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Application of Lambert *W* Function to Planck Spectral Radiance Frequencies

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Abstract

Planck's radiation law provides an equation for the intensity of the electromagnetic radiation from a physical body as a function of frequency and temperature. The frequency that corresponds to the maximum intensity is a function of temperature. At a specific temperature, for the frequencies correspond to much less than the maximum intensity, an equation was derived in the form of the Lambert *W* function. Numerical calculations validate the equation. A new form of solution for the Euler's transcendental equation was derived in the form of the Lambert *W* function with logarithmic argument. Numerical solutions to the Euler's equation were determined iteratively and iterative convergences were investigated. Numerical coincidences with physical constants were explored.

Keywords

Lambert W Function, Planck Radiation, Euler's Equation, Transcendental Equation, Fine Structure Constant

1. Introduction

Lambert *W* function has applications in science [1] [2] [3] [4], especially in physics [5] [6]. The Lambert *W* function has applications in quantum statistics, and it is used to derive Wien's displacement law in connection with the Planck's black body spectral distribution [7] [8] [9], but it has not been used to describe the spectral distribution. Here we present an application to determine the frequencies in the Planck's black body spectral distribution, for a specific intensity much less than maximum intensity, at a temperature.

Euler found the solution for the equation $\mathcal{X}^{\mathcal{Y}} = \mathcal{Y}^{\mathcal{X}}$ in the form of the Lambert *W* function in the 18th century [1]. Recently, an exponential form of this equation was used with iterative technique to find solutions [10] [11], but the iteration progression towards convergence has not been investigated. Here we investigate the iteration progression and found solutions for the Euler's equation for a large range of numbers.

The Lambert W function is defined by $W(\mathcal{X})e^{W(\mathcal{X})} = \mathcal{X}$. For real numbers, when < 0, $W(\mathcal{X})$ is a double valued function.

In the region $W(\mathcal{X}) < -1$, it is denoted as $W_{-1}(\mathcal{X})$, and in the region $W(\mathcal{X}) > -1$, it is denoted as $W_0(\mathcal{X})$.

The plot $W(\mathcal{X})$ vs $\mathcal{X}$ is shown in **Figure 1**, and the plot $\ln(-W(\mathcal{X}))$ vs $\ln(-\mathcal{X})$ for $\mathcal{X} < 0$, is shown in **Figure 2**. **Figure 2** displays more detail description of the Lambert W function in the region 0 to $-\infty$.

2. Lambert *W* Function and Planck's Radiation Law

In the Planck's radiation law, the spectral radiance in terms of frequency [7] is given by

$$B(\nu,T) = \frac{2h\nu^3}{c^2}\frac{1}{e^{\frac{h\nu}{kT}} - 1} \tag{2.1}$$

The frequency $\nu_{\max}$ corresponds to the maximum intensity [8] [9] is given by

$$\nu_{\max} = \frac{kT}{h}\left[3 + W\left(\frac{-3}{e^3}\right)\right] \tag{2.2}$$

In the Planck's radiation curve at a temperature, for any one intensity below the maximum intensity, two different frequencies can be found. Consider ν_1 and ν_2 are two frequencies correspond to one intensity.

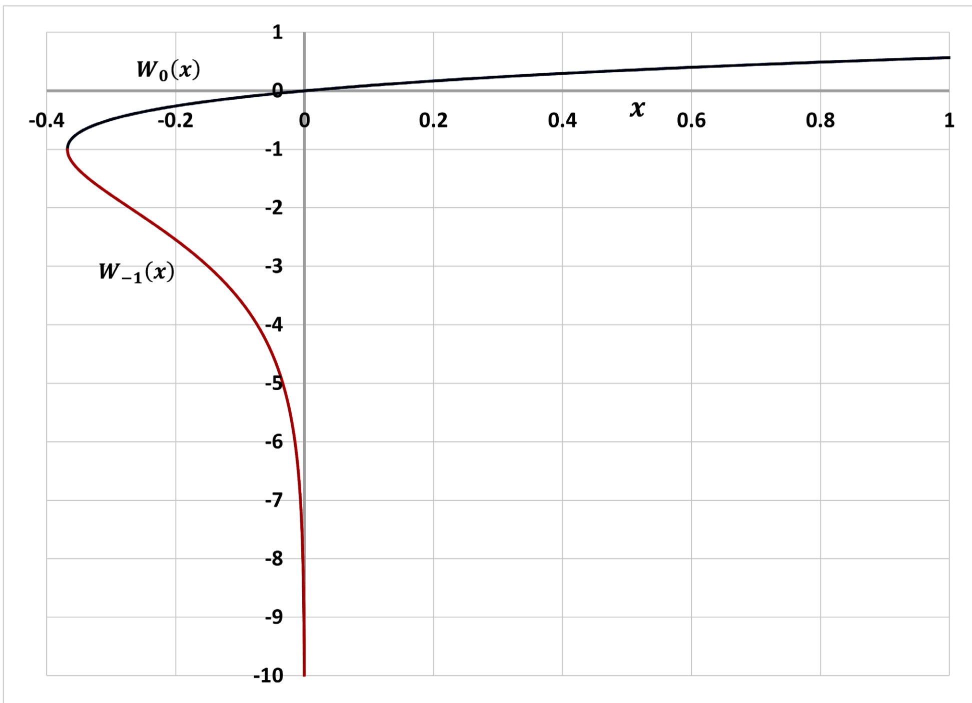

Figure 1. $W(\mathcal{X})$ vs $\mathcal{X}$ plot.

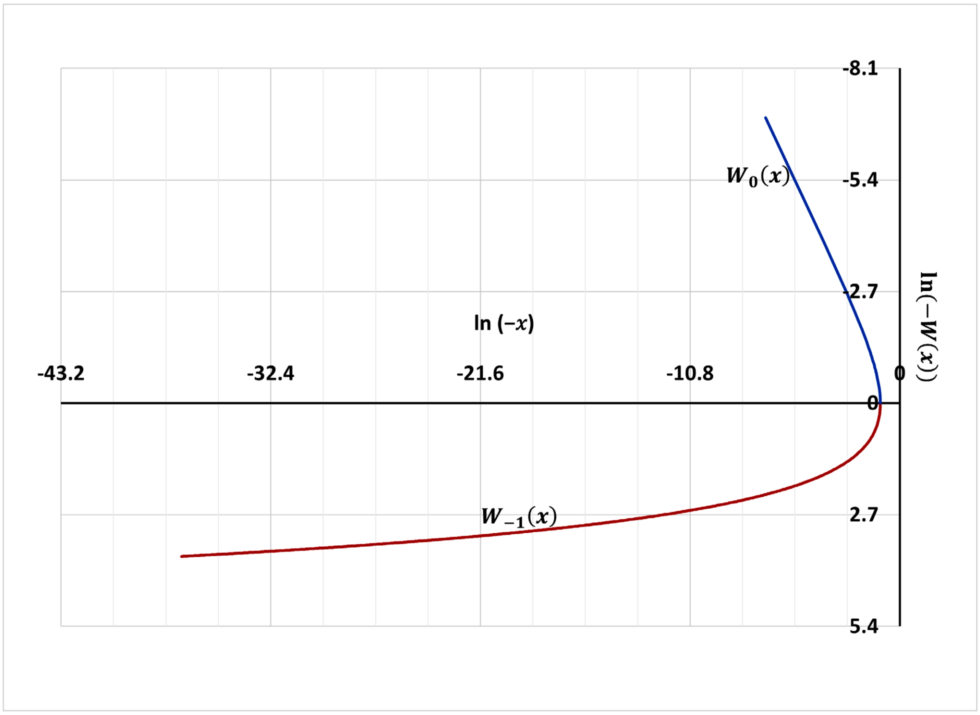

Figure 2. $\ln\left(-W\left(\mathcal{X}\right)\right)$ vs $\ln\left(-\mathcal{X}\right)$ plot.

This implies

$$\frac{\nu_2^3}{e^{\frac{h\nu_2}{kT}}-1}=\frac{\nu_1^3}{e^{\frac{h\nu_1}{kT}}-1} \tag{2.3}$$

$$\left(\frac{\nu_1}{\nu_2}\right)^3 e^{\frac{h\nu_2}{kT}}-\left(\frac{\nu_1}{\nu_2}\right)^3=e^{\frac{h\nu_1}{kT}}-1 \tag{2.4}$$

If $\nu_2 \gg \nu_1$, $\left(\frac{\nu_1}{\nu_2}\right)^3 \ll 1$.

The Equation (2.4) can be written as

$$\left(\frac{\nu_1}{\nu_2}\right)^3 e^{\frac{h\nu_2}{kT}} \simeq e^{\frac{h\nu_1}{kT}}-1 \tag{2.5}$$

For $\frac{h\nu_1}{kT} \ll 1$, region where Raleigh-Jean's law applies, the Equation (2.5) can be written as

$$\left(\frac{\nu_1}{\nu_2}\right)^3 e^{\frac{h\nu_2}{kT}} \simeq \frac{h\nu_1}{kT} \tag{2.6}$$

Let the ratio $\frac{\nu_2}{\nu_1}=r$.

The Equation (2.6) can be written as

$$\frac{h\nu_2}{kT} \simeq \frac{1}{r^2} e^{\frac{h\nu_2}{kT}} \tag{2.7}$$

$$-\left(\frac{h\nu_2}{kT}\right)\mathrm{e}^{-\left(\frac{h\nu_2}{kT}\right)} \simeq -\left(\frac{1}{r^2}\right) \tag{2.8}$$

$$W\left(-\frac{1}{r^2}\right) \simeq -\frac{h\nu_2}{kT} \tag{2.9}$$

$$\nu_2 \simeq \frac{kT}{h}W\left(-\frac{1}{r^2}\right) \text{ and } \nu_1 = \frac{\nu_2}{r} \tag{2.10}$$

This new Equation (2.10) provides the solutions for the frequencies at which the intensities are equal, with the conditions $r \gg 1$ and $\frac{h\nu_1}{kT} \ll 1$. This equation is in the same form as the Equation (2.2) for the $\nu_{\max}$.

Table 1 gives the calculated values for the intensity ratio for the frequencies ν_1 and ν_2. The ratio is close to one for $r \gg 1$, as expected.

3. Euler's Transcendental Equation and Lambert *W* Function

The solution for the equation $\mathcal{X}^{\mathcal{Y}} = \mathcal{Y}^{\mathcal{X}}$ is given by $\mathcal{Y} = \frac{W\left(-\frac{\ln(\mathcal{X})}{\mathcal{X}}\right)}{-\frac{\ln(\mathcal{X})}{\mathcal{X}}}$ derived by Euler in the 18th century [1] [10] [11].

Theorem: The solutions for the series of exponential equations $\mathcal{Y} = \mathcal{X}^{\left(\frac{\mathcal{Y}}{\mathcal{X}^{n+1}}+n\right)}$ is given by $\mathcal{Y} = \mathcal{X}^n \exp\left[-W\left(-\frac{\ln(\mathcal{X})}{\mathcal{X}}\right)\right]$.

Proof:

One form of analytical solutions for the series of exponential equations was derived previously [10] [11].

$$\mathcal{Y} = \mathcal{X}^{\left(\frac{\mathcal{Y}}{\mathcal{X}^{n+1}}+n\right)} \tag{3.1}$$

The solutions derived previously:

$$\mathcal{Y} = \mathcal{X}^{n+1}, \text{ trivial solutions} \tag{3.2}$$

and

Table 1. Intensity ratio for different *r* values with other functions.

r	$-\frac{1}{r^2}$	$-W_{-1}\left(-\frac{1}{r^2}\right) = \frac{h\nu_2}{kT}$	$\frac{h\nu_1}{kT}$	$\frac{\nu_2^3}{\mathrm{e}^{\frac{h\nu_2}{kT}}-1}$	$\frac{\nu_1^3}{\mathrm{e}^{\frac{h\nu_1}{kT}}-1}$	Intensity ratio
1.65E+00	−3.68E−01	1	6.07E−01	5.82E−01	2.68E−01	0.4597
5.45E+00	−3.37E−02	5	9.18E−01	8.48E−01	5.14E−01	0.6062
4.69E+01	−4.54E−04	10	2.13E−01	4.54E−02	4.07E−02	0.8972
4.67E+02	−4.59E−06	15	3.21E−02	1.03E−03	1.02E−03	0.9840
4.93E+03	−4.12E−08	20	4.06E−03	1.65E−05	1.65E−05	0.9980
5.37E+04	−3.47E−10	25	4.66E−04	2.17E−07	2.17E−07	0.9998

$$\mathcal{Y} = \frac{\mathcal{X}^n W\left(-\frac{\ln(\mathcal{X})}{\mathcal{X}}\right)}{\left(-\frac{\ln(\mathcal{X})}{\mathcal{X}}\right)}, \text{ non-trivial solutions} \tag{3.3}$$

The non-trivial solutions can be refined further.

The Equation (3.1) can also be written as

$$\frac{\ln \mathcal{Y}}{\ln \mathcal{X}} = \frac{\mathcal{Y}}{\mathcal{X}^{n+1}} + n \tag{3.4}$$

Rearranging the Equation (3.4)

$$\frac{\ln \mathcal{X}}{\mathcal{X}^{n+1}} = \frac{\ln \mathcal{Y}}{\mathcal{Y}} - \frac{n \ln \mathcal{X}}{\mathcal{Y}} \tag{3.5}$$

Using Equation (3.5), the solution in Equation (3.3) can be written as:

$$\ln \mathcal{Y} - n \ln \mathcal{X} = W\left(-\frac{\ln(\mathcal{X})}{\mathcal{X}}\right) \tag{3.6}$$

Rearranging the Equation (3.6)

$$\ln \mathcal{Y} = \ln \mathcal{X}^n + \ln \exp\left[W\left(-\frac{\ln(\mathcal{X})}{\mathcal{X}}\right)\right] \tag{3.7}$$

Hence the solution for the Equation (3.1) can be written as

$$\mathcal{Y} = \mathcal{X}^n \exp\left[-W\left(-\frac{\ln(\mathcal{X})}{\mathcal{X}}\right)\right] \tag{3.8}$$

If $n = 0$, the Equation (3.1) becomes

$$\mathcal{X} = \mathcal{Y}^{\frac{\mathcal{Y}}{\mathcal{X}}} \tag{3.9}$$

i.e. $\mathcal{X}^{\mathcal{Y}} = \mathcal{Y}^{\mathcal{X}}$ (Euler's equation).

The solution is

$$\mathcal{Y} = \exp\left[-W\left(-\frac{\ln(\mathcal{X})}{\mathcal{X}}\right)\right] \tag{3.10}$$

The $\frac{\ln(\mathcal{X})}{\mathcal{X}}$ is maximum at $\mathcal{X} = \mathrm{e}$. For $1 < \mathcal{X} < \mathrm{e}$, the non-trivial solutions are in terms of $W_0(\mathcal{X})$ and for $\mathcal{X} > \mathrm{e}$, the non-trivial solutions are in terms of $W_{-1}(\mathcal{X})$.

4. Numerical Calculation

The numerical values of the function in Equation (3.10) were calculated using the Equation (3.9), utilizing the iterative technique. The iteration progresses are shown in Figure 3 for few $\mathcal{X}$ values. For $\mathcal{X} > \mathrm{e}$, the iteration converges to the non-trivial solution. For $\mathcal{X} < \mathrm{e}$, the iteration converges to the trivial solution. At $\mathcal{X} = \mathrm{e}$, $\mathcal{Y} = \mathrm{e}$, the trivial and the nontrivial solutions are equal.

The non-trivial solutions in the range of $1 < \mathcal{X} < \mathrm{e}$ were determined, using the $\mathcal{XY}$ symmetry in Equation (3.9). For $\mathcal{X} < \mathrm{e}$, even when the seed value close to the non-trivial solution the iteration is unstable (Figure 4).

The numerical values of the function $\exp\left[-W\left(-\frac{\ln(\mathcal{X})}{\mathcal{X}}\right)\right]$ are given in Table 2.

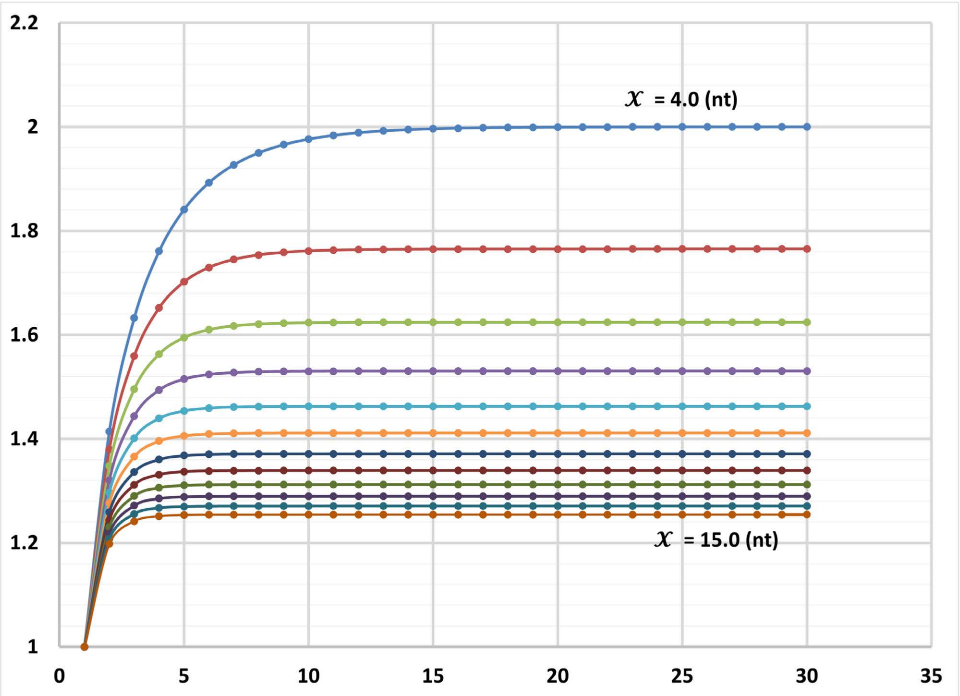

Figure 3. Iteration steps for non-trivial (nt) solutions for $\mathcal{X}$ values from 4 to 15.

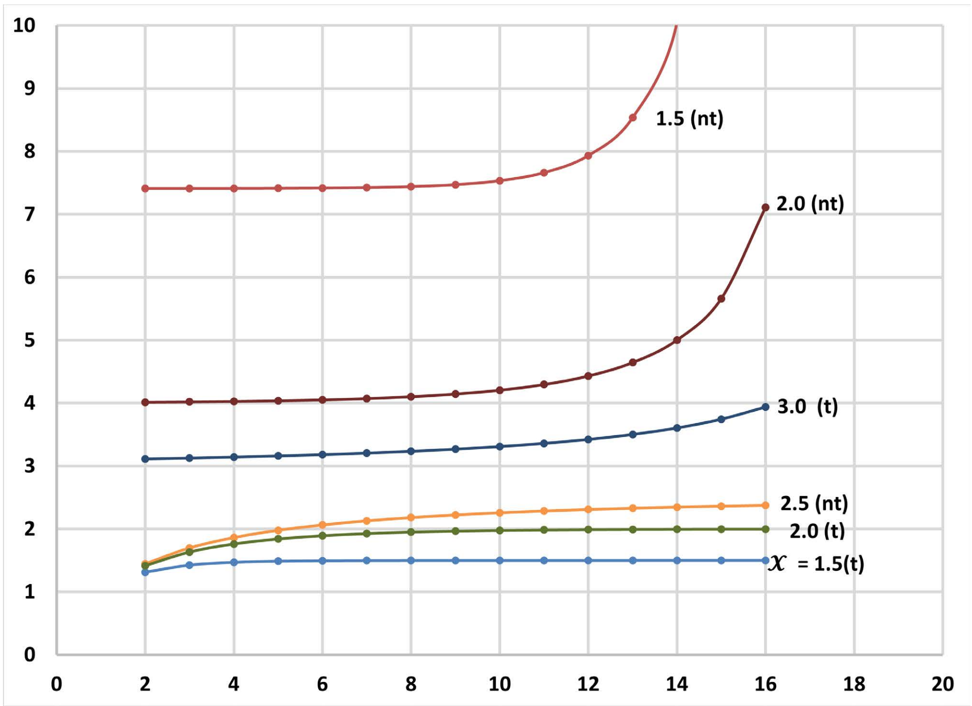

Figure 4. Iteration steps for trivial (t) and non-trivial (nt) solutions for $\mathcal{X}$ values of 1.5, 2 and 3.

Table 2. Calculated values of function $\exp\left[-W\left(-\frac{\ln(\mathcal{X})}{\mathcal{X}}\right)\right]$, given in terms of W_0 and W_{-1}, depending on the range.

$\mathcal{X}$	$\exp\left[-W_0\left(-\frac{\ln(\mathcal{X})}{\mathcal{X}}\right)\right]$	$\mathcal{X}$	$\exp\left[-W_{-1}\left(-\frac{\ln(\mathcal{X})}{\mathcal{X}}\right)\right]$
1.000922	100000	3	2.478052
1.004931	1493.1	4	2.000000
1.00698	1000	5	1.764922
1.012666	500	6	1.624244
1.027597	200	7	1.530140
1.049519	100	8	1.462501
1.066895	70	9	1.411382
1.076203	60	10	1.371289
1.088933	50	11	1.338936
1.107538	40	12	1.312235
1.137669	30	12.9155	1.29155
1.196236	20	13	1.289792
1.254088	15	14	1.270640
1.270640	14	15	1.254088
1.289792	13	20	1.196236
1.312235	12	30	1.137669
1.338936	11	40	1.107538
1.371289	10	50	1.088933
1.411382	9	60	1.076203
1.462501	8	70	1.066895
1.530140	7	100	1.049519
1.624244	6	137.036	1.038
1.764922	5	137.7798	1.03778
2	4	200	1.027597
2.5	2.98	500	1.012666
2.718282	2.718282	1000	1.00698
		1493.1	1.004931

Using the values of the function in Table 2, the plots of the function $\mathcal{X}^n \exp\left[-W\left(-\frac{\ln(\mathcal{X})}{\mathcal{X}}\right)\right]$, for $n = -2, -1, 0, 1$ and 2 are shown in Figure 5. Plot of $-W\left(-\frac{\ln(\mathcal{X})}{\mathcal{X}}\right)$ calculated using the vales in Table 2 and the comparison plot of $\frac{\ln(\mathcal{X})}{\mathcal{X}}$ are shown in Figure 6.

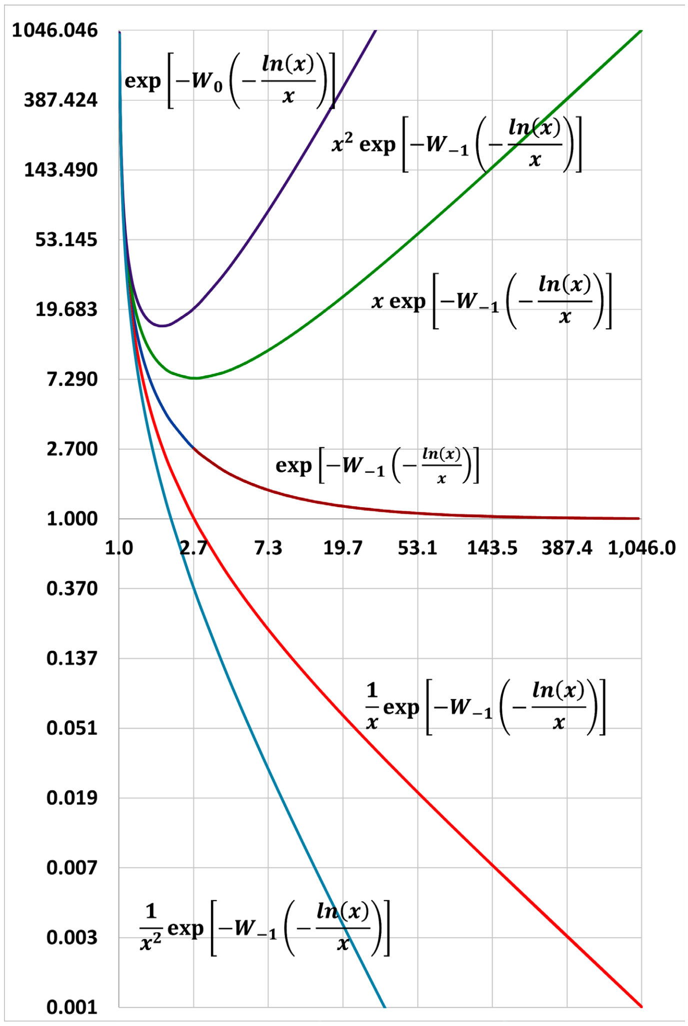

Figure 5. Plots of function $\mathcal{X}^n\exp\left[-W\left(-\frac{\ln(\mathcal{X})}{\mathcal{X}}\right)\right]$ for n of −2, −1, 0, 1, and 2.

For different values of n, $\mathcal{X}$ and $\mathcal{Y}$ in Equation (3.8), using **Table 2**, following numerical equations can be obtained

$$2=\exp\left[-W_{-1}\left(-\frac{\ln(4)}{4}\right)\right] \tag{4.1}$$

$$4=\exp\left[-W_{0}\left(-\frac{\ln(2)}{2}\right)\right] \tag{4.2}$$

$$4^2=2^4 \tag{4.3}$$

$$\exp\left[-W_{0}\left(-\frac{\ln(1.3713)}{1.3713}\right)\right]=10 \tag{4.4}$$

$$\exp\left[-W_{-1}\left(-\frac{\ln(10)}{10}\right)\right]=1.3713 \tag{4.5}$$

$$10^{1.3713}=1.3713^{10}=23.5 \tag{4.6}$$

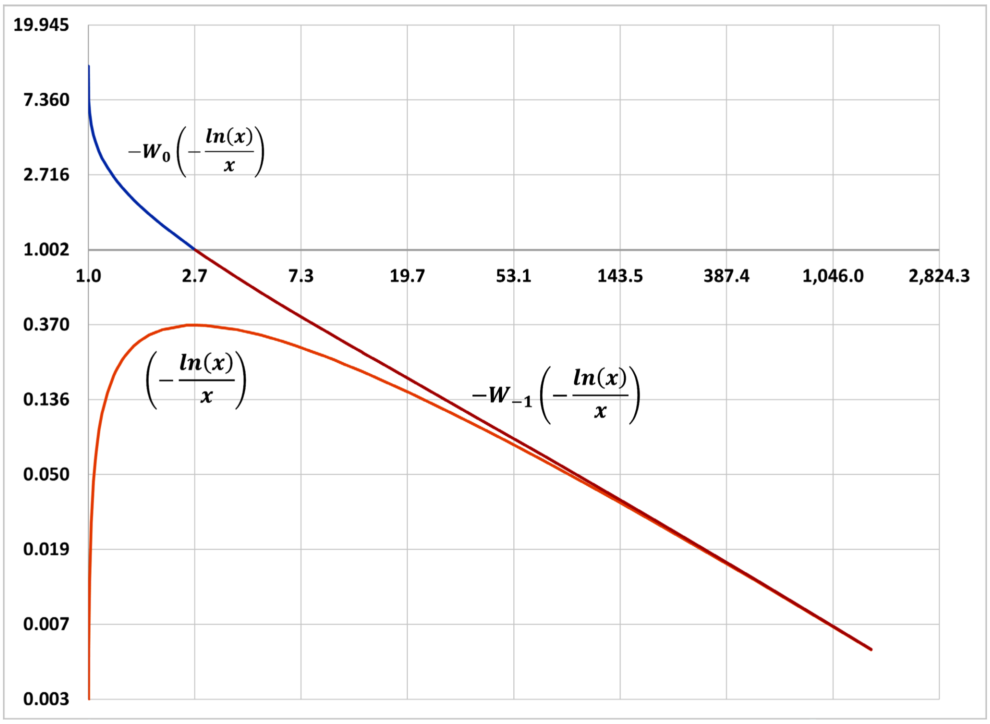

Figure 6. Plots of $-W\left(-\frac{\ln(\mathcal{X})}{\mathcal{X}}\right)$ and $\frac{\ln(\mathcal{X})}{\mathcal{X}}$.

$$\frac{\ln 1.3713}{1.3713} = \frac{\ln 10}{10} = 0.2302 \tag{4.7}$$

$$10^{0.13713} = 1.3713 \tag{4.8}$$

$$10^2 \exp\left[-W_{-1}\left(-\frac{\ln(10)}{10}\right)\right] = 137.129 \tag{4.9}$$

$$\frac{\ln 137.13}{\ln 10} = \frac{137.13}{10^3} + 2 = 2.13713 \tag{4.10}$$

$$\exp\left[-W_{-1}\left(-\frac{\ln(100)}{100}\right)\right] = 1.0495 \tag{4.11}$$

$$100^{1.0495} = 1.0495^{100} \tag{4.12}$$

$$\exp\left[-W_{-1}\left(-\frac{\ln(1000)}{1000}\right)\right] = 1.00698 \tag{4.13}$$

$$1000^{1.00698} = 1.00698^{1000} \tag{4.14}$$

$$12.9155^{1.29155} = 1.2915^{12.9155} \tag{4.15}$$

$$137.78^{1.0378} = 1.0378^{137.78} \tag{4.16}$$

$$1493.1^{1.004931} = 1.004931^{1493.1} \tag{4.17}$$

5. Numerical Coincidences

Consider the Equations (4.9) and (4.16),

$$10^2 \exp\left[-W_{-1}\left(-\frac{\ln(10)}{10}\right)\right] = 137.129$$

$$\exp\left[-W_0\left(-\frac{\ln(1.0378)}{1.0378}\right)\right] = 137.78$$

These solutions are unique. Numerical coincidences for these numbers with physical constants are given below:

The dimensionless electromagnetic fine structure constant

$$\alpha = \frac{e^2}{4\pi\varepsilon_o \hbar c} = 7.2973\times 10^{-3} \quad (5.1)$$

$$\alpha^{-1} = 137.036 \quad (5.2)$$

$$\alpha^{-1} + 10\alpha = 137.11 \quad (5.3)$$

$$\alpha^{-1} + 100\alpha = 137.766 \quad (5.4)$$

$$\alpha^{-1} + \alpha^{1/2} + \alpha = 137.1287 \quad (5.5)$$

The dimensionless gravitational fine structure constant defined using electron mass can be written as

$$\alpha_G^{ee} = \frac{Gm_e^2}{hc} = 7.38\times 10^{-45} \quad (5.6)$$

For convenience hereafter α_G^{ee} will be referred as α_G

$$\alpha_G^{-1} = 1.35\times 10^{44} \quad (5.7)$$

$$\ln \alpha_G^{-1} = 101.6 \quad (5.8)$$

$$\left(\ln \alpha_G^{-1}\right)^{1/2} = 10.08 \quad (5.9)$$

The numerical values of α^{-1} and $\ln \alpha_G^{-1}$ are close and it was suggested that they are related [12].

In Equation (4.9), if 10 is replaced with $\left(\ln \alpha_G^{-1}\right)^{1/2}$, the equation becomes

$$\left(\ln \alpha_G^{-1}\right) \exp\left[-W_{-1}\left(-\frac{\ln\left(\left(\ln \alpha_G^{-1}\right)^{1/2}\right)}{\left(\ln \alpha_G^{-1}\right)^{1/2}}\right)\right] = 136.85 \quad (5.10)$$

6. Conclusion

In the Planck's radiation law equation, for a specific temperature and intensity, the frequencies will be given by $\nu_2 \simeq \frac{kT}{h} W\left(-\frac{1}{r^2}\right)$ and $\nu_1 = \frac{\nu_2}{r}$, with conditions $r \ll 1$, and $\frac{h\nu_1}{kT} \ll 1$. The numerical calculations of the intensity at these frequencies validated the equations.

A new form of solution for the Euler's equation $\mathcal{X}^{\mathcal{Y}} = \mathcal{Y}^{\mathcal{X}}$ was derived in the form of the Lambert W function as, $\mathcal{Y} = \exp\left[-W\left(-\frac{\ln(\mathcal{X})}{\mathcal{X}}\right)\right]$, and the corresponding solutions for the series of exponential equations. Interesting numerical equations were derived and coincidences with electromagnetic fine structure constant were indicated.

"God used beautiful mathematics in creating the world" quote by Paul Dirac.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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