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The Additive Generators of *n*-Dimensional Overlap Functions

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Abstract

In this paper, we introduce the concepts of additive generators and additive generator pair of *n*-dimensional overlap functions, in order to extend the dimensionality of overlap functions from 2 to *n*. We mainly discuss the conditions under which an *n*-dimensional overlap function can be expressed in terms of its generator pair.

Keywords

Overlap Functions, *n*-Dimensional Overlap Functions, Additive Generators, Additive Generator Pair

1. Introduction

Bustince *et al.* introduced overlap functions and their basic properties in [1] [2]. In recent years, as one new case of nonassociative fuzzy logical connective, overlap functions have developed rapidly in theoretical research and practical application. Overlap functions have been applied to some interesting problems, for example image processing [1] [3], classification [4] [5] and decision making [6] [7]. In [8], Dimuro and Bedregal discussed some preliminary ideas on additive generator pair of overlap function. On the basis of [8], Dimuro *et al.* [9] formally introduced the notion of additive generator pair of overlap functions and studied their basic properties. Qiao and Hu [10] mainly investigated the two basic distributive laws of fuzzy implication functions over additively generated overlap and grouping functions. In [11], Gómez *et al.* introduced the definition of *n*-dimensional overlap functions and the conditions under which *n*-dimensional overlap functions are migrative, homogeneous or Lipschitz continuous. The research purpose on the additive generators of overlap functions is to find the constructions of overlap functions by two single-variable functions and addition operation. From the point view of applications, the use of additive generators can provide convenience for the choice of a suitable overlap function and reduce the computational complexity. Current researches on the additive generators of overlap functions are focused on bivariate overlap functions and interval overlap functions. A very natural question is that how to extend the definition of additive generator pair of overlap function from the 2-dimensional to the n-dimensional case ($n \geq 2$). To answer this question, we extend the concept of additive generator pair of overlap functions to the notion of additive generator pair of overlap n-dimensional functions and discuss some basic properties in this paper.

The rest of this paper is organized as follows. In Section 2, we present some basic definitions on overlap functions and n-dimensional overlap functions, and additive generators of overlap functions. In Section 3, the concepts of additive generators and generator pair of n-dimensional overlap functions are introduced. In the final section, we end this paper with some remarks.

2. Preliminaries

In this section, we recall some concepts and properties of bivariate overlap functions, additive generators of overlap functions and n-dimensional overlap functions which shall be needed in the sequel.

Definition 2.1. (See Bustince *et al.* [1]) A bivariate function $O:[0,1]^2 \to [0,1]$ is said to be an overlap function if it satisfies the following conditions:

(O_1) O is commutative;

(O_2) $O(x,y)=0$ iff $xy=0$;

(O_3) $O(x,y)=1$ iff $xy=1$;

(O_4) O is increasing;

(O_5) O is continuous.

Example 2.1. (See Qiao and Hu [12]). (1) For any $p>0$, consider the bivariate function $O_p:[0,1]^2 \to [0,1]$ given by

$$O_p(x,y)=x^p y^p$$

for all $x,y \in [0,1]$. Then it is an overlap function and we call it p-product overlap function, here. It is obvious that 1-product overlap function is the product t-norm. Moreover, for any $p=1$, the p-product overlap function is neither associative nor has 1 as neutral element. Therefore, it is not a t-norm.

(2) The function $O_{DB}:[0,1]^2 \to [0,1]$ given by

$$O_{DB}=\begin{cases} \dfrac{2xy}{x+y}, & \text{if } x+y \neq 0, \\ 0, & \text{if } x+y=0. \end{cases}$$

for all $x,y \in [0,1]$. Then it is an overlap function.

(3) The function $O_{DB}:[0,1]^2 \to [0,1]$ given by

$$O_{Mid}=xy\frac{x+y}{2},$$

for all $x, y \in [0,1]$. Then it is an overlap function.

In the following, we denote the range or image of a function $f : A \to B$ by $Ran(f)$.

Lemma 2.1. (See Dimuro *et al.* [9]). Let $\theta : [0,1] \to [0,\infty]$ be a decreasing function such that

1) $\theta(x) + \theta(y) \in Ran(\theta)$, for $x, y \in [0,1]$ and

2) If $\theta(x) = \theta(0)$ then $x = 0$.

Then $\theta(x) + \theta(y) \geq \theta(0)$ if and only if $x = 0$ or $y = 0$.

Lemma 2.2. (See Dimuro *et al.* [9]). Consider functions $\theta : [0,1] \to [0,\infty]$ and $\vartheta : [0,\infty] \to [0,1]$ such that, for each $x_0 \in [0,1]$, if it holds that

$$\vartheta(\theta(x)) = x_0 \text{ if and only if } x = x_0,$$

then $\theta(x) = \theta(x_0)$ if and only if $x = x_0$.

Lemma 2.3. (See Dimuro *et al.* [9]). Let $\theta : [0,1] \to [0,\infty]$ and $\vartheta : [0,\infty] \to [0,1]$ be continuous and decreasing functions such that

1) $\theta(x) + \theta(y) \in Ran(\theta)$, for $x, y \in [0,1]$;

2) $\vartheta(\theta(x)) = 0$ if and only $x = 0$;

3) $\vartheta(\theta(x)) = 1$ if and only $x = 1$;

4) $\theta(x) + \theta(y) = \theta(1)$ if and only $x = 1$ and $y = 1$.

Then, the function $O_{\theta,\vartheta} : [0,1]^2 \to [0,1]$, defined by

$$O_{\theta,\vartheta}(x, y) = \vartheta(\theta(x) + \theta(y))$$

is an overlap function.

Lemma 2.4. (See Dimuro *et al.* [9]). Let $\theta : [0,1] \to [0,\infty]$ and $\vartheta : [0,\infty] \to [0,1]$ be continuous and decreasing functions such that

1) $\theta(x) = \infty$ if and only if $x = 0$;

2) $\theta(x) = 0$ if and only if $x = 1$;

3) $\vartheta(x) = 1$ if and only if $x = 0$;

4) $\vartheta(x) = 0$ if and only if $x = \infty$.

Then, the function $O_{\theta,\vartheta} : [0,1]^2 \to [0,1]$, defined by

$$O_{\theta,\vartheta}(x, y) = \vartheta(\theta(x) + \theta(y))$$

is an overlap function.

In Lemma 2.3, Dimuro *et al.* only presented the notion of additive generator pair of overlap functions, but did not give the specific definition of additive generator pair of overlap functions. Qiao and Hu [10] introduced the specific definition of additive generator pair of overlap functions by Lemma 2.3.

Definition 2.2. (See Qiao and Hu [10]). Let $\theta : [0,1] \to [0,\infty]$ and $\vartheta : [0,\infty] \to [0,1]$ be two continuous and decreasing functions. If the bivariate function $O_{\theta,\vartheta} : [0,1]^2 \to [0,1]$ defined by

$$O_{\theta,\vartheta}(x, y) = \vartheta(\theta(x) + \theta(y))$$

is an overlap function, then (ϑ, θ) is called an additive generator pair of the overlap function $O_{\theta,\vartheta}$ and $O_{\theta,\vartheta}$ is said to be additively generated by the pair

(ϑ, θ).

Definition 2.3. (See Gómez *et al.* [11]) An n-dimensional aggregation function $\boldsymbol{O}:[0,1]^n \to [0,1]$ is an n-dimensional overlap function if and only if:

($\boldsymbol{O}_1$) $\boldsymbol{O}$ is symmetric.

($\boldsymbol{O}_2$) $\boldsymbol{O}(x_1, x_2, \cdots, x_n) = 0$ if and only if $\prod_{i=1}^{n} x_i = 0$.

($\boldsymbol{O}_3$) $\boldsymbol{O}(x_1, x_2, \cdots, x_n) = 1$ if and only if $x_i = 1$ for all $i \in \{1, \cdots, n\}$.

($\boldsymbol{O}_4$) $\boldsymbol{O}$ is increasing.

($\boldsymbol{O}_5$) $\boldsymbol{O}$ is continuous.

Example 2.2. The following aggregation functions are the most common n-dimensional overlap functions:

1) The product $\boldsymbol{O}(x_1, x_2, \cdots, x_n) = \prod_{i=1}^{n} x_i$ [11].

2) $\boldsymbol{O}_p(x_1, x_2, \cdots, x_n) = \prod_{i=1}^{n} (x_i)^p$, where $p > 0$.

3. Additive Generators of *n*-Dimensional Overlap Functions

In this section, we introduce the notion of additive generator pair for n-dimensional overlap functions and study their basic properties.

Definition 3.1. Let $\theta:[0,1] \to [0,\infty]$ and $\vartheta:[0,\infty] \to [0,1]$ be two continuous and decreasing functions. If the n-dimensional function $\boldsymbol{O}_{\theta,\vartheta}:[0,1]^n \to [0,1]$ defined by

$$\boldsymbol{O}_{\theta,\vartheta}(x_1, x_2, \cdots, x_n) = \vartheta\left(\theta(x_1) + \theta(x_2) + \cdots + \theta(x_n)\right)$$

is an n-dimensional overlap function, then (ϑ, θ) is called an additive generator pair of the n-dimensional overlap function $\boldsymbol{O}_{\theta,\vartheta}$ and $\boldsymbol{O}_{\theta,\vartheta}$ is said to be additively generated by the pair (ϑ, θ).

Proposition 3.1. Let $\theta:[0,1] \to [0,\infty]$ be a decreasing functions such that

1) $\theta(x_1) + \theta(x_2) + \cdots + \theta(x_n) \in Ran(\theta)$, for $x_i \in [0,1]$ ($i = 1, 2, \cdots, n$);

2) If $\theta(x) = \theta(0)$ then $x = 0$.

Then $\theta(x_1) + \theta(x_2) + \cdots + \theta(x_n) \geq \theta(0)$ if and only if $x_1 x_2 \cdots x_n = 0$.

Proof. ($\Rightarrow$) Since θ is decreasing and $\theta(x_1) + \theta(x_2) + \cdots + \theta(x_n) \in Ran(\theta)$, for $x_i \in [0,1]$ ($i = 1, 2, \cdots, n$), then we have that $\theta(x_1) + \theta(x_2) + \cdots + \theta(x_n) \leq \theta(0)$. Hence, if $\theta(x_1) + \theta(x_2) + \cdots + \theta(x_n) \geq \theta(0)$, then one has that $\theta(x_1) + \theta(x_2) + \cdots + \theta(x_n) = \theta(0)$. Suppose that $\theta(0) = 0$, then, since θ is decreasing, it follows that $\theta(x_i) = 0$ for any $x_i \in [0,1]$ ($i = 1, 2, \cdots, n$), which is contradiction with condition 2, and, we have that $\theta(0) > 0$. Now, we suppose that $\theta(0) = 0$ and $\theta(0) = \infty$. Then, since $\theta(0) = 0$, it holds that $\theta(0) + \theta(0) + \cdots + \theta(0) > \theta(0)$, which is contradiction with condition 1. Therefore, one has that $\theta(0) = \infty$, and, hence, since $\theta(x_1) + \theta(x_2) + \cdots + \theta(x_n) = \theta(0)$, it follows that $\theta(x_i) = \infty$ for some $i \in \{1, \cdots, n\}$. Therefore, by condition 2, we have that $x_i = 0$ for some $i \in \{1, \cdots, n\}$, *i.e.*, $x_1 x_2 \cdots x_n = 0$.

($\Leftarrow$) It is straightforward.

Proposition 3.2. Let $\theta:[0,1] \to [0,\infty]$ and $\vartheta:[0,\infty] \to [0,1]$ be continuous and decreasing functions such that

1) $\theta(x_1)+\theta(x_2)+\cdots+\theta(x_n)\in Ran(\theta)$, for $x_i\in[0,1]$ ($i=1,2,\cdots,n$);
2) $\vartheta(\theta(x))=0$ if and only $x=0$;
3) $\vartheta(\theta(x))=1$ if and only $x=1$;
4) $\theta(x_1)+\theta(x_2)+\cdots+\theta(x_n)=\theta(1)$ if and only $x_1=x_2=\cdots=x_n=1$.

Then, the n-dimensional function $\boldsymbol{O}_{\theta,\vartheta}:[0,1]^n\to[0,1]$, defined by

$$\boldsymbol{O}_{\theta,\vartheta}(x_1,x_2,\cdots,x_n)=\vartheta(\theta(x_1)+\theta(x_2)+\cdots+\theta(x_n))$$

is an n-dimensional overlap function.

Proof. We prove that $\boldsymbol{O}_{\theta,\vartheta}$ satisfies the conditions of Definition 2.3. Since $\boldsymbol{O}_{\theta,\vartheta}$ is obviously symmetric and continuous, we only need to prove that $\boldsymbol{O}_{\theta,\vartheta}$ satisfies the conditions ($\boldsymbol{O}_2$), ($\boldsymbol{O}_3$) and ($\boldsymbol{O}_4$).

By condition 1, it holds that $\theta(x_1)+\theta(x_2)+\cdots+\theta(x_n)=\theta(1)$ for some $y\in[0,1]$.

$$\begin{aligned}
&\boldsymbol{O}_{\theta,\vartheta}(x_1,x_2,\cdots,x_n)=0\\
\Leftrightarrow\ &\vartheta(\theta(x_1)+\theta(x_2)+\cdots+\theta(x_n))=0\\
\Leftrightarrow\ &\vartheta(\theta(y))=0 \text{ for some } y\in[0,1] \text{ by condition 1}\\
\Leftrightarrow\ &y=0 \text{ by condition 2}\\
\Leftrightarrow\ &\theta(y)=\theta(0) \text{ by Lemma 2.2}\\
\Leftrightarrow\ &\theta(x_1)+\theta(x_2)+\cdots+\theta(x_n)=\theta(0)\\
\Leftrightarrow\ &x_1x_2\cdots x_n=0 \text{ by Proposition 3.1.}
\end{aligned}$$

Therefore, $\boldsymbol{O}_{\theta,\vartheta}$ satisfies the condition ($\boldsymbol{O}_2$).

Similarly, we have that

$$\begin{aligned}
&\boldsymbol{O}_{\theta,\vartheta}(x_1,x_2,\cdots,x_n)=1\\
\Leftrightarrow\ &\vartheta(\theta(x_1)+\theta(x_2)+\cdots+\theta(x_n))=1\\
\Leftrightarrow\ &\vartheta(\theta(y))=1 \text{ for some } y\in[0,1] \text{ by condition 1.}\\
\Leftrightarrow\ &y=1 \text{ by condition 3.}\\
\Leftrightarrow\ &\theta(y)=\theta(1) \text{ by Lemma 2.2.}\\
\Leftrightarrow\ &\theta(x_1)+\theta(x_2)+\cdots+\theta(x_n)=\theta(1)\\
\Leftrightarrow\ &x_1=x_2=\cdots=x_n=1 \text{ by condition 4.}
\end{aligned}$$

Therefore, $\boldsymbol{O}_{\theta,\vartheta}$ satisfies the condition ($\boldsymbol{O}_3$).

Finally, we prove that $\boldsymbol{O}_{\theta,\vartheta}$ satisfy the condition ($\boldsymbol{O}_4$). Considering $y\in[0,1]$ with $x_i\le y$ ($i\in\{1,2,\cdots,n\}$), then $\theta(x_i)\ge\theta(y)$. It follows that

$$\begin{aligned}
&\boldsymbol{O}_{\theta,\vartheta}(x_1,\cdots,x_{i-1},x_i,x_{i+1},\cdots,x_n)\\
&=\vartheta(\theta(x_1)+\cdots+\theta(x_{i-1})+\theta(x_i)+\theta(x_{i+1})+\cdots+\theta(x_n))\\
&\le\vartheta(\theta(x_1)+\cdots+\theta(x_{i-1})+\theta(y)+\theta(x_{i+1})+\cdots+\theta(x_n))\\
&\le\boldsymbol{O}_{\theta,\vartheta}(x_1,\cdots,x_{i-1},y,x_{i+1},\cdots,x_n)
\end{aligned}$$

Therefore, $\boldsymbol{O}_{\theta,\vartheta}$ satisfies the condition ($\boldsymbol{O}_4$).

Corollary 3.3. Let $\theta:[0,1]\to[0,\infty]$ and $\vartheta:[0,\infty]\to[0,1]$ be continuous and decreasing functions such that

1) $\theta(x)=\infty$ if and only if $x=0$;
2) $\theta(x)=0$ if and only if $x=1$;
3) $\vartheta(x)=1$ if and only if $x=0$;
4) $\vartheta(x)=0$ if and only if $x=\infty$.

Then, the n-dimensional function $\boldsymbol{O}_{\theta,\vartheta}:[0,1]^n\to[0,1]$, defined by

$$\boldsymbol{O}_{\theta,\vartheta}(x_1,x_2,\cdots,x_n)=\vartheta(\theta(x_1)+\theta(x_2)+\cdots+\theta(x_n))$$

is an n-dimensional overlap function.

Proof. It follows from Proposition 3.2.

Proposition 3.4. Let $\theta:[0,1]\to[0,\infty]$ and $\vartheta:[0,\infty]\to[0,1]$ be continuous and decreasing functions such that

1) $\vartheta(x)=1$ if and only if $x=0$;
2) $\vartheta(x)=0$ if and only if $x=\infty$;
3) $0\in Ran(\theta)$;
4) $\boldsymbol{O}_{\theta,\vartheta}(x_1,x_2,\cdots,x_n)=\vartheta(\theta(x_1)+\theta(x_2)+\cdots+\theta(x_n))$ is an n-dimensional overlap function.

Then, the following conditions also hold:

5) $\theta(x)=\infty$ if and only if $x=0$;
6) $\theta(x)=0$ if and only if $x=1$.

Proof. (5) ($\Rightarrow$) If $\boldsymbol{O}_{\theta,\vartheta}$ is an n-dimensional overlap function, then it follows that:

$$\begin{aligned}&\theta(x)=\infty\\ \Rightarrow\ &\theta(x)+\theta(x_1)+\cdots+\theta(x_{n-1})=\infty \ \text{ for some } \textstyle\prod_{i=1}^{n-1}x_i\neq 0\\ \Rightarrow\ &\vartheta(\theta(x)+\theta(x_1)+\cdots+\theta(x_{n-1}))=0 \ \text{ by condition 2}\\ \Rightarrow\ &\boldsymbol{O}_{\theta,\vartheta}(x,x_1,\cdots,x_{n-1})=0\\ \Rightarrow\ &x=0.\end{aligned}$$

($\Leftarrow$) Consider $x_i\in[0,1]$ ($i=1,2,\cdots,n-1$) such that $\theta(x_i)=0$. Then we have that

$$\begin{aligned}&x=0\\ \Rightarrow\ &\boldsymbol{O}_{\theta,\vartheta}(x,x_1,\cdots,x_{n-1})=0\\ \Rightarrow\ &\vartheta(\theta(x)+\theta(x_1)+\cdots+\theta(x_{n-1}))=0\\ \Rightarrow\ &\theta(x)+\theta(x_1)+\cdots+\theta(x_{n-1})=\infty\\ \Rightarrow\ &\theta(x)=\infty.\end{aligned}$$

(6) ($\Rightarrow$) If $\boldsymbol{O}_{\theta,\vartheta}$ is an n-dimensional overlap function, then it follows that:

$$\begin{aligned}&x=1\\ \Rightarrow\ &\boldsymbol{O}_{\theta,\vartheta}\Big(x,\underbrace{1,\cdots,1}_{n-1}\Big)=1\end{aligned}$$

$$\Rightarrow \quad \vartheta\left(\theta(x)+\underbrace{\theta(1)+\cdots+\theta(1)}_{n-1}\right)=1$$

$$\Rightarrow \quad \theta(x)+\underbrace{\theta(1)+\cdots+\theta(1)}_{n-1}=0 \text{ by condition 1}$$

$$\Rightarrow \quad \theta(x)=\theta(1)=0.$$

($\Leftarrow$) By condition 3, one can consider $x_i \in [0,1]$ ($i=1,2,\cdots,n-1$) such that $\theta(x_i)=0$. Then we have that

$$\theta(x)=0$$

$$\Rightarrow \quad \theta(x)+\theta(x_1)+\cdots+\theta(x_{n-1})=0$$

$$\Rightarrow \quad \vartheta(\theta(x)+\theta(x_1)+\cdots+\theta(x_{n-1}))=1$$

$$\Rightarrow \quad \mathcal{O}_{\theta,\vartheta}(x,x_1,\cdots,x_{n-1})=1$$

$$\Rightarrow \quad x=x_1=\cdots=x_{n-1}=1.$$

Proposition 3.5. Let $\theta:[0,1]\to[0,\infty]$ and $\vartheta:[0,\infty]\to[0,1]$ be two continuous and decreasing functions such that

$$\mathcal{O}_{\theta,\vartheta}(x_1,x_2,\cdots,x_n)=\vartheta(\theta(x_1)+\theta(x_2)+\cdots+\theta(x_n))$$

is an n-dimensional overlap function. Then, the following statements hold:

1) $\theta(x)=\infty$ if and only if $x=0$;

2) $\vartheta(x)=0$ if and only if $x=\infty$.

Proof. (1) ($\Rightarrow$) If $x=0$, now we verify that $\theta(x)=\infty$. Otherwise, if $\theta(0)<\infty$, then, for each $x'\in[n\theta(0),\infty]$, one has that

$$\vartheta(x')\le\vartheta(n\theta(0))=\vartheta\left(\underbrace{\theta(0)+\theta(0)+\cdots+\theta(0)}_{n}\right)=\mathcal{O}_{\theta,\vartheta}\left(\underbrace{0,0,\cdots,0}_{n}\right)=0.$$

The function $\theta^{\#}:[0,1]\to[0,\infty]$, defined by $\theta^{\#}(x)=n\theta(x)$ for all $x\in[0,1]$. Since θ is continuous and decreasing, $\theta^{\#}$ is also continuous and decreasing. In the following, we prove that $\vartheta(y)=0$ for all $y\in[0,n\theta(0))$.

Case 1: If $y\in[n\theta(1),n\theta(0))$, then we can have that there exists $z\in(0,1]$ such that $y=\theta^{\#}(z)$. Thus, if we suppose that $\vartheta(y)=0$, then, it follows that

$$\mathcal{O}_{\theta,\vartheta}\left(\underbrace{z,z,\cdots,z}_{n}\right)=\vartheta\left(\underbrace{\theta(0)+\theta(0)+\cdots+\theta(0)}_{n}\right)=\theta^{\#}(z)=\vartheta(y)=0.$$

which is a contradiction with the item ($\mathcal{O}_4$) of Definition 2.3. Therefore, for all $y\in[n\theta(1),n\theta(0))$, one has that $\vartheta(y)=0$.

Case 2: If $y\in[0,n\theta(1))$ and $\vartheta(y)=0$, then, it is obvious that $\vartheta(y)=0$ for all $y\in[n\theta(1),n\theta(0))$. It is a contradiction with $\vartheta(y)=0$.

Hence, it follows that $\vartheta(x')=0$ if and only if $x'\in[n\theta(0),\infty]$.

On the other hand, since $\mathcal{O}_{\theta,\vartheta}$ is an n-dimensional overlap function, we have that for all $y\in[0,1]$, by item ($\mathcal{O}_4$) of Definition 2.3,

$$\vartheta\left(\theta(0)+\underbrace{\theta(y)+\cdots+\theta(y)}_{n-1}\right)=\boldsymbol{O}_{\theta,\vartheta}\left(0,\underbrace{y,\cdots,y}_{n-1}\right)=0.$$

Thus, it holds that $\theta(0)+\underbrace{\theta(y)+\cdots+\theta(y)}_{n-1}\geq n\theta(0)$, *i.e.*, $\theta(y)\geq\theta(0)$. In particular, it follows that $\theta(0)\geq\theta(1)\geq\theta(0)$.

Therefore, one obtains that $\theta(1)=\theta(0)$. Moreover, it holds that

$$\boldsymbol{O}_{\theta,\vartheta}\left(\underbrace{1,\cdots,1}_{n}\right)=\vartheta\left(\underbrace{\theta(1)+\cdots+\theta(1)}_{n}\right)=\vartheta\left(\underbrace{\theta(0)+\cdots+\theta(0)}_{n}\right)=\boldsymbol{O}_{\theta,\vartheta}\left(\underbrace{0,\cdots,0}_{n}\right)=0.$$

which is a contradiction with the item ($\boldsymbol{O}_3$) of Definition 2.3.

Hence, it follows that $\theta(0)=\infty$.

($\Leftarrow$) Since we have proved that $\theta(0)=\infty$ and by item ($\boldsymbol{O}_4$) of Definition 2.3, it holds that

$$\vartheta(\infty)=\vartheta\left(\underbrace{\theta(0)+\cdots+\theta(0)}_{n}\right)=\boldsymbol{O}_{\theta,\vartheta}\left(\underbrace{0,\cdots,0}_{n}\right)=0.$$

Thus, if $\theta(x)=\infty$, then, we verify that $x=0$. Otherwise, if there exists some $x^*\in(0,1]$ such that $\theta(x^*)=\infty$, then, we have that

$$\boldsymbol{O}_{\theta,\vartheta}\left(\underbrace{x^*,\cdots,x^*}_{n}\right)=\vartheta\left(\underbrace{\theta(x^*)+\cdots+\theta(x^*)}_{n}\right)=\theta(\infty)=0. \tag{1}$$

which is a contradiction with the item ($\boldsymbol{O}_4$) of Definition 2.3.

Hence, one has that $\theta(x)=\infty$ if and only if $x=0$.

(2) ($\Leftarrow$) If $x=\infty$, then one has that $\vartheta(x)=0$ by Eq. (1).

($\Rightarrow$) Notice that we have proved that $\theta(x)=\infty$ if and only if $x=0$. If $\vartheta(x)=0$, then, we verify that $x=\infty$. Otherwise, if there exists some $x^*\in[n\theta(1),\infty)$ such that $\vartheta(x^*)=0$, then, from the proof of item 1 above, one has that there exists $z\in(0,1]$ such that $x^*=\theta^{\#}(z)$. Furthermore, one has that

$$\begin{aligned}\boldsymbol{O}_{\theta,\vartheta}\left(\underbrace{z,z,\cdots,z}_{n}\right)&=\vartheta\left(\underbrace{\theta(z)+\theta(z)+\cdots+\theta(z)}_{n}\right)\\&=\vartheta(n\theta(z))=\vartheta(\theta^{\#}(z))=\vartheta(x^*)=0\end{aligned}$$

which is a contradiction with the item ($\boldsymbol{O}_4$) of Definition 2.3. Thus, it holds that for all $y\in[n\theta(1),\infty)$, we have that $\vartheta(y)=0$.

On the other hand, if there exists some $x^*\in[0,n\theta(1))$ such that $\vartheta(x^*)=0$, then, for all $y\in[n\theta(1),\infty)$, it follows that $\vartheta(y)\leq\vartheta(x^*)=0$, which is a contradiction with $\vartheta(y)=0$.

Hence, one has that $\vartheta(x)=0$ if and only if $x=\infty$.

Proposition 3.6. Let $\theta:[0,1]\to[0,\infty]$ and $\vartheta:[0,\infty]\to[0,1]$ be continuous and decreasing functions such that

$$\boldsymbol{O}_{\theta,\vartheta}(x_1,x_2,\cdots,x_n)=\vartheta(\theta(x_1)+\theta(x_2)+\cdots+\theta(x_n))$$

is an n-dimensional overlap function. Then $\vartheta(0)=1$.

Proof. Since ϑ is a decreasing function and $n\theta(1)\ge 0$, it follows that

$$\vartheta(0)\ge\vartheta(n\theta(1))=\vartheta\left(\underbrace{\theta(1)+\cdots+\theta(1)}_{n}\right)=\boldsymbol{O}_{\theta,\vartheta}\left(\underbrace{1,\cdots,1}_{n}\right)=1.$$

Therefore, one has that $\vartheta(0)=1$.

Proposition 3.7. Let $\theta:[0,1]\to[0,\infty]$ and $\vartheta:[0,\infty]\to[0,1]$ be continuous and decreasing functions such that

$$\boldsymbol{O}_{\theta,\vartheta}(x_1,x_2,\cdots,x_n)=\vartheta(\theta(x_1)+\theta(x_2)+\cdots+\theta(x_n))$$

is an n-dimensional overlap function. Then, the following statements are equivalent

1) $\theta(x)=0$ if and only if $x=1$.

2) $\vartheta(x)=1$ if and only if $x=0$.

Proof. (1) $\Rightarrow$ (2) If $x=0$, then one has that $\vartheta(x)=1$ by Proposition 3.6.

Conversely, we know that $\theta(x)=\infty$ if and only if $x=0$ and $\vartheta(x)=0$ if and only if $x=\infty$ from items (1) and (2) of Proposition 3.5. If $\vartheta(x)=1$, then we verify that $x=0$. Otherwise, if there exists some $x^*\in(0,\infty)$ such that $\vartheta(x^*)=1$, then, by item (1) and the proof of item (1) in Proposition 3.5, it holds that there exists $z\in(0,\infty)$ such that $x^*=\theta^{\#}(z)$.

Furthermore, one has that

$$\begin{aligned}\boldsymbol{O}_{\theta,\vartheta}\left(\underbrace{z,z,\cdots,z}_{n}\right)&=\vartheta\left(\underbrace{\theta(z)+\theta(z)+\cdots+\theta(z)}_{n}\right)\\&=\vartheta(n\theta(z))=\vartheta(\theta^{\#}(z))=\vartheta(x^*)=1\end{aligned}$$

which is a contradiction with the item ($\boldsymbol{O}_3$) of Definition 2.3.

Hence, one obtains that $\vartheta(x)=1$ if and only if $x=0$.

(2) $\Rightarrow$ (1) If $x=1$, then it follows that

$$1=\boldsymbol{O}_{\theta,\vartheta}\left(\underbrace{1,1,\cdots,1}_{n}\right)=\vartheta\left(\underbrace{\theta(1)+\theta(1)+\cdots+\theta(1)}_{n}\right)=\vartheta(n\theta(1)).$$

Thus, one has that $n\theta(1)=0$ from item (2). Moreover, we have that $\theta(1)=0$.

Conversely, if $\theta(x)=0$, then, by item (2), it follows that

$$\boldsymbol{O}_{\theta,\vartheta}\left(\underbrace{x,x,\cdots,x}_{n}\right)=\vartheta\left(\underbrace{\theta(x)+\theta(x)+\cdots+\theta(x)}_{n}\right)=\vartheta(n\theta(x))=\vartheta(0)=1$$

Therefore, one has that $x=1$ by item ($\boldsymbol{O}_3$) of Definition 2.3.

Hence, it follows that $\theta(x)=0$ if and only if $x=1$.

Proposition 3.8. Let $\theta:[0,1]\to[0,\infty]$ be a continuous and decreasing function, and $\vartheta:[0,\infty]\to[0,1]$ be a continuous and strictly decreasing function. Then, the following statements are equivalent.

1) (ϑ,θ) is an additive generator pair of the n-dimensional overlap function

$\boldsymbol{O}_{\theta,\vartheta}$.

2) θ and ϑ satisfy the following conditions:

a) $\theta(x)=\infty$ if and only if $x=0$;

b) $\theta(x)=0$ if and only if $x=0$;

c) $\vartheta(x)=1$ if and only if $x=0$;

d) $\vartheta(x)=0$ if and only if $x=\infty$.

Proof. (1) $\Rightarrow$ (2) On the one hand, items (a) and (d) hold immediately from Proposition 3.6. On the other hand, since ϑ is strictly decreasing, by Proposition 3.6, it follows that $\vartheta(x)=1$ if and only if $x=0$. Furthermore, item (b) follows immediately from Proposition 3.7.

(2) $\Rightarrow$ (1) It follows immediately from Lemma 2.4.

4. Conclusion

In this paper, we mainly discuss the conditions under which two unary functions ϑ and θ can generate an *n*-dimensional overlap function $\boldsymbol{O}_{\theta,\vartheta}$. As application of the additively generated overlap functions, in [10], Qiao and Hu studied the distributive laws of fuzzy implication functions over additively generated overlap functions, *i.e.*, $I(x,O_1(y,z))=O_2(I(x,y),I(x,z))$, where O_1,O_2 are additively generated overlap functions, *I* is a fuzzy implication function. In future works, we will research the following distributive law

$$I\left(x,\boldsymbol{O}_1\left(x_1,x_2,\cdots,x_n\right)\right)=\boldsymbol{O}_2\left(I\left(x,x_1\right),I\left(x,x_2\right),\cdots,I\left(x,x_n\right)\right)$$

where $\boldsymbol{O}_1,\boldsymbol{O}_2$ are *n*-dimensional overlap functions, and *I* is a fuzzy implication function.

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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