $d=3$. According to Claim 3.3 (2), u is adjacent to at least two 4^{+}-vertex. Then by (R4), $w^{*}(u)=3-\frac{37}{12}+2\times\frac{1}{24}=0$.

(3) $d=4$. If u is adjacent to 2-vertex, then by Claim 3.2 (2), $d_{4^{+}}(u)=3$. By (R3), $w^{*}(u)=4-\frac{37}{12}-\frac{11}{24}=\frac{11}{24}$. If u is not adjacent to 2-vertex, then by (R4), $w^{*}(u)\geq 4-\frac{37}{12}-4\times\frac{1}{24}=\frac{3}{4}$.

(4) $d\geq 5$. It is trivial that $s(u)\leq d_3(u)$. Hence we have:

$$\begin{aligned} w^{*}(u) &\geq d-\frac{37}{12}-\frac{13}{12}d_{b2}(u)-\frac{15}{24}\left(d_{def}(u)-d_{b2}(u)\right) \\ &\quad -\frac{13}{24}d_{g2}(u)-\frac{11}{48}s(u)-\frac{1}{24}d_3(u) \\ &\geq d-\frac{37}{12}-\frac{15}{24}d_{def}(u)-\frac{11}{24}d_{b2}(u)-\frac{13}{24}d_{g2}(u)-\frac{13}{48}d_3(u). \end{aligned} \tag{1}$$

(4.1) $d=5$. According to Claim 3.3 (4), $d_2(u)\leq 2$. By Claim 3.3 (3), $d_{b2}(u)\leq 1$.

Suppose that $d_2(u)=2$. By Claim 3.3 (4.2), $d_{def}(u)\leq 1$. Furthermore, if $d_{def}(u)=1$, then $d_3(u)=0$. Suppose that $d_{def}(u)=1$. Then $d_3(u)=0$, $d_{b2}(u)\leq 1$ and $d_{g2}(u)=1$. Thus, $w^{*}(u)\geq 5-\frac{37}{12}-\frac{15}{24}-\frac{11}{24}-\frac{13}{24}=\frac{7}{24}$ by (1). Suppose that $d_{def}(u)=0$. Then $d_{b2}(u)\leq d_{def}=0$, $d_{g2}(u)=2$ and $d_3(u)=3$. Thus, $w^{*}(u)\geq 5-\frac{37}{12}-2\times\frac{13}{24}-3\times\frac{13}{48}=\frac{1}{48}$ by (1).

Suppose that $d_2(u)=1$. If $d_{def}(u)=1$, then $s(u)=0$ and $d_{b2}(u)\le 1$ by Claim 3.3 (3) (4.1). Thus, $w^*(u)\ge 5-\frac{37}{12}-\frac{15}{24}-\frac{11}{24}-4\times\frac{1}{24}=\frac{2}{3}$. If $d_{def}(u)=0$, then $d_{b2}(u)\le d_{def}=0$, $d_{g2}(u)=1$ and $d_3(u)=4$. Thus,
$w^*(u)\ge 5-\frac{37}{12}-\frac{13}{24}-4\times\frac{13}{48}=\frac{7}{24}$ by (1).

Suppose that $d_2(u)=0$. Then $w^*(u)\ge 5-\frac{37}{12}-5\times\frac{13}{48}=\frac{9}{16}$ by (1).

(4.2) $d=6$. According to Claim 3.3 (5), $d_{def}(u)\le 1$.

Suppose that $d_{def}(u)=1$. Then we have $d_{b2}(u)\le d_{def}(u)=1$. By Claim 3.3 (5.2), $d_{g2}(u)\le 1$. Thus, $w^*(u)\ge 6-\frac{37}{12}-\frac{15}{24}-\frac{11}{24}-\frac{13}{24}-4\times\frac{13}{48}=\frac{5}{24}$ by (1).

Suppose that $d_{def}(u)=0$. If $d_{g2}(u)\ge 5$, then $d_{3^-}(u)=5$. We have $d_{b2}(u)\le d_{def}(u)=0$. Thus, $w^*(u)\ge 6-\frac{37}{12}-5\times\frac{13}{24}=\frac{5}{24}$ by (1). If $d_{g2}(u)\le 4$, then $w^*(u)\ge 6-\frac{37}{12}-4\times\frac{13}{24}-2\times\frac{13}{48}=\frac{5}{24}$ by (1).

(4.3) $d\ge 7$. By Claim 3.3 (3), $d_{b2}(u)\le 1$.

(4.3.1) Suppose that $\Delta(G)\ge 13$.

Suppose that $7\le d\le\left\lceil\frac{\Delta(G)}{2}\right\rceil$. Then by Claim 3.4 (1) and (2), we have $d_2(u)\le d-1$ and $d_{def}(u)=0$. Then $d_{b2}(u)\le d_{def}(u)=0$ and $d_{g2}(u)\le d-1$. Thus, $w^*(u)\ge d-\frac{37}{12}-(d-1)\times\frac{13}{24}-\frac{13}{48}=\frac{11}{24}d-\frac{45}{16}>0$.

Suppose that $\left\lceil\frac{\Delta(G)}{2}\right\rceil+1\le d\le\Delta(G)-1$. Since $\Delta(G)\ge 13$, we have $d\ge 7$. Then by Claim 3.4 (3), we have $d_{def}(u)\le\left\lceil\frac{d}{2}\right\rceil-1$. If $d_{def}(u)=\left\lceil\frac{d}{2}\right\rceil-1$, then $d_{g2}(u)\le 1$, $d_{b2}(u)\le 1$. Thus,

$$\begin{aligned}w^*(u)&\ge d-\frac{37}{12}-\frac{15}{24}\left(\left\lceil\frac{d}{2}\right\rceil-1\right)-\frac{11}{24}-\frac{13}{24}-\frac{13}{48}\times\left(d-\left\lceil\frac{d}{2}\right\rceil\right)\\&=\frac{35}{48}d-\frac{17}{48}\left\lceil\frac{d}{2}\right\rceil-\frac{83}{24}\ge\frac{35}{48}d-\frac{17}{48}\left(\frac{d}{2}+\frac{1}{2}\right)-\frac{83}{24}=\frac{53}{96}d-\frac{349}{96}>0\end{aligned}$$

by (1). If $d_{def}(u)\le\left\lceil\frac{d}{2}\right\rceil-2$, then $d_2(u)\le d-3$ and $d_{b2}(u)\le 1$. Thus,

$$\begin{aligned}w^*(u)&\ge d-\frac{37}{12}-\frac{15}{24}\left(\left\lceil\frac{d}{2}\right\rceil-2\right)-\frac{11}{24}-\frac{13}{24}\left(d-3-\left\lceil\frac{d}{2}\right\rceil+2\right)-\frac{13}{48}\times 3\\&=\frac{11}{24}d-\frac{1}{12}\left\lceil\frac{d}{2}\right\rceil-\frac{123}{48}\ge\frac{11}{24}d-\frac{1}{12}\left(\frac{d}{2}+\frac{1}{2}\right)-\frac{123}{48}=\frac{5}{12}d-\frac{125}{48}>0\end{aligned}$$

by (1).

Suppose that $d=\Delta(G)\ge 13$. By Claim 3.5, we know $d_{def}(u)\le\Delta(G)-1$. If $d_{def}(u)=\Delta(G)-1$, then $d_{3^-}(u)=\Delta(G)-1$. Thus,
$w^*(u)\ge\Delta(G)-\frac{37}{12}-\frac{15}{24}(\Delta(G)-1)-\frac{11}{24}=\frac{3}{8}\Delta(G)-\frac{35}{12}>0$ by (1). If

$1 \le d_{def}(u) \le \Delta(G) - 2$, then by (1), we have

$$w^*(u) \ge \Delta(G) - \frac{37}{12} - \frac{15}{24} d_{def}(u) - \frac{11}{24} d_{b2}(u) - \frac{13}{24} d_{g2}(u) - \frac{13}{48} d_3(u)$$
$$\ge \Delta(G) - \frac{37}{12} - \frac{15}{24} d_2(u) - \frac{11}{24} d_{b2}(u) - \frac{13}{48}(\Delta(G) - d_2(u))$$
$$\ge \Delta(G) - \frac{37}{12} - \frac{15}{24}(\Delta(G) - 2) - \frac{11}{24} - \frac{13}{48} \times 2 = \frac{3}{8}\Delta(G) - \frac{17}{6} > 0$$

. If $d_{def}(u) = 0$, then $w^*(u) \ge \Delta(G) - \frac{37}{12} - \frac{13}{24}\Delta(G) = \frac{11}{24}\Delta(G) - \frac{37}{12} > 0$.

(4.3.1) Suppose that $8 \le \Delta(G) \le 12$. We have $d \ge 7 \ge \left\lceil \frac{\Delta(G)}{2} \right\rceil + 1$. According to Claim 3.4 (3) and Claim 3.5, the derivation process is completely analogous to 4.3.1.

4. Conclusion

In this paper, we have studied neighbor sum distinguishing index of sparse graphs. However, there are still many graphs which are not covered here. So, for further research, we will consider the neighbor sum distinguishing edge coloring of planar graphs.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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