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The Differential Transform

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Abstract

One of the methods of mathematical analysis in many cases makes it possible to reduce the study of differential operators, pseudo-differential operators and certain types of integral operators and the solution of equations containing them, to an examination of simpler algebraic problems. The development and systematic use of operational calculus began with the work of O. Heaviside (1892), who proposed formal rules for dealing with the differentiation operator d/dt and solved a number of applied problems. However, he did not give operational calculus a mathematical basis; this was done with the aid of the Laplace transform; J. Mikusiński (1953) put operational calculus into algebraic form, using the concept of a function ring [1]. Thereupon I'm suggesting here the use of the integration operator dt to make an extension for the systematic use of operational calculus. Throughout designing and analyzing a control system, we need to treat the functionality of the system with respect to time. The reaction of the system instructs us how to stable it by amplifiers and feedbacks [2], thereafter the Differential Transform is a good tool for doing this, and it's a technique to frustrate difficulties we may bump into, also it has the methods to find the immediate reaction of the system with respect to infinitesimal (tiny) time which spares us from the hard work in finding the time dependent function, this could be done by producing finite series.

Keywords

Operational Calculus, Time Domain, Differential Domain, Serieses, Difference to Differential Equation

1. Introduction

The Differential Transform shifts differential system from time plane to Cartesian plane that depends on the operator dt, this new presentation makes it easier to be solved. The Differential Transform is very useful in solution of problems for linear ordinary differential equations [3], analyzing electric systems [4] and solving physical problems [5].

Here we define the Differential Transform as a theme and present several examples and calculate the Transform for the basic functions. It may be regarded as a nice exercise from the mathematical point of view, also it might have some applications to engineers.

In this article we'll attempt to interpret the Differential Transform by presenting its definition and properties. The examples accompanying, demonstrate the convenient usage of this transform and improve understanding its concept.

The uniqueness of this Transform refers to the direct inversion which gives the series of the functions. The inversion is easy and in case of a complicated function produces a finite series of the function which will be regarded as its abbreviation.

The main advantages of this Transform are in Digital Signal Processing (DSP) [6] where a difference equation of a filter could be treated the same as a differential equation.

2. The Main Theorem

Here is the essence of the Differential Transform presented by this definition:

Let the Differential Transform of $f_{(t)}$ be denoted as $df_{(t)}$ and define it as

$$df_{(t)} = \frac{1}{dt}\int_0^\infty e^{\frac{-\tau}{dt}} f_{(\tau)} d\tau \tag{0}$$

where the operator dt represents the integrator $\int_0^t dt$.

3. The Properties

In our all next discussion we'll regard $G_{(dt)} = df_{(t)}$ and $F_{(s)}$ the Laplace Transform [7] of $f_{(t)}$.

3.1. Integrating

For $f_{(t)} = \int_0^t f'_{(t)} dt + f_{(0)}$ the Differential Transform is

$$df_{(t)} = df'_{(t)} dt + f_{(0)} \tag{1}$$

3.2. Differentiating

The Differential Transform of $f'_{(t)} = \frac{d}{dt} f_{(t)}$ is

$$df'_{(t)} = \frac{df_{(t)} - f_{(0)}}{dt} = \frac{df\big|_0^{dt}}{dt} \tag{2}$$

3.3. Linearity

Simply by switching $G_{(dt)} = df_{(t)}$ from t to the ct, c = constant

$$df_{(ct)} = G_{(d_{ct})} = G_{(cdt)} \tag{3}$$

3.4. Splitting

For complex functions $G_{(dt)} = df_{(t)}$ the $d\left(\text{Real}(f)\right) = \text{Real}(G)$ and the

$$d\left(\text{Imaginary}(f)\right) = \text{Imaginary}(G) \tag{4}$$

3.5. Limiting

$$f_{(0)} = G_{(0)} \tag{5}$$

3.6. Shifting

Having $G_{(dt)} = df_{(t)}$ with $t \geq \tau$ we get $df_{(t-\tau)} = G_{(dt)} e^{-\frac{\tau}{dt}}$. (6)

3.7. Convolution

If $dh_{(t)} = df_{(t)} dt dy_{(t)}$ then

$$h_{(t)} = f(t) * y(t) = \int_0^t f(t-\tau) y(\tau) d\tau \tag{7}$$

3.8. Low Shaking

$$d\left(t^n \frac{d^n}{dt} f_{(t)}\right) = d_t^n \frac{d^n}{ddt} df \tag{8}$$

$\frac{d^n}{ddt} df$ is the n-th derivation of df dependent to ddt.

3.9. Inverse Low Shaking

$$d\left(\int_0^t \frac{f_{(t)}\big|_0^t}{t} dt\right) = \int_0^{dt} \frac{df\big|_0^{dt}}{dt} ddt = \int_0^{dt} df' ddt \tag{9}$$

$\int_0^{dt} df' ddt$ is the integral of df' dependent to ddt.

3.10. High Shaking

$$d\left(\frac{d^n}{dt} t^n f_{(t)}\right) = \frac{d^n}{ddt} d_t^n df \tag{10}$$

$\frac{d^n}{ddt} d_t^n df$ is the n-th derivation of $d_t^n df$ dependent to ddt.

3.11. Inverse High Shaking

$$d\left(\frac{1}{t}\int_0^t f_{(t)} dt\right) = \frac{1}{dt}\int_0^{dt} dfddt \tag{11}$$

$\int_0^{dt} dfddt$ is the integral of df dependent to ddt.

3.12. Particular Low Shaking

$$d\left(t^n f_{(t)}\right)=d_t^n \frac{d^n}{ddt}\left(d_t^n df\right) \tag{12}$$

$\frac{d^n}{ddt}d_t^n df$ is the n-th derivation of $d_t^n df$ dependent to ddt.

4. Implementation Examples

Now let's consider DT as an abbreviation for Differential Transform and make some DT'es using the properties:

1) Knowing that $t=\int_0^t dt$ leads to

$$d(t)=dt \tag{13}$$

2) Let $f_{(t)}=\frac{t^2}{2}$, $f'_{(t)}=t$ following Equation (1) $\frac{t^2}{2}=\int_0^t tdt$ and its DT would be:

$$d\left(\frac{t^2}{2}\right)=dtdt=d_t^2 \Rightarrow d\left(\frac{t^2}{2}\right)=d_t^2 \tag{14}$$

3) Let $f_{(t)}=\frac{t^3}{3!}$, $f'_{(t)}=\frac{t^2}{2!}$ and following Equation (1) $\frac{t^3}{3!}=\int_0^t \frac{t^2}{2!}dt$ and its DT would be:

$$d\left(\frac{t^3}{3!}\right)=d\left(\frac{t^2}{2!}\right)dt=d_t^2 dt=d_t^3 \Rightarrow d\left(\frac{t^3}{3!}\right)=d_t^3 \tag{15}$$

4) Proceeding so on using Equation (1) we'll come $f_{(t)}=\frac{t^n}{n!}$, $f'_{(t)}=\frac{t^{n-1}}{(n-1)!}$ with $\frac{t^3}{3!}=\int_0^t \frac{t^2}{2!}dt$ its DT would be

$$d\left(\frac{t^n}{n!}\right)=d\left(\frac{t^{n-1}}{(n-1)!}\right)dt=d_t^{n-1}dt=d_t^n \Rightarrow d\left(\frac{t^n}{n!}\right)=d_t^n \tag{16}$$

5) Substituting Equation (12) in Equation (2) $f_{(t)}=t$, $f'_{(t)}=1$ so $1=\frac{d}{dt}t$ its DT would be

$$d(1)=\frac{d(t)-f(0)}{dt}=\frac{dt|_0^t}{dt}=\frac{dt-0}{dt}=1 \Rightarrow d(1)=1 \tag{17}$$

6) The DT of a constant *c*, using Equation (16) and (3) is

$$d(c)=cd(1)=c\cdot 1=c \quad d(c)=c \tag{18}$$

7) Now we can make $d\left(e^t\right)$, $e^t=1+t+\frac{t^2}{2}+\frac{t^3}{3!}+\cdots+\frac{t^n}{n!}$ $n\to\infty$ so

$$d\left(e^t\right)=d(1)+d(t)+d\left(\frac{t^2}{2}\right)+d\left(\frac{t^3}{3!}\right)+\cdots+d\left(\frac{t^n}{n!}\right) \Rightarrow$$

$$d\left(e^t\right)=1+d_t+d_t^2+d_t^3+\cdots+d_t^n$$

Here we have a finite series sum cause dt is infinitesimal

$$d\left(e^t\right)=\frac{1}{1-dt} \tag{19}$$

8) The DT of $e^{iwt}=\cos(\omega t)+i\sin(\omega t)$ is $de^{iwt}=d\cos(\omega t)+id\sin(\omega t)$ and by applying

Linearity, Equation (3), on $d\left(e^{t}\right)=\frac{1}{1-dt}$ via replacing t with $i\omega t$ gives:

$$de^{iwt}=\frac{1}{1-d_{i\omega t}}=\frac{1}{1-i\omega d_t}=\frac{1+i\omega d_t}{1+\omega^2 d_t^2}=\frac{1}{1+\omega^2 d_t^2}+i\frac{\omega d_t}{1+\omega^2 d_t^2}$$

now splitting (Equation (4)) gives $d\cos(\omega t)=\frac{1}{1+\omega^2 d_t^2}$ and the imaginary part

$$d\sin(\omega t)=\frac{\omega dt}{1+\omega^2 d_t^2} \tag{20}$$

9) Let's calculate the DT of te^{t} by shaking (Equation (11)):

$$d\left(te^{t}\right)=dt\frac{d}{ddt}dtde^{t}=dt\left(\frac{dt}{1-dt}\right)'=\frac{dt}{(1-dt)^2}.$$

5. The DT of Some Basic Function Is Shown in Table 1

See Table 1.

6. Solving Differential Equations

6.1. First Order Differential Equation

Differential equations of the form $mf'+nf=x$ with the initial $f(0)=a$.

df' as to Equation (2) is $df'=\frac{df-f(0)}{dt}=\frac{df-a}{dt}=\frac{df}{dt}-\frac{a}{dt}$ and the DT of $mf'+nf=x$ is

$$m\frac{df}{dt}-m\frac{a}{dt}+ndf=dx$$

$$df(m+ndt)=dxdt+ma \quad\Rightarrow$$

$$df=\frac{dxdt+ma}{ndt+m} \tag{21}$$

Example: solve $f'+f=e^{-t}$ with the initial $f_{(0)}=0$.
We'll substitute $m=1, n=1, x=e^{-t}, a=0$ in Equation (20):

$df=\frac{de^{-t}dt+1\cdot 0}{dt+1}$ and since $de^{-t}=\frac{1}{dt+1}$ we obtain $df=\frac{dt}{(dt+1)^2}$.

Now looking in Table 1 we'll found that $f_{(t)}=te^{-t}$.

6.2. Second Order Differential Equation

Differential equations of the form $kf''+mf'+nf=x$ with the initials $f(0)=a, f'(0)=b$ df' as to Equation (2) is $df'=\frac{df-f(0)}{dt}$ and df'' is

$$df''=\frac{df'-f'(0)}{dt}=\frac{df-f(0)-f'(0)dt}{d_t^2}.$$

Table 1. The DT'es of the basic function.

	The function	The Differential Transform	Where
1	$f_{(t)}$	$df_{(t)} = \frac{1}{dt}\int_0^\infty e^{\frac{-\tau}{dt}} f_{(\tau)} d\tau$	
2	$h(t) = f(t)*y(t) = \int_0^t f_{(t-\tau)} y_{(\tau)} d\tau$	$dh_{(t)} = df_{(t)} dt dy_{(t)}$	
3	$f_{(t-\tau)},\quad t \ge \tau$	$e^{-\frac{\tau}{dt}} df_{(t)}$	
4	$e^{-at} f(t)$	$\frac{1}{1+dat} G\left(\frac{dt}{1+dat}\right),\quad G = df$	
5	unit impulse $\delta_{(t)}$	$1/dt$	
6	$t^n/n!$	d_t^n	
7	Constant: C	C	
8	$e^{-\alpha t}$	$1/(1+d\alpha t)$	
9	$\sin(\omega t)$	$d\omega t/(1+d_{\omega t}^2)$	
10	$\cos(\omega t)$	$1/(1+d_{\omega t}^2)$	
11	$\sinh(\omega t)$	$d\omega t/(1-d_{\omega t}^2)$	
12	$t\sinh\omega t$	$\frac{2d_{\omega t}^2}{\omega(1-d_{\omega t}^2)^2}$	
13	$\cosh(\omega t)$	$1/(1-d_{\omega t}^2)$	
14	$t\cosh\omega t$	$dt\frac{1+d_{\omega t}^2}{(1-d_{\omega t}^2)^2}$	
15	$t^n e^{-\alpha t}$	$d_t^n/(1+\alpha dt)^{n+1}$	
16	$t\sin\omega t$	$\frac{2d_{\omega t}^2}{\omega(1+d_{\omega t}^2)^2}$	
17	$\sin\omega t/t$	$\arctan(d\omega t/dt)$	
18	$t\cos\omega t$	$dt\frac{1-d_{\omega t}^2}{(1+d_{\omega t}^2)^2}$	
19	$\frac{1-\cos\omega t}{t}$	$\frac{\ln\sqrt{1+d_{\omega t}^2}}{dt}$	
20	$e^{-\alpha t}\sin\omega t$	$\frac{\omega dt}{d_{\omega_n t}^2 + 2\alpha dt + 1}$	$\alpha + i\omega = \omega_n e^{i\theta}$
21	$e^{-\alpha t}\cos\omega t$	$\frac{1+\alpha dt}{d_{\omega_n t}^2 + 2\alpha dt + 1}$	$\omega_n = \sqrt{\alpha^2 + \omega^2}$
22	$\frac{e^{-\alpha t}}{\sin\theta}\sin(\theta - \omega t)$	$\frac{1}{d_{\omega_n t}^2 + 2\alpha dt + 1}$	$\theta = \arctan(\omega/\alpha)$
23	$1-\frac{e^{-\alpha t}}{\sin\theta}\sin(\omega t + \theta)$	$\frac{d_{\omega_n t}^2}{d_{\omega_n t}^2 + 2\alpha d\omega_n t + 1}$	$\alpha = \omega_n\cos\theta$
24	$\omega_n e^{-\alpha t}\cos(\omega t - \theta)$	$\frac{\omega_n^2 dt + \alpha}{d_{\omega_n t}^2 + 2\alpha dt + 1}$	$\omega = \omega_n\sin(\theta)$

The DT of $kf'' + mf' + nf = x$ becomes

$$k\frac{df}{d_t^2} - k\frac{a}{d_t^2} - k\frac{b}{dt} + m\frac{df}{dt} - m\frac{a}{dt} + ndf = dx$$

$$df\left(k + mdt + nd_t^2\right) = dxd_t^2 + ka + kbdt + madt$$

$$df = \frac{dxd_t^2 + (ma + kb)dt + ka}{nd_t^2 + mdt + k} \tag{22}$$

Example: solve $f'' + f = 0$ with the initials $f(0) = 4, f'(0) = -1$.

We'll substitute $k = 1, m = 0, n = 1, x = 0, a = 4, b = -1$ in Equation (21):

$$df = \frac{0 + (0-1)dt + 4}{d_t^2 + 1} = \frac{4}{1 + d_t^2} - \frac{dt}{1 + d_t^2}$$

Now looking in Table 1 we'll found that

$$f_{(t)} = 4\cos(t) - \sin(t).$$

7. Electrical Circuits

Here we transferred the electrical circuit elements to the differential domain as shown in Figure 1.

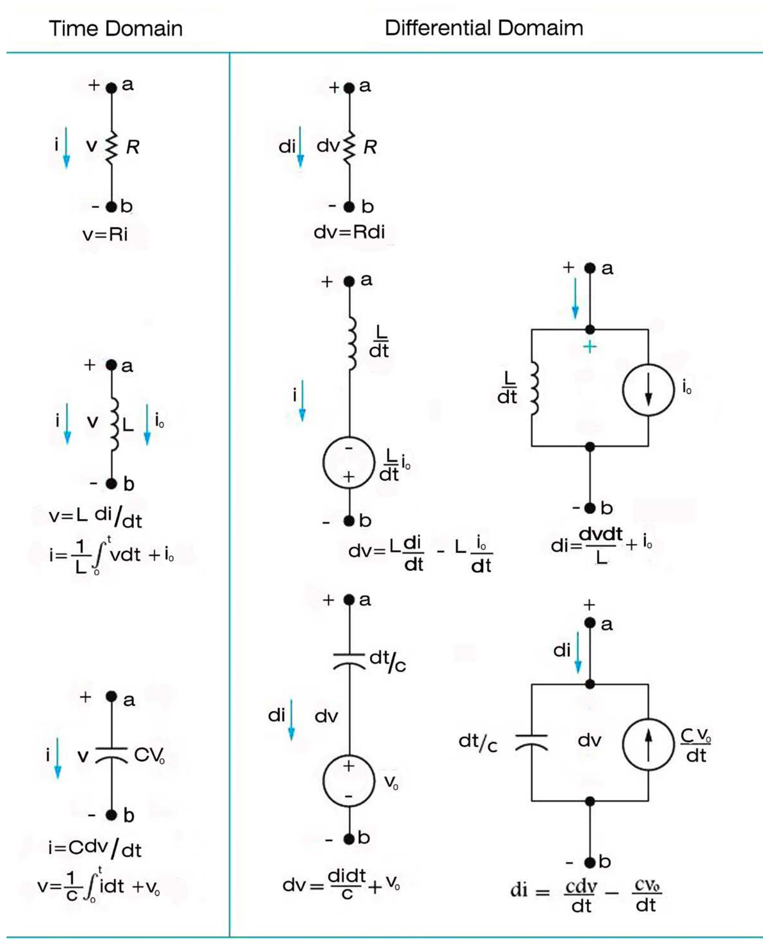

Figure 1. The transformed circuits the differential domain.

7.1. RC Circuit

The transformed circuit in Figure 2 is shown in Figure 3.

The circuit is fed with $v(t)=v$ DC volts and we need to find $i(t)$ using Kirchoff's law:

$$v(t)=Ri(t)+\frac{1}{c}\left(\int_0^t i(t)\,dt+q_0\right) \quad \text{so} \quad dv=Rdi+\frac{1}{c}didt+\frac{q_0}{c}$$

$$di=\frac{v-\frac{q_0}{c}}{R+\frac{dt}{c}}=\frac{v-\frac{q_0}{c}}{R}\cdot\frac{1}{1+d\frac{t}{RC}} \;\Rightarrow\; i(t)=\frac{v-\frac{q_0}{c}}{R}e^{-\frac{t}{RC}}.$$

7.2. RL Circuit

Applying the above steps to the circuit in shown in Figure 4:

$$v(t)=Ri(t)+L\frac{d}{dt}i(t) \;\Rightarrow\; \text{the DT is } dv=v=Rdi+L\frac{di-i_0}{dt}=Rdi+L\frac{di}{dt}-L\frac{i_0}{dt}$$

$$di=\frac{v+L\,i_0/dt}{R+L/d}=\frac{v}{R+L/dt}+\frac{L\,i_0/dt}{R+L/dt}=\frac{v}{R}\cdot\frac{d\frac{R}{L}t}{1+d\frac{R}{L}t}+i_0\frac{1}{1+d\frac{R}{L}t}$$

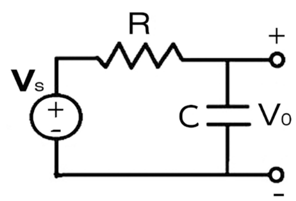

Figure 2. RC circuit.

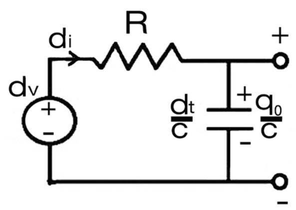

Figure 3. The DT of the circuit in Figure 2.

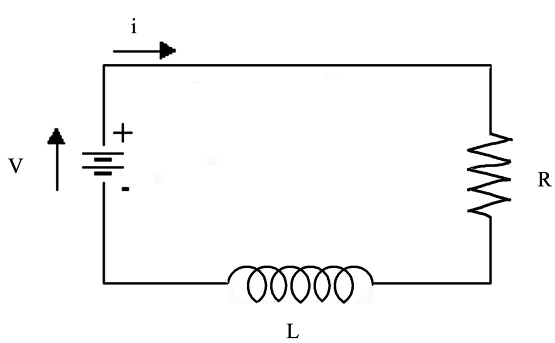

Figure 4. RL circuit.

$$i(t)=\frac{v}{R}\left(1-e^{-\frac{R}{L}t}\right)+i_0e^{-\frac{R}{L}t}=\frac{v}{R}+\left(i_0-\frac{v}{R}\right)e^{-\frac{R}{L}t}$$

$$i(t)=\frac{v}{R}+\left(i_0-\frac{v}{R}\right)e^{-\frac{R}{L}t}.$$

8. Physical Problems

Harmonic Motion

The differential equation of the physics harmonic motion discribed in Figure 5.

$$ma+kx=0$$

where the m is the mass, $a=x''$ is the acceleration, k is the spring constant, $A=x_0$ is the amplitude with the initials $x(0)=A, x'(0)=v_0=0$.

The DT of $mdx''+kdx=0$ where $dx''=\frac{dx-x_0}{d_t^2}=\frac{dx}{d_t^2}-\frac{A}{d_t^2}$ is

$m\frac{dx}{d_t^2}-m\frac{A}{d_t^2}+kdx=0 \quad \Rightarrow \quad mdx-mA+kdxd_t^2=0$ giving $dx=\frac{A}{1+\frac{k}{m}d_t^2}$ and

by Linearity (Equation (3)) $dx=\frac{A}{1+\frac{k}{m}d_t^2}=\frac{A}{1+d^2_{t\sqrt{\frac{k}{m}}}}$ $\quad x(t)=A\cos\left(t\sqrt{\frac{k}{m}}\right)$.

9. Conversion between Laplace Transform (LT) [7] and Differential Transform (DT)

$$G=\frac{1}{dt}F_{\left(\frac{1}{dt}\right)} \quad \text{and} \quad F=\frac{1}{s}G_{\left(\frac{1}{s}\right)}$$

1) Let's convert the DT of $f_{(t)}=\sin(t)$ to LT:

$$G=df=d\sin t=\frac{dt}{1+d_t^2} \quad \text{therefore} \quad F=\frac{1}{s}G_{\left(\frac{1}{s}\right)}=\frac{1}{s}\frac{\frac{1}{s}}{1+\left(\frac{1}{s}\right)^2}=\frac{1}{1+s^2}$$

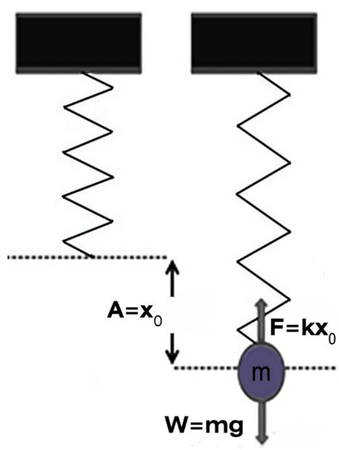

Figure 5. Harmonic motion.

2) Let's convert the LT of $f_{(t)} = e^{-t}$ to DT:

$$F_{(s)} = \frac{1}{s+1} \quad \text{so} \quad G = \frac{1}{dt} F_{\left(\frac{1}{dt}\right)} = \frac{1}{dt} \frac{1}{\frac{1}{dt}+1} = \frac{1}{1+dt}$$

10. The DT Inversion

The inversion of $df_{(t)}$ is $f_{(t)}$, it means that we transfer the function from the differential domain to the time domain.

1) Inversion via Convolution (Equation (6)).

Example: Find $f(t)$ if $df = \frac{dt}{-12d_t^2 - dt + 1}$.

$$df = \frac{dt}{-12d_t^2 - dt + 1} = \frac{dt}{(1+3dt)(1-4dt)} = \frac{1}{(1+d3t)} dt \frac{1}{(1-d4t)}$$

$$= de^{-3t} dt de^{4t} = d\left(e^{-3t} * e^{4t}\right)$$

see **Table 2**.

$$f_{(t)} = e^{-3t} * e^{4t} = \int_0^t e^{-3\tau} e^{4(t-\tau)} d\tau = e^{4t} \int_0^t e^{-7\tau} d\tau = e^{4t} \left. \frac{e^{-7\tau}}{-7} \right|_0^t = \frac{1}{7} e^{4t} - \frac{1}{7} e^{-3t}$$

$$f_{(t)} = \frac{1}{7} e^{4t} - \frac{1}{7} e^{-3t}.$$

Table 2. Series of some basic functions.

	The function	The series
1	$f_{(t)}$	A_n
2	αe^{t}	α
3	te^{t}	n
4	$\alpha e^{\beta t}$	$\alpha \beta^n$
5	$te^{\alpha t}$	$n\alpha^n$
6	$\sin(\omega_0 t)$	$\omega_0^n \sin\left(n\frac{\pi}{2}\right)$
7	$\cos(\omega_0 t)$	$\omega_0^n \cos\left(n\frac{\pi}{2}\right)$
8	$\int_0^t f_{(t)} d_t$	A_{n-1}, $n-1 \geq 0$
9	$\int_0^t \overset{m}{\cdots} \int_0^t f_{(t)} d_t \overset{m}{\cdots} d_t$, m times integration	A_{n-m}, $n-m \geq 0$
10	$\frac{1}{t} \int_0^t f_{(t)} d_t$	$\frac{1}{n+1} A_n$
11	$\int_0^t \left(f_{(t)}/t\right) d_t$	$\frac{1}{n} A_n$
12	$t^m \frac{d^m}{d_t^m} f(t)$, m times derivation	$\frac{n!}{(n-m)!} A_n$

2) The direct inversion

We know from algebra that the sum of the series $r(x)=\sum_{k=0}^{\infty}x^k=1+x+x^2+x^3+\cdots$ for $|x|<1$ is $sum(r(x))=\frac{1}{1-x}$, and will regard this formula as "The sum formula".

Hence $d(\sin t)=\frac{dt}{1+d_t^2}$ the sum formula is $\frac{1}{1+d_t^2}=1-d_t^2+d_t^4-d_t^6+\cdots$ and $d\sin t=\frac{dt}{1+d_t^2}=dt\left(1-d_t^2+d_t^4-\cdots\right)=dt-d_t^3+d_t^5-\cdots$ now inverting this equation gives $\sin t=1-\frac{t^2}{2!}+\frac{t^4}{4!}-\frac{t^6}{6!}+\cdots$.

2) A good exercise is to invert $df(t)=\frac{1}{1-d_t^{24}}$ according to the sum formula $df(t)=\frac{1}{1-d_t^{24}}=1+d_t^{24}+d_t^{48}+\cdots$, and the series of $f(t)$ becomes $f(t)=1+\frac{t^{24}}{24!}+\frac{t^{48}}{48!}+\cdots$.

Notice that dealing with the series of $f_{(t)}$ is easier than dealing directly with $f_{(t)}$ specially when we are working with values of t near the zero.

Means that the function behaves like $f(t)_{t\to 0}\approx 1+\frac{t^{24}}{24!}+\frac{t^{48}}{48!}$ and under these conditions the finite series of the function could be regarded as its abbreviation and we can be satisfied with $f(t)^*=1+\frac{t^{24}}{24!}+\frac{t^{48}}{48!}$.

3) A function of variable approaching the zero *i.e.* the infinitesimal function of $f(t)$ is $f^*(t)=f(t)\big|_{t\to 0}$.

Recall that: $g(t)=\sin t=t-\frac{t^3}{3!}+\frac{t^5}{5!}-\frac{t^7}{7!}+\cdots \Rightarrow g^*(t)=\sin t\big|_{t\to 0}=t$.

Example: To find $f^*(t)$ where $df(t)=\frac{(1+adt)^n}{(1+bdt)^m}$ and $t\to 0$ we can replace $(1+adt)^n$ with $1+andt$ and $\frac{1}{(1+bdt)^m}$ with $1-amdt$ and get

$df^*(t)=(1+andt)(1-bmdt)=1+(an-bm)dt-abnmd_t^2$,

$f^*(t)=1+(an-bm)t-0.5abnmt^2$.

11. Solving Differential Equations via Series Method

11.1. Taylor Series Form [8]

$$f(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\frac{f'''(a)}{3!}(x-a)^3+\cdots+\frac{f^{(n)}(a)}{n!}(x-a)^n+\cdots$$

And for $a=0$:

$$f(x)=f(0)+f'(0)x+f''(0)\frac{x^2}{2!}+f'''(0)\frac{x^3}{3!}+f''''(0)\frac{x^4}{4!}+\cdots$$

The DT of Tylor series:

$$df = f(0) + f'(0)dx + f''(0)d_x^2 + f'''(0)d_x^3 + f''''(0)d_x^4 + \cdots \tag{23}$$

$$df = \sum_{n=0}^{\infty} K_n d_t^{(n)}$$

where the series

$$K_n = \left\{ f(0), f'(0), f''(0), f'''(0), f''''(0), \cdots, f^{(n)}(0) \right\}$$

11.2. The Differentiation of f Derived from the DT of Taylor Series

$$df' = \frac{df - f(0)}{dx} = f'(0) + f''(0)d_x + f'''(0)d_x^2 + f''''(0)d_x^3 + \cdots = A_n d_t^{(n)}$$

$$A_n = \left\{ f'(0), f''(0), f'''(0), f''''(0), \cdots, f^{(n)}(0) \right\}$$

$$df'' = \frac{df - f(0) - f'(0)dx}{d_x^2} = f''(0) + f'''(0)d_x + f''''(0)d_x^2 + \cdots = B_n d_t^{(n)}$$

$$B_n = \left\{ f''(0), f'''(0), f''''(0), \cdots, f^{(n)}(0) \right\}$$

$$df''' = \frac{df - f(0) - f'(0)dx - f''(0)d_x^2}{d_x^3}$$

$$= f'''(0) + f''''(0)d_x + f^{(5)}(0)d_x^2 \cdots = C_n d_t^{(n)}$$

$$C_n = \left\{ f'''(0), f''''(0), f^{(5)}(0), \cdots, f^{(n)}(0) \right\}$$

The general form of linear differential equations order m is $A_m f_{(t)}^{(m)} = x(t)$ with the initial conditions

$$K_m = \left\{ f(0), f'(0), f''(0), f'''(0), f''''(0), \cdots, f^{(m-1)}(0) \right\}$$

And $x(t) = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}$. So we can perform the DT for the equation:

$$A_0 df + A_1\left[\frac{df}{d_t} - \frac{K_0}{d_t}\right] + A_2\left[\frac{df}{d_t^2} - \frac{K_0}{d_t^2} - \frac{K_1}{d_t}\right] + A_3\left[\frac{df}{d_t^3} - \frac{K_0}{d_t^3} - \frac{K_1}{d_t^2} - \frac{K_2}{d_t}\right]$$

$$+ \cdots + A_m\left[\frac{df}{d_t^m} - \frac{K_0}{d_t^m} - \frac{K_1}{d_t^{m-1}} - \cdots - \frac{K_{m-1}}{d_t}\right] = d_t^m dx$$

In series method solution the initials series for $n > m-1$ $K_n = 0$ so it doesn't effect finding.

The solution $f_{(t)} = \sum_{n=0}^{\infty} A_n \frac{t^n}{n!}$ $n = 0, 1, 2, \cdots, \infty$ that we have $A_n = \left\{ f(0), f'(0), f''(0), f'''(0), f''''(0), \cdots, f^{(m-1)}(0), \cdots \right\}$ and will start finding A_n for $n \geq m$ so the relevant DT donated RTD is:

$$A_0 d_t^m df + A_1 d_t^{m-1} df + A_2 d_t^{m-2} df + A_3 d_t^{m-3} df + \cdots + A_m df = d_t^m dx \tag{24}$$

11.3. Deriving Series from the DT of Taylor Series

Having

$$df = \sum_{n=0}^{\infty} K_n d_t^{(n)},$$

$$K_n = \left\{ f(0), f'(0), f''(0), f'''(0), f''''(0), \cdots, f^{(n)}(0) \right\}$$

$$G = d_t^m df = \sum_{n=m}^{\infty} C_{n-m} d_t^{(n)} \tag{25}$$

where $C_{n-m} = 0$ for $(n-m) < 0$.

And recalling (Equation (7)) $H = d\left(t^m \frac{d^m}{dt} f_{(t)} \right) = d_t^m \frac{d^m}{ddt} df$.
Gives

$$G = \sum_{n=m}^{\infty} \frac{n!}{(n-m)!} K_n d_t^{(n)} \tag{26}$$

Example 11.3.1: solve $f'' + f = 0$ with the initials $f(0) = 4, f'(0) = -1$ $df'' + df = 0$.

The solution would be of the form $f_{(t)} = \sum_{n=0}^{\infty} A_n \frac{t^n}{n!}$ so we should find $A_n = ?$.

1) Regular method:

$$\frac{df - f(0) - f'(0)d_t}{d_t^2} + df = 0 \rightarrow df - f(0) - f'(0)d_t + d_t^2 df = 0$$

$$df\left(1 + d_t^2\right) = f(0) + f'(0)d_t \rightarrow df = \frac{f(0) + f'(0)d_t}{1 + d_t^2}$$

$$\rightarrow df = \frac{4 - d_t}{1 + d_t^2} \quad \dots \text{(sol1)}$$

And the DT inversion of *df*: $f(x) = 4\cos(t) - \sin(t)$.

2) Seriese method: the relevant transform RTD of the differential equation is $\rightarrow d_t^2 df + df = 0$.

And as to Equation (24). $A_n d_t^n + A_{n-2} d_t^n = 0 \rightarrow A_n = -A_{n-2}$.

The initials gives $A_0 = 4$, $A_1 = -1 \rightarrow A_2 = -A_0 = -4$, $A_3 = -A_1 = 1$.

Example 11.3.2: Solve $t^2 f'' + f = e^t$.

We recall that $e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!} \rightarrow de^t = \sum_{n=0}^{\infty} d_t^n = \sum_{n=0}^{\infty} C_n d_t^n$ for $C_n = 1$.

The solution would be of the form $f_{(t)} = \sum_{n=0}^{\infty} A_n \frac{t^n}{n!}$ and we should find $A_n = ?$.

According to Equation (25): $\frac{n!}{(n-2)!} A_n + A_n = C_n = 1$ so $A_n = \frac{1}{n^2 - n + 1}$,

$A_0 = \frac{1}{0^2 - 0 + 1} = 1$, $A_1 = \frac{1}{1^2 - 1 + 1} = 1$, $A_2 = \frac{1}{2^2 - 2 + 1} = \frac{1}{3}$, $A_3 = \frac{1}{3^2 - 3 + 1} = \frac{1}{7}$, …

Therefore we find $f(t) = \sum_{n=0}^{\infty} \frac{1}{n^2 - n + 1} * \frac{t^n}{n!}$.

11.4. Table 2 Gives the Series of Some Basic Functions Concerning That $f_{(t)} = \sum_{n=0}^{\infty} A_n \frac{t^n}{n!}$

See Table 2.

12. A New Interpretation for Z Transform

We can shift between the Z transform [9] and differential transform by substituting $d_t = Z^{-1}$.

Hence we resolve a difference equation [10] of digital filter via Z transform and getting the solution as a series y_n. Means that the difference equation isn't other than a differential equation of a physical filter and it's solution is $f_{(t)} = \sum_{n=0}^{\infty} y_n \frac{t^n}{n!}$.

Example: Consider the difference equation $y_{n+1} - 2y_n = 7$ with the initial $y_0 = 3$.

Applying Z transform $zY_{(z)} - 3z - 2Y_{(z)} = \frac{7z}{z-1} \rightarrow Y_{(z)} = \frac{10z}{z-2} - \frac{7z}{z-1}$ gives $y_n = 10 * 2^n - 7$.

Actually this is the solution of the differential equation $f' - 2f = 7$ which is:

$$f_{(t)} = \sum_{n=0}^{\infty} y_n \frac{t^n}{n!} = \sum_{n=0}^{\infty} \left(10 * 2^n - 7\right) \frac{t^n}{n!}$$

and from Table 2: $f_{(t)} = 10e^{2t} - 7e^t$.

13. Finalization

We can learn from this paper that the Differential Transform is not a replacement for the existing methods, it could be useful in treating differential problems and for deriving the series of the solution. Here we found that difference equation which represents a digital filter could be treated the same as a differential equation of a physical filter. And a high potential is still embodied in this Transform.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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