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Minimal Time of Null Controllability for the 1D Heat Equation by a Strategic Zone Profile

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Abstract

The aim of this work is to improve the minimum time of null controllability of the 1D heat equation by using the notion of strategic zone actuators. In fact, motivated by the work of Khodja on the null controllability of the heat equation and of El Jai on the controllability by the use of strategic zone actuators, we managed, in this work, to improve the minimal time of null controllability to the 1D heat equation. However, the restrictions and difficulties to establish the inequality of coercivity of the parabolic operator, require to seek other methods of internal control. Thus in this paper, a mixed method combining the method of moments and the notion of strategic profile was used to find a better minimal time of null controllability of the 1D heat equation.

Keywords

Control, Null Controllability, Moment Methods, Estimations, Strategic Actuator

1. Introduce the Problem

One of the objectives of the control theory of partial differential equations of evolution is to be interested in how to act on such dynamic systems. So, the exact controllability of distributed systems has attracted a lot of interest in recent years. And this thanks to one of the pioneers J.L. Lions [1] [2] who developed the HUM method (Hilbert Uniqueness Methods). It is based essentially on the properties of uniqueness of the homogeneous equation by a particular choice of controls, the construction of a hilbert space and of a continuous linear application of this hilbert space in its dual which is, in fact, an isomorphism that establishes exact controllability.

For hyperbolic problems, this method has given important results (Lions [2], Niane [3] [4], Seck *et al.* [5] [6]).

Although when the controls have a small support (Niane [4], Seck [6]), it seems to be ineffective, even when for technical reasons, the multiplier method does not give results, see Niane [5].

As for the parabolic equations, there are the results of Russel [7] first; Later G. Lebeau-L. Robbiano [8] and Imanuvilov-Fursikov [9] who have proven with different methods but **very technical and long by using Carleman's Inequalies** in the exact null controllability of the Heat equation.

Also, the harmonic method is also ineffective for this type of equation.

More recently, Khodja *et al.* [10] have shown that there is a minimal time T_0 of controllability below which null controllability is not achievable for a parabolic operator. Thus, by Khodja [11], Tucsnak [12] and Avdonin [13], a means of calculating a minimum cost associated with this minimum time of null controllability has been established.

2. Problem Statement

In this work, to circumvent certain constraints linked to estimates in the work of G. Lebeau [8], Imanuvilov-Fursikov [9] notably the Carleman's inequalities, we show that a new method solves some of these difficulties. It is based on a fusion of the moments method used by Khodja [11], Tucsnak [12], Avdonin [13] and the use of strategic actuators zones El. Jai [14] [15] to solve the problem of null controllability of the heat equation with a minimum time T_0^{μ} controllability less than the minimal time of null controllability T_0 provided by Khodja *et al.* [10] [11].

There are two types of criteria:

1) A criterion for constructing a functional space F_T containing H_0^1 and its dual F_T^* contained in H^{-1} thus making L^2 a pivotal space Brezis [16];

2) A criterion of non-degeneration of a strategic zone profile which stems from the parabolic nature of the operator and the regularity of control Hörmander [17];

In both cases these criteria allowed us to obtain a better minimum time of controllability.

This method opens wide perspectives to the theory of null controllability in general, as well as to the theory of exact controllability by zone strategic actuators and will allow for parabolic equations 1D (and 2D), Schrödinger, plates and of Navier-Stocks linearized to solve many questions thus opening many perspectives for the improvement of the minimum times of controllability.

3. Concept of Strategic Zone Actuators

3.1. Notations and Definition

Definition 1. *A function* $\mu: I \to \overline{\mathbb{R}}$ *square integrable is said strategic if it verify*, for all $y_0 \in L^2(I)$, the solution y of the heat equation

$$\begin{cases} y_t(t,x) - \partial_{xx} y(t,x) = 0 \text{ in } Q_T =]0,+\infty[\times I \\ \gamma y(t,x) = 0 \text{ in } \Sigma_T =]0,+\infty[\times \partial I \\ y(0) = y_0 \text{ in } I \end{cases} \tag{1}$$

$$\forall t > 0, \int_I \mu(x) y(t,x) \mathrm{d}x = 0 \text{ then } y_0 = 0. \tag{2}$$

Let $I =]0,\pi[$ an interval of $\mathbb{R}$, let A the operator defined by

$$D(A) = \{ y \in H_0^1(I) / -\partial_{xx} y \in L^2(I) \}, Ay = -\partial_{xx} y, \forall y \in D(A) \tag{3}$$

According to the spectral theory, A admits a Hilbertian base of $L^2(I)$ of eigenfunctions $(w_k)_{k\ge1}$ whose associated eigenvalues are $(\lambda_k)_{k\ge1}$ rows in the ascending direction where

$$\begin{cases} w_k(x) = \sqrt{\dfrac{2}{\pi}} \sin(kx) \\ \lambda_k = k^2 \end{cases} \tag{4}$$

Remark

1) It suffices that the relation (2) is verified over an interval $]0,T[$ for it to be true on $]0,+\infty[$ because of the analyticity of $t \to \int_I \mu(x) y(t,x) \mathrm{d}x$ on $\mathbb{R}_+^*$.

2) Here Ω is a bounded open of $\mathbb{R}^2$, of regular border; $L^2(I)$ is, a priori, the state space and T define the time horizon considered for the exact controllability of the system (1).

Proposition 1. *There are strategic actuators with support contained in any interval* $]\alpha,\beta[$ *such that* $0 < \alpha < \beta < \pi$.

Proof. We can first notice that μ is strategic if and only if: $\forall k \in \mathbb{N}^*, \mu_k \neq 0$.

Let $\alpha, \beta \in]0,\pi[$ such that $\alpha < \beta$ and posing that: $\mu = \chi_{]\alpha,\beta[}$.

Then, we have

$$\mu_k = \int_0^\pi \chi_{]\alpha,\beta[}(x) \frac{\sqrt{2}}{\sqrt{\pi}} \sin(kx) \mathrm{d}x \tag{5}$$

$$= -\frac{\sqrt{2}}{k\sqrt{\pi}} [\cos k(\beta) - \cos k(\alpha)] \tag{6}$$

$$= -\frac{\sqrt{2}}{k\sqrt{\pi}} 2 \sin\left(\frac{k(\beta-\alpha)}{2}\right) \sin\left(\frac{k(\beta+\alpha)}{2}\right) \tag{7}$$

We have $\mu_k = 0$ if and only if

$$\begin{cases} \dfrac{k(\beta-\alpha)}{2} = l, \quad l \in \mathbb{Z} \\ \dfrac{k(\beta+\alpha)}{2} = r, \quad r \in \mathbb{Z} \end{cases} \tag{8}$$

Therefore, for that $\mu_k \neq 0$ it is sufficient that:

$$\beta - \alpha \notin \mathbb{Q} \text{ and } \beta + \alpha \notin \mathbb{Q}.$$

So, if we take $\alpha \in \mathbb{Q}$ and $\beta = \alpha + r$ where $r \notin \mathbb{Q}$ then $\mu = \chi_{]\alpha,\beta[}$ is strategic ■

Remark. *Obviously, other strategic actuators can be built without great diffi-*

culty see Jai [5] [6].

3.2. Notations, Definition and Functional Spaces

Let $T > 0$, consider the following Hilberts spaces and their respective dual:

$$F_T = \left\{ u = \sum_{k=1}^{+\infty} \mu_k w_k \Big/ \sum_{k=1}^{+\infty} \lambda_k \mu_k^2 e^{2\lambda_k T} < +\infty \right\} \tag{9}$$

and

$$F_T^* = \left\{ u = \sum_{k=1}^{+\infty} \mu_k w_k \Big/ \sum_{k=1}^{+\infty} \lambda_k \mu_k^2 e^{-2\lambda_k T} < +\infty \right\} \tag{10}$$

We equip F_T with the following scalar product

$$(x, y)_{F_T} = \sum_{k=1}^{+\infty} \mu_k w_k \Big/ \sum_{k=1}^{+\infty} \lambda_k \mu_k^2 e^{2\lambda_k T} < +\infty \tag{11}$$

and, the associeted norm $\|.\|_{F_T}$.

The dual of F_T is F_T^* provided the scalar product

$$(x, y)_{F_T^*} = \sum_{k=1}^{+\infty} \mu_k w_k \Big/ \sum_{k=1}^{+\infty} \lambda_k \mu_k^2 e^{-2\lambda_k T} < +\infty \tag{12}$$

and, associeted norm $\|.\|_{F_T^*}$.

If $x \in F_T$ and $y \in F_T^*$ we have:

$$(y, x)_{F_T^*, F_T} = \sum_{k=1}^{+\infty} x_k y_k \tag{13}$$

Let us define now the setting that we will deal in the sequel and assume that

$$\sum_{k=1}^{+\infty} \frac{1}{\lambda_k} < +\infty$$

Definition 2. The condensation index of sequences $\Lambda = (\lambda_k)_{k \geq 1}$ is defined as

$$I(\Lambda) = \lim_{k \to +\infty} \sup \frac{-\ln |E'(\lambda_k)|}{\lambda_k} \tag{14}$$

where the function E is defined by

$$E(x) := \prod_{k=1}^{+\infty} \left(1 - \frac{x^2}{\lambda_k^2} \right) \tag{15}$$

To apply the moment method, let us define the concept of biorthogonal family.

Definition 3. Let $\sigma = (\sigma_k)_{k \in \mathbb{N}}$ be a real sequence and $T > 0$. We say that the family of functions $(q_k)_{k \in \mathbb{N}} \subset L^2(0, T; \mathbb{R})$ is a biorthogonal family to the exponentials associated with σ if for any $k, j \in \mathbb{N}$

$$\int_0^T e^{-\sigma_j t} q_k(t) \, dt = \delta_{k,j} \tag{16}$$

Also assume a fundamental lemma we need in the sequel for the proof of the main result.

Lemma 2. See Khodja [7] or Tucsnak [17]

Let $T>0$ and let $\sigma=(\sigma_k)_{k\in\mathbb{N}}$ be a ordered sequence such that $\sum_{k\geq 1}\frac{1}{|\sigma_k|}<+\infty$. Then, there exists a biorthogonal family $(q_k)_{k\in\mathbb{N}}$ to the exponentials associeted with σ such that for any $\varepsilon>0$ there exists a constant $C_\varepsilon>0$ such that

$$\|q_k\|_{L^2(0,T;\mathbb{R})}\leq C_\varepsilon e^{\sigma_k(I(\sigma)+\varepsilon)}, \tag{17}$$

for k sufficiently large, where $I(\sigma)$ is the condensation index of the sequence σ.

4. Main Result of Null Controllability of the 1D Heat Equation

4.1. Main Theorem

Theorem 3. If μ is a strategic actuator on $[0,\pi]$, $u(.)$ a control and $T>0$ a strictly positive real; for all $y_0\in F_T^*$, there exist $\beta\in L^2(]0,T[)$ and $T_0^\mu<T$ such that if y is solution of

$$\begin{cases} y_t-\partial_{xx}y=\beta(t)\mu(x)u(t) \text{ in }]0,T[\times I \\ \gamma y=0 \text{ in }]0,T[\times\partial I \\ y(0)=y_0 \text{ in } I \end{cases} \tag{18}$$

then $y(T)=0$.

Proof. Let be the heat equation with an internal strategic zone profile $\mu(.)$ and a control $u(.)$ defined by:

$$\begin{cases} y_t-\partial_{xx}y=\beta(t)\mu(x)u(t) \text{ in } Q_T=]0,+\infty[\times I \\ y(0;t)=y(\pi;t)=0 \text{ in }]0,T[\\ y(0)=y_0 \end{cases} \tag{19}$$

Let $B=\beta(t)u(t)$ be a linear control operator, then the previous Equation (19) becomes:

$$\begin{cases} y_t-\partial_{xx}y=B\mu(x) \text{ in } Q_T \\ y(0;t)=y(\pi;t)=0 \\ y(0)=y_0 \end{cases} \tag{20}$$

Then the solution of the previous Equation (20) is given by:

$$y(t)=e^{t\Delta}y_0+\int_0^t e^{(t-s)\Delta}B\mu(x)\,ds. \tag{21}$$

The Equation (19) is null controllable at time $T>0$ if $y(T)=0$ which is equivalent to

$$-e^{T\Delta}y_0=\int_0^T e^{(T-s)\Delta}B\mu(x)\,ds. \tag{22}$$

Based on the definition of the following spaces previously defined:

$$F_T=\left\{u=\sum_{k=1}^{+\infty}\mu_k w_k \middle/ \sum_{k=1}^{+\infty}\lambda_k\mu_k^2 e^{2\lambda_k T}<+\infty\right\} \tag{23}$$

$$F_T^* = \left\{ u = \sum_{k=1}^{+\infty} \mu_k w_k \Big/ \sum_{k=1}^{+\infty} \lambda_k \mu_k^2 \mathrm{e}^{-2\lambda_k T} < +\infty \right\} \tag{24}$$

and, if the solution $y \in L^2(0;T)$, then we have $y(t;x) = \sum_{k\geq 1} y_k(t) w_k$ and $y_0 = \sum_{k\geq 1} y_k^0 w_k$.

Likewise B is in $L^2(]0,T[)$ and is written:

$$B = \sum_{k\geq 1} b_k(t) w_k;\ \text{with}\ w_k = \sqrt{\frac{2}{\pi}} \sin(kx). \tag{25}$$

basis of eigenfunctions.

Then the Equation (20) becomes:

$$\begin{cases} y_k' + k^2 y_k = b_k(t)\mu(x); \quad k \geq 1 \\ y_k\big|_{t=0} = y_k^0 \end{cases} \tag{26}$$

Therefore the solution becomes:

$$y_k(t) = \mathrm{e}^{-k^2 t} y_k^0 + \int_0^t \mathrm{e}^{-k^2(t-s)} b_k(t)\mu(x)\mathrm{d}s; \quad k \geq 1 \ \Rightarrow \tag{27}$$

$$y_k(t) = \mathrm{e}^{-k^2 t} y_k^0 + b_k(t)\int_0^t \mathrm{e}^{-k^2(t-s)} \mu(x)\mathrm{d}s; \quad k \geq 1 \tag{28}$$

(20) is null controllable at time $T > 0$ if and only if $y_k(T) = 0$ which means that

$$b_k(T)\int_0^T \mathrm{e}^{-k^2(T-s)} \mu(x)\mathrm{d}s = -\mathrm{e}^{-k^2 T} y_k^0 \tag{29}$$

We have $b_k(T) = \beta_k(T)u(T)$, and (3.11) becomes:

$$\beta_k(T)u(T)\int_0^T \mathrm{e}^{-k^2(T-s)} \mu(x)\mathrm{d}s = -\mathrm{e}^{-k^2 T} y_k^0 \tag{30}$$

Let's do the following variable change (to have the backward problem):

$$v(t) = u(T-t)$$

we have then and $v(t) = \sum_{k\geq 1} v_k q_k(t)$ with $\{q_k\}_{k\geq 1}$ a bi-orthogonal family of $\left\{\mathrm{e}^{-k^2 t}\right\}_{k\geq 1}$ in $L^2(0;T)$ which satisfy the condition:

$$v_k = -\frac{\mathrm{e}^{-\lambda_k T}}{b_k}\left(y_k^0, \omega_k\right), \quad \forall k \geq 1 \tag{31}$$

this is to say

$$v(t) = -\sum_{k\geq 1} \frac{\mathrm{e}^{-\lambda_k T}}{b_k}\left(y_k^0, \omega_k\right) q_k(t), \quad \forall k \geq 1 \tag{32}$$

Therefore by estimation, Khodja [7] and Tucsnak [17], we have

$$\|q_k\|_{L^2} \leq C_\varepsilon \mathrm{e}^{\varepsilon k^2}, \quad \forall \varepsilon > 0 \tag{33}$$

where C_ε a constant depending only on ε and (3.12) becomes

$$\beta_k(T)v(T)\int_0^T \mathrm{e}^{-k^2 t} \mu(x)\mathrm{d}t = -\mathrm{e}^{-k^2 T} y_k^0.$$

If $\left\{\mathrm{e}^{-k^2 t}\right\}_{k\geq 1}$ admits a bi-orthogonal family $\{q_k\}_{k\geq 1}$, then $\int_0^T \mathrm{e}^{-k^2 t} q_l = \delta_{kl}$ (Kronecker symbol), which finally gives

$$v(T) = \frac{\sum_{k\geq 1} e^{-k^2 T} y_k^0 \cdot q_k}{\beta_k(T)\mu(x)} \tag{34}$$

now the system (20) is null controllable if $v \in L^2(0;T)$ and if and only if $\beta_k \neq 0$ because $\mu(x) \neq 0$ (μ is strategic on I).

Let's take a look at norm of $v(T)$?

$$\|v(T)\|^2 = \left\| \frac{e^{-k^2 T} y_k^0 \cdot q_k}{\beta_k(T)\mu(x)} \right\|_{L^2(I)}^2 = \frac{e^{-2k^2 T}}{\beta_k^2 \cdot \mu^2(x)} \|y_k^0\|^2 \cdot \|q_k\|^2 \tag{35}$$

And we had according to the theorem 4.1 of Khodia [7],

$$\|q_k\|_{L^2} \leq C_\varepsilon \frac{e^{\varepsilon \cdot k^2}}{|E'(k^2)|} \tag{36}$$

where

$$E(x) = \prod_{k\geq 1}\left(1 - \frac{x^2}{\lambda_k^2}\right); x \in \mathbb{R} \text{ denotes the associated interpolation function} \tag{37}$$

then the inequality (3.16) becomes:

$$\frac{e^{-2k^2 T}}{\beta_k^2 \cdot \mu^2(x)} \|y_k^0\|^2 \cdot \|q_k\|^2 \leq C_\varepsilon \frac{e^{-2k^2 T}}{\beta_k^2 \cdot \mu^2(x)} \|y_k^0\|^2 \cdot \frac{e^{2\varepsilon \cdot k^2}}{|E'(k^2)|^2} \tag{38}$$

$$\leq C_\varepsilon \|y_k^0\|_{L^2}^2 e^{-2k^2(T-\varepsilon)} \cdot e^{\log\left(\frac{1}{\beta_k^2} \cdot \frac{1}{\mu^2(x)} \cdot \frac{1}{|E'(k^2)|^2}\right)} \tag{39}$$

$$\frac{e^{-2k^2 T}}{\beta_k^2 \cdot \mu^2(x)} \|y_k^0\|_{L^2}^2 \cdot \|q_k\|_{L^2}^2 \leq C_\varepsilon e^{-2k^2\left(T - \varepsilon - \frac{\log\frac{1}{\beta_k} + \log\frac{1}{\mu(x)} + \log\frac{1}{|E'(k^2)|}}{k^2}\right)} \tag{40}$$

Now let's pose

$$T_0^\mu = \operatorname{limsup}_{k\geq 1} \frac{\log\left(\frac{1}{\beta_k}\right) + \log\left(\frac{1}{\mu(x)}\right) + \log\left(\frac{1}{|E'(k^2)|}\right)}{k^2} \tag{41}$$

the minimal time of null controllabilty of system (20), then we obtain

$$T_0^\mu = T_0 + \frac{\log\left(\frac{1}{\mu(x)}\right)}{k^2} \text{ with } T_0 = \operatorname{lim\,sup}_{k\geq 1} \frac{\log\frac{1}{\beta_k} + \log\frac{1}{|E'(k^2)|}}{k^2}.$$

Then

$$\|v(T)\|_{L^2}^2 = \sum_{k\geq 1} \frac{e^{-2k^2 T}}{\beta_k^2 \cdot \mu^2(x)} \|y_k^0\|^2 \cdot \|q_k\|^2 \leq \sum_{k\geq 1} C_\varepsilon e^{-2k^2\left(T - \varepsilon - T_0^\mu\right)}$$

so, (20) is null controllable if and only if $T > T_0^\mu$ and $\mu(x) > 1$. □

4.2. Controllability on the F_T Space

The spaces F_T et F_T^* have been defined previously; and the same calculations will be repeated on these spaces.

Remark. *We can thus notice that by construction:*

i) $H_0^1 \subset F_T \subset L^2$

ii) $L^2 \subset F_T^* \subset H^{-1}$

iii) What we can summarize on the following diagram (see Figure 1).

Taking back the following system (20):

$$\begin{cases} y_t - \partial_{xx} y = \beta(t)\mu(x)u(t) \text{ in } Q_T \\ y(0;t) = y(\pi;t) = 0 \\ y(0) = y_0 \end{cases} \tag{42}$$

By setting $B = \beta(t)u(t)$ as a linear control operator and we resume the calculations on the spaces F_T and F_T^*; then the previous Equation (20) becomes:

$$\begin{cases} y_t - \partial_{xx} y = B\mu(x) \text{ in } Q_T \\ y(0;t) = y(\pi;t) = 0 \\ y(0) = y_0 \end{cases} \tag{43}$$

Then the solution of Equation (20) is done:

$$y(t) = \mathrm{e}^{t\Delta} y_0 + \int_0^t \mathrm{e}^{(t-s)\Delta} B\mu(x)\,\mathrm{d}s, \tag{44}$$

Knowing that $\partial_{xx} y = \Delta y$ The Equation (20) is null controllable at time $T > 0$ in F_T if $y(T) = 0$ which equals

$$-\mathrm{e}^{T\Delta} y_0 = \int_0^T \mathrm{e}^{(T-s)\Delta} B\mu(x)\,\mathrm{d}s. \tag{45}$$

If the solution $y \in F_T(0;T)$, we have $y(t;x) = \sum_{k\ge1} y_k(t) w_k$ et $y_0 = \sum_{k\ge1} y_k^0 w_k$. Likewise B is in F_T and is written:

$$B = \sum_{k\ge1} b_k(t) w_k; \text{ with } w_k = \sqrt{\frac{2}{\pi}} \sin(k\pi x). \tag{46}$$

So (20) becomes:

$$\begin{cases} y_k' + k^2 y_k = b_k(t)\mu(x);\ k \ge 1 \\ y_k|_{t=0} = y_k^0 \end{cases} \tag{47}$$

Hence the solution becomes:

$$y_k(t) = \mathrm{e}^{-k^2 t} y_k^0 + b_k(t) \int_0^t \mathrm{e}^{-k^2(t-s)} \mu(x)\,\mathrm{d}s; \quad k \ge 1 \tag{48}$$

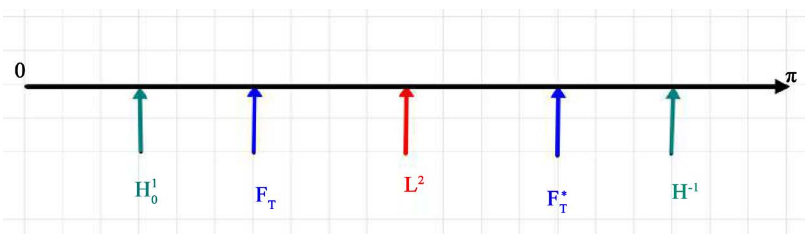

Figure 1. Pivot space L^2.

(20) is null controllable at time $T>0$ if and only if $y_k(T)=0 \Leftrightarrow$

$$b_k(t)\int_0^T \mathrm{e}^{-k^2(T-s)}\mu(x)\mathrm{d}s = -\mathrm{e}^{-k^2T}y_k^0 \tag{49}$$

We have $b_k(T)=\beta_k(T)u(T)$, and (2.23) becomes:

$$\beta_k(T)u(T)\int_0^T \mathrm{e}^{-k^2(T-s)}\mu(x)\mathrm{d}s = -\mathrm{e}^{-k^2T}y_k^0 \tag{50}$$

Let's change the variable

$$v(t)=u(T-t)$$

and then we have $v(t)=\sum_{k\ge1} v_k q_k(t)$ with $\{q_k\}_{k\ge1}$ a bi-orthogonal family of $\left\{\mathrm{e}^{-k^2t}\right\}_{k\ge1}$ in $F_T(0;T)$ which satisfy the condition:

$$\|q_k\|_{F_T} \le C_{1;\varepsilon}\mathrm{e}^{\varepsilon k^2} \tag{51}$$

with $C_{1;\varepsilon} \le C_\varepsilon$ where C_ε was the constant of inequality (2.15) because we have $F_T \subset L^2(I)$ by taking the same calculations, we end up with:

$$\|v(T)\|_{F_T} = \sum_{k\ge1} \frac{\mathrm{e}^{-2k^2T}}{\beta_k^2\cdot\mu^2(x)}\left\|y_k^0\right\|_{F_T}^2 \cdot \|q_k\|_{F_T}^2 \le \sum_{k\ge1} C_{1;\varepsilon}\left\|y_k^0\right\|_{F_T}^2 \mathrm{e}^{-2k^2\left(T-\varepsilon-T_0^\mu\right)} \tag{52}$$

So the system (20) is null controllable if

$$\sum_{k\ge1}\mathrm{e}^{-2k^2\left(T-\varepsilon-T_0^\mu\right)} < \infty \tag{53}$$

that is to say if $T>T_0^\mu$ and $\mu(x)>1$. ■

Remark. 1) There is no uniqueness of the control profile bringing the system of the initial condition y_0 to the final state (the set of strategic profiles is a closed affine subspace: we can naturally choose a norm control minimal on F_T as being the projection of 0 on this convex).

2) In Khodja *et al.* [7], it has been shown that there is a minimal time $T_0<T$ and that if $T>T_0$, the Equation (20) is null controllable (*i.e.* $y(T)=0$ otherwise not controllable).

3) In this theorem 3, we show that there is a minimal time T_0^μ to this T_0 *i.e.* $T_0^\mu < T_0$ for which we have null controllability.

4) This result of theorem 3.1 was obtained under the condition that the strategic profile zone $\mu>1$.

5) Indeed, if a profile μ is strategic over an interval then $C\cdot\mu$ (where C a constant) is still strategic.

6) Thereby the set $\mathcal{A}=\left\{\mu\in L^2(I)/\mu \text{ strategic actuator}\right\}$ is not empty.

5. Conclusions and Perspectives

In the literature, the controllability of the heat equation has been established since the mid-90s by Lebeau and Fursikov. In all these works and others more recent, there is always a time T_0 from which the control is realizable. Our aim was to find a better minimum time to carry out this control.

So, in this work, we were based on the work of Khodja [7] *et al.* and Tusnack [17] *et al.* to find a better time to achieve the null controllability of the 1-D heat equation. This goal was achieved with another simpler approach and the addition of a strategic profile assumption.

Another work is being finalized to find a minimum cost linked to this minimum time to obtain the null controllability of the heat equation.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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