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# Positive Solution for a Singular Fourth-Order Differential System

**B. Kovács**

Institute of Mathematics, University of Miskolc, Miskolc, Hungary
Email: matmn@uni-miskolc.hu

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## Abstract

In this paper, we investigate the existence of positive solutions for the singular fourth-order differential system $u^{(4)} = \varphi u + f(t,u,u'',\varphi)$, $0<t<1$, $-\varphi'' = \mu g(t,u,u'')$, $0<t<1$, $u(0)=u(1)=u''(0)=u''(1)=0$, $\varphi(0)=\varphi(1)=0$; where $\mu>0$ is a constant, and the nonlinear terms $f,g$ may be singular with respect to both the time and space variables. The results obtained herein generalize and improve some known results including singular and non-singular cases.

## Keywords

Positive Solutions, Fixed Point Theorem, Singular Solutions, Bending of an Elastic Beam, Cone, Boundary Value Problem, Existence, Multiplicity

## 1. Introduction

It is well known that the bending of an elastic beam can be described with fourth-order boundary value problems. An elastic beam with its two ends simply supported, can be described by the fourth-order boundary value problem

$$u^{(4)}(t) = f(t,u(t),u''(t)), \quad 0<t<1, \tag{1}$$

$$u(0)=u(1)=u''(0)=u''(1)=0. \tag{2}$$

Existence of solutions for problem (1) was established for example by Gupta [1] [2], Liu [3], Ma [4], Ma *et al.* [5], Ma and Wang [6], Aftabizadeh [7], Yang [8], Del Pino and Manasevich [9], RP Agarwal *et al.* [10] [11] [12] (see also the references therein). All of those results are based on the Leray-Schauder continuation method, topological degree and the method of lower and upper solutions.

Recently, Wang and An [13] studied the existence of positive solutions for the second-order differential system by using the fixed point theorem of cone expansion and compression.

By applying the cone compression and expansion fixed point theorem, Cui and Zou [14] showed that a fourth-order singular boundary problem has a unique positive solution.

By constructing a new type of cone and using fixed point index theory, López-Somoza and Minhós [15] investigated existence of solutions for the Hammerstein equations.

In [16], the authors use a mixed monotone operator method to investigate the existence of positive solution to a fourth-order boundary value problem which describes the deflection of an elastic beam.

In this paper we shall discuss the existence of positive solutions for the fourth-order boundary value problem

$$\begin{gathered} u^{(4)} = \varphi u + f\left(t,u,u'',\varphi\right),\ \ 0<t<1 \\ -\varphi'' = \mu g\left(t,u,u''\right),\ \ 0<t<1 \\ u(0)=u(1)=u''(0)=u''(1)=0, \\ \varphi(0)=\varphi(1)=0, \end{gathered} \tag{3}$$

where $\mu$ is a positive parameter and $f\left(t,u,v,\varphi\right):(0,1)\times[0,\infty)\times(-\infty,0]\times[0,\infty)\to(0,\infty)$ is continuous. In fact as we will see below one could consider in Section 2 and 3 $f\left(t,u,v,\varphi\right)\le f_1(t)f_2\left(t,u,v,\varphi\right)$ with $f_2\left(t,u,v,\varphi\right):[0,1]\times[0,\infty)\times(-\infty,0]\times[0,\infty)\to(0,\infty)$ and $f_1:(0,1)\to(0,+\infty)$ is continuous provided

$$\int_0^1\int_0^1 K(\tau,\tau)K(\tau,s)f_1(s)\,\mathrm{d}s\,\mathrm{d}\tau<+\infty;$$

here $K$ is as defined in Section 2. Moreover, our hypotheses allow but do not require $g\left(t,u,v\right):[0,1]\times[0,\infty)\times(-\infty,0)\to(0,\infty)$ to be singular at $v=0$.

## 2. Preliminaries

Let $Y=C[0,1]$ and

$$Y_+=\left\{u\in Y:u(t)\ge 0,t\in[0,1]\right\}.$$

It is well known that $Y$ is a Banach space equipped with the norm $\|u\|_0=\sup_{t\in[0,1]}|u(t)|$. We denote the norm $\|u\|_2$ by

$$\|u\|_2=\max\left\{\|u\|_0,\|u''\|_0\right\}.$$

It is easy to show that $C^2[0,1]$ is complete with the norm $\|u\|_2$ and $\|u\|_2\le\|u\|_0+\|u''\|_0\le 2\|u\|_2$.

Suppose that $K(t,s)$ is the Green function associated with

$$-u''=f(t),\ \ u(0)=u(1)=0, \tag{4}$$

which is explicitly expressed by

$$K(t,s)=\begin{cases} t(1-s), & 0\le t\le s\le 1 \\ s(1-t), & 0\le s\le t\le 1 \end{cases}$$

We need the following lemmas.

**Lemma 1.** $K(t,s)$ has the following properties:

1) $K(t,s)>0, \forall t,s\in(0,1)$;

2) $K(t,s)\le K(s,s), \forall t,s\in[0,1]$;

3) $K(t,s)\ge K(t,t)K(s,s), \forall t,s\in[0,1]$;

4) $|K(t_1,s)-K(t_2,s)|\le |t_1-t_2|$, for all $t_1,t_2,s\in[0,1]$

**Lemma 2.** ([17]) Let $E$ be a real Banach space and let $P\subset E$ be a cone in $E$. Assume $\Omega_1,\Omega_2$ are open subset of $E$ with $\theta\in\Omega_1$, $\bar{\Omega}_1\subset\Omega_2$, and let $Q: P\cap(\bar{\Omega}_2\setminus\Omega_1)\to P$ be a completely continuous operator such that either

1) $\|Qu\|\le\|u\|, u\in P\cap\partial\Omega_1$ and $\|Qu\|\ge\|u\|, u\in P\cap\partial\Omega_2$; or

2) $\|Qu\|\ge\|u\|, u\in P\cap\partial\Omega_1$ and $\|Qu\|\le\|u\|, u\in P\cap\partial\Omega_2$.

Then $Q$ has a fixed point in $P\cap(\bar{\Omega}_2\setminus\Omega_1)$.

The boundary value problem

$$-\varphi''=\mu g(t,u(t),u''(t)),\ \varphi(0)=\varphi(1)=0,$$

can be solved by using the Green's function, namely,

$$\varphi(t)=\mu\int_0^1 K(t,v)g(v,u(v),u''(v))\mathrm{d}v,\ 0<t<1. \tag{5}$$

Thus inserting (5) into the first equation of (3), we have

$$\begin{aligned} u^{(4)} &= \mu u(t)\int_0^1 K(t,v)g(v,u(v),u''(v))\mathrm{d}v \\ &\quad + f\left(t,u,u'',\mu\int_0^1 K(t,v)g(v,u(v),u''(v))\mathrm{d}v\right), \\ & u(0)=u(1)=u''(0)=u''(1)=0. \end{aligned} \tag{6}$$

Now we consider the existence of a positive solution of (6). The function $u\in C^4(0,1)\cap C^2[0,1]$ is a positive solution of (6), if $u\ge 0$, $t\in[0,1]$, and $u\ne 0$.

Then the solution of (6) can be expressed as

$$\begin{aligned} u(t) &= \mu\int_0^1\int_0^1 K(t,\tau)K(\tau,s)u(s)\int_0^1 K(s,v)g(v,u(v),u''(v))\mathrm{d}v\mathrm{d}s\mathrm{d}\tau \\ &+\int_0^1\int_0^1 K(t,\tau)K(\tau,s)f\left(s,u(s),u''(s),\mu\int_0^1 K(s,v)g(v,u(v),u''(v))\mathrm{d}v\right)\mathrm{d}s\mathrm{d}\tau \end{aligned} \tag{7}$$

and the second-order derivative $u''$ can be expressed by

$$\begin{aligned} u''(t) &= -\mu\int_0^1 K(t,s)u(s)\int_0^1 K(s,v)g\left(v,u(v),u''(v)\mu\int_0^1 K(s,v)g(v,u(v),u''(v))\mathrm{d}v\right)\mathrm{d}v\mathrm{d}s \\ &-\int_0^1 K(t,s)f(s,u(s),u''(s))\mathrm{d}s. \end{aligned} \tag{8}$$

Set

$$\begin{aligned} P=\{u\in C^2[0,1] &: u(0)=u(1)=0, u(t)\ge K(t,t)\|u\|_0, \\ & -u''(t)\ge K(t,t)\|u''\|_0, t\in[0,1]\}. \end{aligned}$$

Note $P$ is a cone in $C^2[0,1]$. For $R>0$, write $B_R=\{u\in C^2[0,1]:\|u\|_2<R\}$.

We now define a mapping $T:P\to C^2[0,1]$ by

$$\begin{aligned}Tu(t)&=\mu\int_0^1\int_0^1 K(t,\tau)K(\tau,s)u(s)\int_0^1 K(s,v)g(v,u(v),u''(v))\,\mathrm{d}v\,\mathrm{d}s\,\mathrm{d}\tau\\&+\int_0^1\int_0^1 K(t,\tau)K(\tau,s)f\left(s,u(s),u''(s),\mu\int_0^1 K(s,v)g(v,u(v),u''(v))\,\mathrm{d}v\right)\mathrm{d}s\,\mathrm{d}\tau.\end{aligned}\tag{9}$$

It is easy to see that if $u\in P$ than

$$-u''(t)\ge\sigma\|u''\|_0,\ t\in\left[\frac{1}{4},\frac{3}{4}\right],\tag{10}$$

where $\sigma=\dfrac{3}{16}$.

**Lemma 3.** Let $w\in P$. Then the following relations hold:

1) $(Tw)(t)\ge K(t,t)\|Tw\|_0$ for $t\in[0,1]$, and

2) $-(Tw)''(t)\ge K(t,t)\|Tw''\|_0$ for $t\in[0,1]$.

**Proof.** For simplicity we denote

$$\begin{aligned}I&=\mu\int_0^1\int_0^1 K(\tau,\tau)K(\tau,s)w(s)\int_0^1 K(s,v)q(v)\,\mathrm{d}v\,\mathrm{d}s\,\mathrm{d}\tau\\&\quad+\int_0^1\int_0^1 K(\tau,\tau)K(\tau,s)h(s)\,\mathrm{d}s\,\mathrm{d}\tau,\end{aligned}$$

$$J=\mu\int_0^1 K(s,s)w(s)\int_0^1 K(s,v)q(v)\,\mathrm{d}v\,\mathrm{d}s+\int_0^1 K(s,s)h(s)\,\mathrm{d}s,$$

and

$$q(v)=g(v,w(v),w''(v)),$$

$$h(s)=f\left(s,w(s),w''(s),\mu\int_0^1 K(s,v)g(v,w(v),w''(v))\,\mathrm{d}v\right).$$

From Lemma 1 it is easy to see that

$$K(t,t)I\le Tw(t)\le I \text{ and } t\in[0,1]\tag{11}$$

$$K(t,t)J\le-(Tw)''(t)\le J,\ t\in[0,1]\tag{12}$$

Using (11-12), we have

$$\|Tw\|_0\le I \text{ and } \left\|-(Tw)''\right\|_0\le J,$$

hence

$$(Tw)(t)\ge K(t,t)\|Tw\|_0 \text{ for } t\in[0,1] \text{ and}$$
$$-(Tw)''(t)\ge K(t,t)\|Tw''\|_0 \text{ for } t\in[0,1].$$

(H1) Throughout this paper, we assume additionally that the function $f(t,u,v,\varphi)$ satisfies

$$f(t,u,v,\varphi)\le f_1(t)f_2((u+|v|)\varphi),\ t\in(0,1),\ u\ge0,\ v\le0,$$

where $f_1:(0,1)\to(0,+\infty)$ and $f_2:[0,+\infty)\to(0,+\infty)$ is continuous provided

$$\int_0^1\int_0^1 K(\tau,\tau)K(\tau,s)f_1(s)\,\mathrm{d}s\,\mathrm{d}\tau<+\infty.$$

Moreover the function $g(t,u,v):[0,1]\times[0,+\infty)\times(-\infty,0)\to(0,+\infty)$ satisfies.

(H2) There exists an $a > 0$ such that $g(t,u,v)$ is non-decreasing in $v \le a$ for each fixed $t \in [0,1]$ and $u \in R^+ = [0,+\infty)$, *i.e.* if $-a \le v_2 \le v_1 < 0$ then $g(t,u,v_1) \ge g(t,u,v_2)$.

(H3) For each fixed $0 < r \le a$

$$0 < \int_0^1 g(s,u,-rK(s,s))\mathrm{d}s < \infty, \quad \forall u \in [0,+\infty).$$

Let us introduce the following notations

$$D_1 = \int_0^1\int_0^1 K(\tau,\tau)K(\tau,s)\mathrm{d}s\mathrm{d}\tau,$$

$$D_2 = \int_0^1\int_0^1 K(\tau,\tau)K(\tau,s)f_1(s)\mathrm{d}s\mathrm{d}\tau,$$

$$D_3 = \int_0^1 K(\tau,\tau)\mathrm{d}\tau,$$

$$D_4 = \int_0^1 K(s,s)f_1(s)\mathrm{d}s,$$

$$D_5 = \int_{\frac{1}{4}}^{\frac{3}{4}}\int_{\frac{1}{4}}^{\frac{3}{4}} K\left(\frac{1}{2},\tau\right)K(\tau,s)\mathrm{d}s\mathrm{d}\tau.$$

**Lemma 4.** Let (H1), (H2) and (H3) hold. Then for all $u \in P \cap \overline{B}_R / B_r$ where $r < a < R$ the following hold

$$(Tu)(t) \le \mu D_3 \|u\|_0 M$$
$$+ D_4 \sup_{s\in(0,1)} f_2\left(\mu\left[\int_0^1 K(s,v)g(v,u(v),u''(v))\mathrm{d}v\right](u(s)+|u''(s)|)\right), t \in (0,1),$$

and

$$-(Tu)''(t) \le \mu D_3 \|u\|_0 M$$
$$+ D_4 \sup_{s\in(0,1)} f_2\left(\mu\left[\int_0^1 K(s,v)g(v,u(v),u''(v))\mathrm{d}v\right](u(s)+|u''(s)|)\right), t \in (0,1),$$

where

$$M = \sup_{w\in[0,R]} \int_0^1 K(v,v)g(v,w,-rK(v,v))\mathrm{d}v + \sup_{w\in[0,R]}\sup_{z\in[a,R]} \int_0^1 K(v,v)g(v,w,z)\mathrm{d}v.$$

**Proof.** It is easy to see that $D_1 \le D_3$ and $D_2 \le D_4$. Let $u \in P \cap \overline{B}_R / B_r$, then by Lemma 6, $\|u\|_0 \le \|u''\|_0$ and by Corollary 7, $\|u\|_2 = \|u''\|_0$. Thus $r \le \|u''\|_0 \le R$. Also since $u \in P$ we have $-u''(t) \ge K(t,t)\|u''\|_0$, $t \in [0,1]$.

By Lemma 1. and (H1)-(H3) we have

$$Tu(t) = \mu\int_0^1\int_0^1 K(t,\tau)K(\tau,s)u(s)\int_0^1 K(s,v)g(v,u(v),u''(v))\mathrm{d}v\mathrm{d}s\mathrm{d}\tau$$
$$+\int_0^1\int_0^1 K(t,\tau)K(\tau,s)f\left(s,u(s),u''(s),\mu\int_0^1 K(s,v)g(v,u(v),u''(v))\mathrm{d}v\right)\mathrm{d}s\mathrm{d}\tau$$
$$= \mu\int_0^1\int_0^1 K(t,\tau)K(\tau,s)u(s)\int_{|u''(v)|\le a} K(s,v)g(v,u(v),u''(v))\mathrm{d}v\mathrm{d}s\mathrm{d}\tau$$
$$+ \mu\int_0^1\int_0^1 K(t,\tau)K(\tau,s)u(s)\int_{|u''(v)|\ge a} K(s,v)g(v,u(v),u''(v))\mathrm{d}v\mathrm{d}s\mathrm{d}\tau$$
$$+\int_0^1\int_0^1 K(t,\tau)K(\tau,s)f\left(s,u(s),u''(s),\mu\int_0^1 K(s,v)g(v,u(v),u''(v))\mathrm{d}v\right)\mathrm{d}s\mathrm{d}\tau$$

$$\le \mu\int_0^1\int_0^1 K(t,\tau)K(\tau,s)\|u\|_0 \sup_{w\in[0,R]}\int_{|u''(v)|\le a} K(v,v)g(v,w,-rK(v,v))\mathrm{d}v\mathrm{d}s\mathrm{d}\tau$$

$$+\mu\int_0^1\int_0^1 K(t,\tau)K(\tau,s)\|u\|_0 \sup_{w\in[0,R]}\sup_{z\in[a,R]}\int_{|u''(v)|\ge a} K(v,v)g(v,w,z)\mathrm{d}v\mathrm{d}s\mathrm{d}\tau$$

$$+\int_0^1\int_0^1 K(\tau,\tau)K(\tau,s)f_1(s)f_2$$

$$\left(\mu\left[\int_0^1 K(s,v)g(v,u(v),u''(v))\mathrm{d}v\right](u(s)+|u''(s)|)\right)\mathrm{d}s\mathrm{d}\tau$$

$$\le \mu\int_0^1\int_0^1 K(\tau,\tau)K(\tau,s)\|u\|_0 \sup_{w\in[0,R]}\int_0^1 K(v,v)g(v,w,-rK(v,v))\mathrm{d}v\mathrm{d}s\mathrm{d}\tau$$

$$+\mu\int_0^1\int_0^1 K(\tau,\tau)K(\tau,s)\|u\|_0\left[\sup_{w\in[0,R]}\sup_{z\in[a,R]}\int_0^1 K(v,v)g(v,w,z)\mathrm{d}v\right]\mathrm{d}s\mathrm{d}\tau$$

$$+\int_0^1\int_0^1 K(\tau,\tau)K(\tau,s)f_1(s)\mathrm{d}s\mathrm{d}\tau$$

$$\sup_{s\in(0,1)} f_2\left(\mu\left[\int_0^1 K(s,v)g(v,u(v),u''(v))\mathrm{d}v\right](u(s)+|u''(s)|)\right)$$

$$\le \mu D_1 M\|u\|_0 + D_2\sup_{s\in(0,1)} f_2\left(\left[\int_0^1 K(s,v)g(v,u(v),u''(v))\mathrm{d}v\right](\mu u(s)+|u''(s)|)\right)$$

$$\le \mu D_3 M\|u\|_0 + D_4\sup_{s\in(0,1)} f_2\left(\left[\int_0^1 K(s,v)g(v,u(v),u''(v))\mathrm{d}v\right](\mu u(s)+|u''(s)|)\right),$$

and similarly we also have

$$-(Tu)''(t)\le \mu\int_0^1\int_0^1 K(\tau,\tau)K(\tau,s)\mathrm{d}s\mathrm{d}\tau\|u\|_0 M$$

$$+\int_0^1 K(s,s)f_1(s)\mathrm{d}s\mathrm{d}\tau\sup_{s\in(0,1)} f_2\left(\left[\int_0^1 K(s,v)g(v,u(v),u''(v))\mathrm{d}v\right](\mu u(s)+|u''(s)|)\right)$$

$$\le \mu D_3\|u\|_0 M + D_4\sup_{s\in(0,1)} f_2\left(\mu\left[\int_0^1 K(s,v)g(v,u(v),u''(v))\mathrm{d}v\right](u(s)+|u''(s)|)\right).$$

**Lemma 5.** $T(P)\subset P$ and $T:P\to P$ is completely continuous.

**Proof.** Let $u\in P$, then we define mapping $T:P\to C^2[0,1]$ by (9). Then for any $u\in P$, it is clear that

$$\begin{aligned}(Tu)''(t)&=-\mu\int_0^1 K(t,s)u(s)\int_0^1 K(s,v)g(v,u(v),u''(v))\mathrm{d}v\mathrm{d}s\\&-\int_0^1 K(t,s)f\left(s,u(s),u''(s),\mu\int_0^1 K(s,v)g(v,w(v),w''(v))\mathrm{d}v\right)\mathrm{d}s\le 0.\end{aligned}\tag{13}$$

By Lemma 3,

$$Tu(t)\ge K(t,t)\|Tu\|_0,\ t\in[0,1]$$

and

$$-(Tu)''(t)\ge K(t,t)\left\|(Tu)''\right\|_0,\ t\in[0,1].$$

Hence $T(P)\subset P$.

Let $V\subset P$ be a bounded set. Then there exists a $d>0$, such that $\sup\{\|u\|_2 : u\in V\}=d$.

First we prove $T(V)$ is bounded. Since $\|u\|_2=\max\{\|u\|_0,\|u''\|_0\}$, and $\|u\|_0\le\|u''\|_0$ we have

$$\mu\left[\int_0^1 K(s,v)g\left(v,u(v),u''(v)\right)\mathrm{d}v\right]\left(u(t)+\left|u''(t)\right|\right)$$
$$\le \mu M\left(\|u\|_0+\|u''\|_0\right)\le 2dM\mu,\ \text{ for all } t\in[0,1].$$

Let $M_d=\sup\left\{f_2(w):w\in[0,2dM\mu]\right\}$. Now from Lemma 4 we have for any $u\in V$ and $t\in[0,1]$ that

$$\begin{aligned}
&|Tu(t)|=\Big|\mu\int_0^1\int_0^1 K(t,\tau)K(\tau,s)u(s)\int_0^1 K(s,v)g\left(v,u(v),u''(v)\right)\mathrm{d}v\mathrm{d}s\mathrm{d}\tau\\
&+\int_0^1\int_0^1 K(t,\tau)K(\tau,s)f\left(s,u(s),u''(s),\mu\int_0^1 K(s,v)g\left(v,w(v),w''(v)\right)\mathrm{d}v\right)\mathrm{d}s\mathrm{d}\tau\Big|\\
&\le \mu D_1\|u\|_0 M+D_2\sup_{s\in(0,1)} f_2\left(\mu\left[\int_0^1 K(s,v)g\left(v,u(v),u''(v)\right)\mathrm{d}v\right]\left(u(s)+\left|u''(s)\right|\right)\right)\\
&\le \mu D_1 dM+M_d D_2.
\end{aligned}\tag{14}$$

We have a similar type inequality for

$$\left|(Tu)''(t)\right|.$$

Therefore $T(V)$ is bounded.

Next we prove that $T(V)$ is equicontinuous. Now from Lemma 4, we have for any $u\in V$ and any $t_1,t_2\in[0,1]$ that

$$\begin{aligned}
&\left|(Tu)(t_1)-(Tu)(t_2)\right|\\
&\le \mu\int_0^1\int_0^1\left|K(t_1,\tau)-K(t_2,\tau)\right|K(\tau,s)u(s)\int_0^1 K(s,v)g\left(v,u(v),u''(v)\right)\mathrm{d}v\mathrm{d}s\mathrm{d}\tau\\
&+\int_0^1\int_0^1\left|K(t_1,\tau)-K(t_2,\tau)\right|K(\tau,s)\\
&\quad f\left(s,u(s),u''(s),\mu\int_0^1 K(s,v)g\left(v,w(v),w''(v)\right)\mathrm{d}v\right)\mathrm{d}s\mathrm{d}\tau\\
&\le \mu\int_0^1\int_0^1\left|K(t_1,\tau)-K(t_2,\tau)\right|K(\tau,s)\mathrm{d}s\mathrm{d}\tau\|u\|_0 M\\
&+\int_0^1\int_0^1\left|K(t_1,\tau)-K(t_2,\tau)\right|K(\tau,s)f_1(s)\\
&\quad f_2\left(\mu\left[\int_0^1 K(s,v)g\left(v,u(v),u''(v)\right)\mathrm{d}v\right]\left(u(s)+\left|u''(s)\right|\right)\right)\mathrm{d}s\mathrm{d}\tau\\
&\le \mu|t_1-t_2|\left|\int_0^1\int_0^1 K(s,s)K(s,v)\mathrm{d}v\mathrm{d}s\right|\|u\|_0 M+M_d|t_1-t_2|\left|\int_0^1\int_0^1 K(s,s)f_1(s)\mathrm{d}s\mathrm{d}\tau\right.\\
&\le \left(\mu D_3 dM+M_d D_4\right)|t_1-t_2|.
\end{aligned}$$

We have a similar type inequality for $\left|(Tu)''(t_1)-(Tu)''(t_2)\right|$.

Therefore $T(V)$ is equicontinuous.

Next we prove that $T$ is continuous. Suppose $u_n,u\in P$ and $\|u_n-u\|_2\to 0$ which implies that $u_n(t)\to u(t),u_n''(t)\to u''(t)$ uniformly on $[0,1]$. Similarly for $f(t,u,v)\le f_1(t)f_2(|u|+|v|)$, $f_2\left(|u_n(t)|+|u_n''(t)|\right)\to f_2\left(|u(t)|+|u''(t)|\right)$ uniformly on $[0,1]$ and $g(t,u_n(t))\to g(t,u(t))$ uniformly on $[0,1]$. The assertion follows from the estimate

$$\begin{aligned}
&\left|Tu_n(t)-Tu(t)\right|\\
&\le \mu\int_0^1\int_0^1 K(t,\tau)K(\tau,s)\Big|u_n(s)\int_0^1 K(s,v)g\left(v,u_n(v),u_n''(v)\right)\mathrm{d}v\\
&-u(s)\int_0^1 K(s,v)g\left(v,u(v),u''(v)\right)\mathrm{d}v\Big|\mathrm{d}s\mathrm{d}\tau
\end{aligned}$$

$$+\int_0^1\int_0^1 K(t,\tau)K(\tau,s)|f_1(s)|\Big|f_2\Big(\mu\Big[\int_0^1 K(s,v)g(v,u_n(v),u_n''(v))\mathrm{d}v\Big](u_n(s)$$

$$+|u_n''(s)|)\Big)-f_2\Big(\mu\Big[\int_0^1 K(s,v)g(v,u(v),u''(v))\mathrm{d}v\Big](u(s)+|u''(s)|)\Big)\Big|\mathrm{d}s\mathrm{d}\tau$$

and the similar estimate for

$$\left|(Tu_n)''(t)-(Tu)''(t)\right|$$

by an application of the standard theorem on the convergence of integrals.

The Ascoli-Arzela theorem guarantees that $T:P\to P$ is completely continuous.

**Lemma 6.** If $u(0)=u(1)=0$ and $u\in C^2[0,1]$, then $\|u\|_0\le\|u''\|_0$, and so, $\|u\|_2=\|u''\|_0$.

**Proof.** Since $u(0)=u(1)$, there is a $\alpha\in(0,1)$ such that $u'(\alpha)=0$, and so $u'(t)=\int_\alpha^t u''(s)\mathrm{d}s$, $t\in[0,1]$. Hence $|u'(t)|\le\int_\alpha^t|u''(s)|\mathrm{d}s\le\int_0^1|u''(s)|\mathrm{d}s\le\|u''\|_0$, $t\in[0,1]$. Thus $\|u'\|_0\le\|u''\|_0$. Since $u(0)=0$, we have $u(t)=\int_0^t u'(s)\mathrm{d}s$, $t\in[0,1]$, and so $|u(t)|\le\int_0^1|u'(s)|\mathrm{d}s\le\|u'\|_0$. Thus $\|u\|_0\le\|u'\|_0\le\|u''\|_0$. Since $\|u\|_2=\max\{\|u\|_0,\|u''\|_0\}$ and $\|u\|_0\le\|u''\|_0$, we obtain that $\|u\|_2=\|u''\|_0$.

**Corollary 7.** Let $r>0$ and let $u\in\partial B_r\cap P$. Then $\|u\|_2=\|u''\|_0=r$.

## 3. Main Results

**Theorem 1.** Let (H1), (H2) and (H3) hold. Assume that the following condition holds

(H4)

$$\limsup_{w\to0^+}\frac{f_2(w)}{w}=0,$$

and

$$\liminf_{|v|\to\infty}\min_{t\in\left[\frac14,\frac34\right]}\inf_{u\in[0,+\infty)}\inf_{\varphi\in[0,+\infty)}\frac{f(t,u,v,\varphi)}{|v|}=\infty.$$

If $\mu\in\left(0,\frac{1}{4D_1M_r}\right)$, then problem (3) has at least one positive solution.

**Proof.** Let us choose $0<c_1\le\frac{1}{4D_2}$. Then by (H4), there exist $0<r<a$ such that

$$f_2(w)\le c_1w,\ \ w\in[0,r]. \tag{15}$$

Let $u\in\partial B_r\cap P$, then by Corollary 7, $\|u\|_2=\|u''\|_0=r$ and $u(0)=u(1)=0$. Also since $\|u\|_0\le\|u''\|_0$ we have $u(t)\le\|u\|_0\le r$, $|u''(t)|\le\|u''\|_0=r$, $\forall t\in[0,1]$.

Let $M_r=\sup_{w\in[0,r]}\int_0^1 K(v,v)g(v,w,-rK(v,v))\mathrm{d}v$.

We now show that

$$0 \le \mu \int_0^1 K(s,v)\, g(v,u(v),u''(v))\,\mathrm{d}v\left(u(s)+|u''(s)|\right) \le r,\ \ s\in[0,1],\ \text{if}\ \mu \le \frac{1}{2M_r}.$$

To see this, let $\mu \le \frac{1}{2M_r}$, then we have

$$\begin{aligned}&\mu \int_0^1 K(s,v)\, g(v,u(v),u''(v))\,\mathrm{d}v\left(u(s)+|u''(s)|\right)\\ &\le \mu \sup_{w\in[0,r]} \int_0^1 K(v,v)\, g(v,w,-rK(v,v))\,\mathrm{d}v\left(\|u\|_0+\|u''\|_0\right) = \mu M_r 2r \le r.\end{aligned}$$

Thus using by (15) we obtain

$$\begin{aligned}&f_2\left(\mu \int_0^1 K(s,v)\, g(v,u(v),u''(v))\,\mathrm{d}v\left(u(s)+|u''(s)|\right)\right)\\ &\le c_1 \mu \int_0^1 K(s,v)\, g(v,u(v),u''(v))\,\mathrm{d}v\left(u(s)+|u''(s)|\right) \le c_1 r.\end{aligned}$$

Thus, by Lemma 4, and (H1-H2) we have

$$\begin{aligned}(Tu)(t) &\le \mu \int_0^1\int_0^1 K(\tau,\tau)K(\tau,s)u(s)\,\mathrm{d}s\mathrm{d}\tau \int_0^1 K(v,v)\, g(v,u(v),-rK(v,v))\,\mathrm{d}v\\ &+\int_0^1\int_0^1 K(\tau,\tau)K(\tau,s) f_1(s) f_2\left(\mu\int_0^1 K(s,v)\, g(v,u(v),u''(v))\,\mathrm{d}v\left(u(s)+|u''(s)|\right)\right)\mathrm{d}s\mathrm{d}\tau\\ &\le \mu \int_0^1\int_0^1 K(\tau,\tau)K(\tau,s)\,\mathrm{d}s\mathrm{d}\tau\, \|u\|_0 M_r\\ &+\int_0^1\int_0^1 K(\tau,\tau)K(\tau,s) f_1(s) c_1 \mu\int_0^1 K(s,v)\, g(v,u(v),u''(v))\,\mathrm{d}v\left(u(s)+|u''(s)|\right)\mathrm{d}s\mathrm{d}\tau\\ &\le \mu D_1 \|u\|_0 M_r + c_1 D_2 r \le \frac{1}{4}\|u\|_0 + \frac{1}{4}\|u\|_2\\ &\le \frac{1}{4}\|u\|_2 + \frac{1}{4}\|u\|_2 \le \frac{1}{2}\|u\|_2,\ \ \forall u\in \partial B_r \cap P,\ t\in[0,1].\end{aligned}$$

Consequently,

$$\|Tu\|_0 \le \frac{1}{2}\|u\|_2,\ \ \forall u \in \partial B_r \cap P. \tag{16}$$

Similarly we also have

$$\begin{aligned}(Tu)''(t) &= -\mu \int_0^1 K(t,s)u(s)\int_0^1 K(s,v)\, g(v,u(v),u''(v))\,\mathrm{d}v\mathrm{d}s\\ &\quad -\int_0^1 K(t,s) f(t,u(s),u''(s))\,\mathrm{d}s.\end{aligned}$$

Hence

$$\begin{aligned}\left|(Tu)''(t)\right| &\le \mu \int_0^1 K(s,s)\,\mathrm{d}s \|u\|_0 M_r\\ &+\int_0^1 K(s,s) f_1(s) f_2\left(\mu \int_0^1 K(s,v)\, g(v,u(v),u''(v))\,\mathrm{d}v\left(u(s)+|u''(s)|\right)\right)\mathrm{d}s\\ &\le \mu D_3 \|u\|_0 M_r + c_1 D_4 r \le \mu D_1\|u\|_0 M_r + c_1 D_2 r\\ &\le \frac{1}{4}\|u\|_0 + \frac{1}{4}\|u\|_2 \le \frac{1}{2}\|u\|_2,\ \ \forall u\in\partial B_r\cap P, t\in[0,1].\end{aligned}$$

Consequently,

$$\left\|(Tu)''\right\|_0 \le \frac{1}{2}\|u\|_2,\ \ \forall u\in \partial B_r\cap P. \tag{17}$$

Using (16) and (17) we have

$$\|Tu\|_2 \leq \|Tu\|_0 + \left\|(Tu)''\right\|_0 \leq \|u\|_2, \quad \forall u \in \partial B_r \cap P. \tag{18}$$

Let us choose $c_2 \geq \frac{1}{\sigma D_5}$. Then by condition (H4), there exists $R_1 > 0$ such that

$$f(t,u,v,\varphi) \geq c_2|v|, \quad \forall u \in R^+, \quad \forall \varphi \in R^+, \quad |v| \geq R_1, \quad t \in \left[\frac{1}{4}, \frac{3}{4}\right].$$

Let $R > \max\left\{\frac{R_1}{\sigma}, a\right\}$. Let $u \in \partial B_R \cap P$, *i.e.* $\|u''\|_0 = R$. Thus by using (10) we have

$$\min_{t\in\left[\frac{1}{4},\frac{3}{4}\right]} |u''(t)| \geq \sigma \|u''\|_0 = \sigma R > R_1, \quad \forall u \in \partial B_R \cap P.$$

Then, by Lemma 1, we have

$$\begin{aligned}
(Tu)\left(\frac{1}{2}\right) &\geq \mu \int_{\frac{1}{4}}^{\frac{3}{4}} \int_{\frac{1}{4}}^{\frac{3}{4}} \int_{\frac{1}{4}}^{\frac{3}{4}} K\left(\frac{1}{2},\tau\right) K(\tau,s) u(s) K(s,v) g\left(v,u(v),u''(v)\right) \mathrm{d}v\mathrm{d}s\mathrm{d}\tau \\
&+ \int_{\frac{1}{4}}^{\frac{3}{4}} \int_{\frac{1}{4}}^{\frac{3}{4}} K\left(\frac{1}{2},\tau\right) K(\tau,s) f\left(t,u(s),u''(s),\varphi(s)\right) \mathrm{d}s\mathrm{d}\tau \\
&\geq c_2 \int_{\frac{1}{4}}^{\frac{3}{4}} \int_{\frac{1}{4}}^{\frac{3}{4}} K\left(\frac{1}{2},\tau\right) K(\tau,s) |u''(s)| \mathrm{d}s\mathrm{d}\tau \\
&\geq c_2\sigma \int_{\frac{1}{4}}^{\frac{3}{4}} \int_{\frac{1}{4}}^{\frac{3}{4}} K\left(\frac{1}{2},\tau\right) K(\tau,s) \mathrm{d}s\mathrm{d}\tau \|u''\|_0 \geq \|u''\|_0
\end{aligned}$$

so

$$(Tu)\left(\frac{1}{2}\right) \geq \|u''\|_0 = \|u\|_2, \quad \forall u \in \partial B_R \cap P.$$

Consequently,

$$\|u\|_2 \leq \|Tu\|_0 \leq \|Tu\|_2, \quad \forall u \in \partial B_R \cap P.$$

Then due to Lemma 2, by (18) and the above inequality we see that the problem (3) has at least one positive solution.

**Theorem 2.** Let (H1), (H2) and (H3) hold. Assume that the following conditions hold

(H5)

$$\liminf_{|u|+|v|\to 0^+} \min_{t\in\left[\frac{1}{4},\frac{3}{4}\right]} \inf_{\varphi\in[0,+\infty)} \frac{f(t,u,v,\varphi)}{|u|+|v|} = \infty,$$

and

$$\liminf_{|v|\to\infty} \min_{t\in\left[\frac{1}{4},\frac{3}{4}\right]} \inf_{u\in[0,+\infty)} \inf_{\varphi\in[0,+\infty)} \frac{f(t,u,v,\varphi)}{|v|} = \infty.$$

(H6) there exists $0 < \rho < a$ such that

$$\sup_{w\in[0,a]} f_2(w) \leq \frac{\rho}{4D_4}. \tag{19}$$

If $\mu \in \left(0, \frac{1}{4D_3 M}\right)$, then problem (3) has at least two positive solutions.

We note for the argument below that $D_1 \le D_3$ and $D_2 \le D_4$.

**Proof.** By condition (H6) there exists $0 < \rho < a$ such that (19) is fulfilled. Let $u \in \partial B_\rho \cap P$, by Corollary 7, $\|u''\|_0 = \rho$, $u(0) = u(1) = 0$. Also since $\|u\|_0 \le \|u''\|_0$ we have $u(t) \le \|u\|_0 \le \rho$, $|u''(t)| \le \|u''\|_0 = \rho$, $\forall t \in [0,1]$.

Let $M_\rho = \sup_{w\in[0,r]} \int_0^1 K(v,v) g(v, w, -\rho K(v,v)) \mathrm{d}v$.

We now show that

$$0 \le \mu \int_0^1 K(s,v) g(v, u(v), u''(v)) \mathrm{d}v \left(u(s) + |u''(s)|\right) \le \rho,\ s \in [0,1],\ \text{if } \mu \le \frac{1}{2M_\rho}.$$

To see this, let $\mu \le \frac{1}{2M_\rho}$, then we have

$$\begin{aligned}
&\mu \int_0^1 K(s,v) g(v, u(v), u''(v)) \mathrm{d}v \left(u(s) + |u''(s)|\right) \\
&\le \mu \sup_{w\in[0,r]} \int_0^1 K(v,v) g(v, w, -\rho K(v,v)) \mathrm{d}v \left(\|u\|_0 + \|u''\|_0\right) = \mu M_\rho 2\rho \le \rho.
\end{aligned}$$

By condition (H6), $\forall u \in \partial B_\rho \cap P$ and $t \in [0,1]$, we have

$$\begin{aligned}
(Tu)(t) &\le \mu \int_0^1 \int_0^1 K(\tau,\tau) K(\tau,s) \mathrm{d}s \mathrm{d}\tau \|u\|_0 M \\
&+ \int_0^1 \int_0^1 K(\tau,\tau) K(\tau,s) f_1(s) f_2\left(\mu \int_0^1 K(s,v) g(v, u(v), u''(v)) \mathrm{d}v \left(u(s) + |u''(s)|\right)\right) \mathrm{d}s \mathrm{d}\tau \\
&\le \mu D_1 \|u\|_0 M + \frac{\rho}{4D_4} \int_0^1 \int_0^1 K(\tau,\tau) K(\tau,s) f_1(s) \mathrm{d}s \\
&\le \frac{1}{4}\|u\|_0 + \frac{1}{4}\rho = \frac{1}{4}\|u\|_0 + \frac{1}{4}\|u''\|_0 = \frac{1}{4}\|u\|_0 + \frac{1}{4}\|u\|_2 \\
&\le \frac{1}{4}\|u\|_2 + \frac{1}{4}\|u\|_2 = \frac{1}{2}\|u\|_2,\ \forall u \in \partial B_\rho \cap P,\ t \in [0,1].
\end{aligned}$$

Consequently, we get

$$\|Tu\|_0 \le \|u\|_2,\ \forall u \in \partial B_\rho \cap P. \tag{20}$$

Similarly we also have

$$\begin{aligned}
(Tu)''(t) &= -\mu \int_0^1 K(t,s) u(s) \int_0^1 K(s,v) g(v, u(v), u''(v)) \mathrm{d}v \mathrm{d}s \mathrm{d}\tau \\
&\quad - \int_0^1 K(t,s) f(s, u(s), u''(s), \varphi(s)) \mathrm{d}s \mathrm{d}\tau.
\end{aligned}$$

Hence

$$\begin{aligned}
\left|(Tu)''(t)\right| &\le \mu \int_0^1 K(s,s) \mathrm{d}s \|u\|_0 M \\
&+ \int_0^1 K(s,s) f_1(s) f_2\left(\mu \int_0^1 K(s,v) g(v, u(v), u''(v)) \mathrm{d}v \left(u(s) + |u''(s)|\right)\right) \mathrm{d}s \\
&\le \mu D_3 \|u\|_0 M + \frac{\rho}{4D_4} \int_0^1 K(s,s) f_1(s) \mathrm{d}s \\
&\le \frac{1}{4}\|u\|_0 + \frac{1}{4}\rho = \frac{1}{4}\|u\|_0 + \frac{1}{4}\|u\|_2 \\
&\le \frac{1}{4}\|u\|_2 + \frac{1}{4}\|u\|_2 \le \frac{1}{2}\|u\|_2,\ \forall u \in \partial B_\rho \cap P,\ t \in [0,1].
\end{aligned}$$

Consequently,

$$\left\|(Tu)''\right\|_0 \le \frac{1}{2}\|u\|_2, \ \forall u \in \partial B_\rho \cap P. \tag{21}$$

Using (20) and (21) we have

$$\|Tu\|_2 \le \|Tu\|_0 + \left\|(Tu)''\right\|_0 \le \|u\|_2, \ \forall u \in \partial B_\rho \cap P. \tag{22}$$

Let us choose $c_3 \ge \dfrac{1}{\sigma D_5}$. The by condition (H5), there exists $0 < r < \rho$ such that

$$f(t,u,v,\varphi) \ge c_3(u+|v|), \ \forall u \in [0,r], \ \forall \varphi \in [0,\infty), \ \forall |v| \in [0,r], \ t \in \left[\frac{1}{4},\frac{3}{4}\right].$$

Let $u \in \partial B_r \cap P$, by Corollary 7, $\|u''\|_0 = r$, $u(0) = u(1) = 0$. Also since $\|u\|_0 \le \|u''\|_0$ we have

$$0 \le u(t) \le \|u\|_0 \le r,$$

$$|u''(t)| \le \|u''\|_0 = \|u\|_2 = r, \ \forall u \in \partial B_r \cap P.$$

Thus by using (10) we have

$$\min_{t\in\left[\frac{1}{4},\frac{3}{4}\right]} |u''(t)| \ge \sigma \|u''\|_0 = \sigma r, \ \forall u \in \partial B_r \cap P.$$

The estimate for $(Tu)\left(\frac{1}{2}\right)$ is similar to that in the proof of Theorem 1 *i.e.* from Lemma 1 and (H5) we have

$$(Tu)\left(\frac{1}{2}\right) \ge c_3 \int_{\frac{1}{4}}^{\frac{3}{4}} \int_{\frac{1}{4}}^{\frac{3}{4}} K\left(\frac{1}{2},\tau\right) K(\tau,s)\left(u(s)+|u''(s)|\right) \mathrm{d}s\mathrm{d}\tau$$

$$\ge c_3\sigma \int_{\frac{1}{4}}^{\frac{3}{4}} \int_{\frac{1}{4}}^{\frac{3}{4}} K\left(\frac{1}{2},\tau\right) K(\tau,s) \mathrm{d}s\mathrm{d}\tau \|u''\|_0 \ge \|u''\|_0.$$

Thus

$$(Tu)\left(\frac{1}{2}\right) \ge \|u''\|_0 = \|u\|_2, \ \forall u \in \partial B_r \cap P.$$

Consequently,

$$\|u\|_2 \le \|Tu\|_0 \le \|Tu\|_2, \ \forall u \in \partial B_r \cap P.$$

Finally we show that for sufficiently large $R > a$, it holds

$$\|Tu\|_2 \ge \|u\|_2, \ \forall u \in \partial B_R \cap P.$$

To see this, we choose $c_2 \ge \dfrac{1}{\sigma D_5}$. Due to condition (H5), there exist $R_1 > 0$ such that

$$f(t,u,v,\varphi) \ge c_2|v|, \ \forall u \in R^+, \ \forall \varphi \in R^+, \ |v| \ge R_1, \ t \in \left[\frac{1}{4},\frac{3}{4}\right].$$

Let $R > \max\left\{\dfrac{R_1}{\sigma}, a\right\}$. Let $u \in \partial B_R \cap P$, by Corollary 7, $\|u''\|_0 = R$. Thus by using (10) we have

$$\min_{t\in\left[\frac{1}{4},\frac{3}{4}\right]}\left|u''(t)\right| \ge \sigma\left\|u''\right\|_0 = \sigma R > R_1,\ \ \forall u\in\partial B_R\cap P.$$

Then, by Lemma 1, (H1) and (H4), we have

$$\begin{aligned}(Tu)\left(\frac{1}{2}\right) &\ge c_2\int_{\frac{1}{4}}^{\frac{3}{4}}\int_{\frac{1}{4}}^{\frac{3}{4}}K\left(\frac{1}{2},\tau\right)K(\tau,s)\left|u''(s)\right|\mathrm{d}s\mathrm{d}\tau \\ &\ge c_2\sigma\int_{\frac{1}{4}}^{\frac{3}{4}}\int_{\frac{1}{4}}^{\frac{3}{4}}K\left(\frac{1}{2},\tau\right)K(\tau,s)\mathrm{d}s\mathrm{d}\tau\left\|u''\right\|_0 \ge \left\|u''\right\|_0\end{aligned}$$

so

$$(Tu)\left(\frac{1}{2}\right) \ge \left\|u''\right\|_0 = \left\|u\right\|_2,\ \ \forall u\in\partial B_R\cap P.$$

Consequently,

$$\left\|u\right\|_2 \le \left\|Tu\right\|_0 \le \left\|Tu\right\|_2,\ \ \forall u\in\partial B_R\cap P.$$

Then by Lemma 2, we know that $T$ has at least two fixed points in $\left(\overline{B}_R\setminus B_\rho\right)\cap P$ and $\left(\overline{B}_\rho\setminus B_r\right)\cap P$, *i.e.* problem (3) has at least two positive solutions.

## 4. Conclusion

This paper investigates the existence of positive solutions for a singular fourth-order differential system using a fixed point theorem of cone expansion and compression type. The nonlinear terms may be singular with respect to both the time and space variables. The problem comes from the deformation analysis of an elastic beam in the equilibrium state, whose two ends are simply supported. The results obtained herein generalize and improve some known results including singular and non-singular cases.

## Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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